The Adjusted Winner Procedure: Characterizations and Equilibria

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Abstract

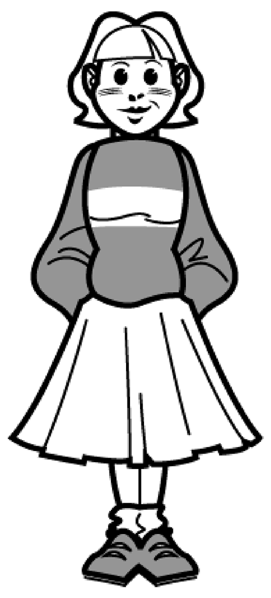
The Adjusted Winner procedure is an important mechanism proposed by Brams and Taylor for fairly allocating goods between two agents. It has been used in practice for divorce settlements and analyzing political disputes. Assuming truthful declaration of the valuations, it computes an allocation that is envy-free, equitable and Pareto optimal. We show that Adjusted Winner admits several elegant characterizations, which further shed light on the outcomes reached with strategic agents. We find that the procedure may not admit pure Nash equilibria in either the discrete or continuous variants, but is guaranteed to have ε-Nash equilibria for each ε > 0. Moreover, under informed tie-breaking, exact pure Nash equilibria always exist, are Pareto optimal, and their social welfare is at least 75% of the optimal.

Background: Adjusted Winner

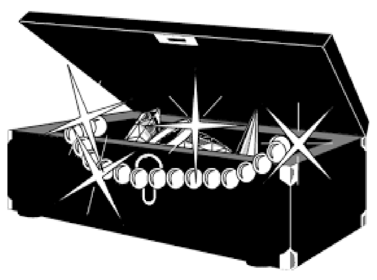
Algorithm for allocating multiple goods among two parties (treatment in several books – Brams and Taylor, 1996; 2000)

Adjusted Winner guarantees very desirable fairness properties of the outcome: envy-free, equitable, and Pareto efficient.

Advocated for settling inheritances and divorces, international border conflicts, political issues, real estate disputes, water disputes, deciding debate formats, various negotiation settings, …


Alice


Bob



Preferences over goods given by vectors $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$. Alice and Bob get points that they can use to acquire items; discrete and continuous settings.

Allocation : assignment of fractions of items (bundles) to players; say $W_A = (w_A^1, \ldots, w_A^m) \in [0,1]^m$ and $W_B = (w_B^1, \ldots, w_B^m) \in [0,1]^m$;

Utility : additive over the items received.

Procedure: Mediator asks the players to state their values, then:

Phase 1 : For every item i, if $a_i > b_i$, then give the item to Alice, else to Bob.
Let (W_A, W_B) be the resulting allocation; w.l.o.g. $u_a(W_A) \geq u_b(W_B)$.

Phase 2 : Order the items won by Alice increasingly by the ratio a_i/b_i : $a_{k1}/b_{k1} \leq \ldots \leq a_{km}/b_{km}$. From left to right, continuously transfer fractions of items from Alice to Bob, until an allocation where the players have the same utility is reached: $u_a(W'_A) = u_b(W'_B)$

Alternative view of the algorithm

An allocation is ***ordered*** if it can be produced by sorting the items in decreasing order of valuation ratios a_i/b_i and placing a boundary line somewhere (possibly splitting an item)

$$\underbrace{\frac{a_{k_1}}{b_{k_2}} \geq \frac{a_{k_2}}{b_{k_2}} \geq \cdots \geq \frac{a_{k_i}}{b_{k_i}} \geq}_{\text{Alice's allocation}} \Big| \geq \underbrace{\frac{a_{k_{i+1}}}{b_{k_{i+1}}} \geq \cdots \geq \frac{a_{k_m}}{b_{k_m}}}_{\text{Bob's allocation}}$$

Adjusted Winner produced an ordered allocation with the property that the boundary line is placed appropriately to guarantee equitability

Characterizations

Theorem: The Adjusted Winner procedure is the only Pareto efficient, equitable, and minimally fractional protocol. Any ordered equitable allocation can be produced by Adjusted Winner under some tie-breaking rule.

Theorem: The Adjusted Winner procedure is equivalent to the protocol that always outputs a maxmin and minimally fractional allocation.

Equilibrium existence

Prior to this work, there were examples of manipulations, but nothing solved beyond examples; Brams and Taylor also computed a one-step best reply starting from the true valuation profile.

Alice and Bob can report arbitrary valuation vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ instead of the true valuations; (x, y) are their strategies.

Strategy profile (x, y) is an *ε-Nash equilibrium* if no player can increase its utility by more than ε by deviating to a different (pure) strategy. For ε = 0, we obtain a *pure Nash equilibrium*.

In a book review for Brams and Taylor (1996), Nalebuff writes :

…thus we have to hypothesize how they (the players) would have played the game and where they would have ended up.

Theorem: the existence of pure Nash equilibria (exact and approximate) of Adjusted Winner is given by the following table.

Continuous Procedure	Lexicographic tie-breaking	Informed tie-breaking
pure Nash	✗	✓
ε-Nash	✓	✓

Discrete Procedure	Lexicographic tie-breaking	Informed tie-breaking
pure Nash	✗	✓
ε-Nash	✓(*)	✓

In fact, the equilibria have surprisingly good properties…

Theorem: All the exact equilibria of Adjusted Winner are envy-free and Pareto efficient with respect to the true valuations. Moreover, their social welfare is 75% of the optimal welfare.