The Big Match in Small Space

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Abstract

In this paper we study how to play (stochastic) games optimally using little space. We focus on repeated games with absorbing states, a type of two-player, zero-sum concurrent mean-payoff games. The prototypical example of these games is the well known Big Match of Gillette (1957). These games may not allow optimal strategies but they always have $\varepsilon$-optimal strategies. In this paper we design $\varepsilon$-optimal strategies for Player 1 in these games that use only $O(\log \log T)$ space. Furthermore, we construct strategies for Player 1 that use space $s(T)$, for an arbitrary small unbounded non-decreasing function $s$, and which guarantee an $\varepsilon$-optimal value for Player 1 in the limit superior sense. The previously known strategies use space $\Omega(\log T)$ and it was known that no strategy can use constant space if it is $\varepsilon$-optimal even in the limit superior sense. We also give a complementary lower bound.

Furthermore, we also show that no Markov strategy, even extended with finite memory, can ensure value greater than 0 in the Big Match, answering a question posed by Abraham Neyman.

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1 Introduction

In game theory there has been considerable interest in studying the complexity of strategies in infinitely repeated games. A natural way how to measure the complexity of a strategy is by the number of states of a finite automaton implementing the strategy. A common theme is to consider what happens when some or all players are restricted to play using a strategy given by an automaton of a certain bounded complexity.

Asymptotic view. Previous works have mostly been limited to dichotomy results: either there is a good strategy implementable by finite automaton or there is no such strategy. Our goal here is to refine this picture. We do this by taking the asymptotic view: measuring the complexity as a function of the number of rounds played in the game. Now when the strategy no longer depends just on a finite amount of information about the history of the game it could even be a computationally difficult problem to decide the next move of the strategy. But we focus on investigating how much information a good strategy must store about the play so far to decide on the next move; in other words, we study how much space the strategy needs.

Game classes. The class of games we study is that of repeated zero-sum games with absorbing states. These form a special case of undiscounted stochastic games. Stochastic games were introduced by Shapley [14], and they constitute a very general model of games proceeding in rounds. We consider the basic version of two-player zero-sum stochastic games with a constant number of states and a constant number of actions. In a given round $t$ the two players simultaneously choose among a number of different actions depending on the current state. Based on the choice of the pair $(i,j)$ of actions as well as the current state $k$, Player 1 receives a reward $r_t = a^t_{ij}$ from Player 2, and the game proceeds to the next state $\ell$ according to probabilities $p^k_{ij}$.

Limit-average rewards. In Shapley’s model, in every round the game stops with non-zero probability, and the payoff assigned to Player 1 by a play is simply the sum of rewards $r_t$. The stopping might be viewed as discounting later rewards by a discounting factor $0 < \beta < 1$. Gillette [5] considered the more general model of undiscounted stochastic games where all plays are infinite. He is interested in the average reward $\frac{1}{T} \sum_{t=1}^{T} r_t$ to Player 1 as $T$ tends to infinity. As the limit may not exist one needs to consider lim inf, lim sup, or some Banach limit [15] of the sums. In many cases the particular choice of the limit does not matter much, but it turns out that for our results it has interesting consequences. For this reason we consider both $\liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t$ and $\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t$.

Note that both these notions have natural interpretations. For instance, the lim inf notion suits the setup where the infinite repeated game actually models a game played repeatedly for an unspecified (but large) number of rounds, where one thus desires a guarantee on the average reward after a certain number of rounds. The lim sup notion on the other hand models the ability to always recover from arbitrary losing streaks in the repeated game.

The Big Match. A prototypical example of an undiscounted stochastic game is the well-known Big Match of Gillette [5] (see Figure 1 for an illustration of the Big Match). This game fits also into an important special subclass of undiscounted stochastic games: the repeated games with absorbing states, defined by Kohlberg [11]. In a repeated game with absorbing states there is only one state that can be left; all the other states are absorbing, i.e., the probability of leaving them is zero regardless of the actions of the players. Even in these games, as for general undiscounted stochastic games, there might not be an optimal strategy for the players [5]. On the other hand there always exist $\varepsilon$-optimal strategies [11], which are strategies guaranteeing the value of the game up to an
additive term \( \varepsilon \). The Big Match provides such an example: the value of the game is \( 1/2 \), but Player 1 does not have an optimal strategy, and must settle for an \( \varepsilon \)-optimal strategy \([2]\). On the other hand, it is known that such \( \varepsilon \)-optimal strategies in the Big Match must have a certain level of complexity. More precisely, for any \( \varepsilon < \frac{1}{2} \), an \( \varepsilon \)-optimal strategy can neither be implemented by a finite automaton nor take the form of a Markov strategy (a strategy whose only dependence on the history is the number of rounds played) \([16]\).

In this paper we consider the Big Match in particular and then generalize our results to general repeated games with absorbing states.

**The model under consideration.** We are interested in the space complexity of \( \varepsilon \)-optimal strategies in repeated games with absorbing states. A general strategy of a player in a game might depend on the whole history of the play up to the current time step. Moreover the decision about the next move might depend arbitrarily on the history. This provides the strategies with lots of power. There are two natural ways how to restrict the strategies: one can put computational restrictions on how the next move is decided based on the history of the play, or one can put a limit on how much information can the strategy remember about the history. One can also combine both types of restrictions, which leads to an interactive Turing machine based model, modelling a dynamic algorithm.

In this paper we mainly focus on restricting the amount of information the strategy can remember. This restriction is usually studied in the form of how large size a finite automaton (transducer) for the strategy has to be, and we follow this convention. By the size of a finite automaton we mean the number of states. The automats we consider can make use of probabilistic transitions, and we will not consider the description of these probabilities as part of the size of the automaton. We do address these separately, however.

**History of the model.** The idea of measuring complexity of strategies in repeated games in terms of automata was proposed by Aumann \([1]\). The survey by Kalai \([10]\) further discuss the idea in several settings of repeated games. However in this line of research the finite automata is assumed to be fixed for the duration of the game. This represents a considerable restriction as for many games there is no good strategy that could be described in this setting. Hence we consider strategies in which the automata can grow with time. To be more precise we consider infinite automata and measure how many different states we could have visited during the first \( T \) steps of the play. The logarithm of this number corresponds to the amount of space one would need to keep track of the current state of the automaton. We are interested in how this space grows with the number of rounds of the play.

**Comparison of our model with a Turing machine based model.** To impose also computational restrictions on the model, one can consider the usual Turing machine with one-way input and output tapes that work in lock-step and that record the play: whenever the machine writes its next action on the output tape it advances the input head to see the corresponding move of the other player. The space usage of the model is then the work space used by the machine, growing with the number of actions processed. The Turing machine can be randomized to allow for randomized strategies. The main differences between this model and the automaton based model we focus on in this paper is that in the case of infinite automata the strategy can be *non-uniform* and use arbitrary probabilities on its transitions whereas the Turing machine is *uniform* in the sense that it has a finite program that is fixed for the duration play and in particular, all transition probabilities are explicitly generated by the machine.

**Bounds for strategies with deterministic update.** Trivially, any strategy needs space at most
since such memory would suffice to remember the whole history of the play. It is not hard to see (cf. [9, Chap. 3.2.1]) that if a strategy is not restricted to a finite number of states, then the number of reachable states by round \( T \) must be at least \( T \). This means that the space needed by any such strategy is \( \Omega(\log T) \). However this provides only worst-case answer to our question, since for randomized strategies it might happen that only negligible fraction of the states can be reached with reasonable probability. Indeed, it might be that with probability close to 1 the strategy reaches only a very limited number of states. This is the setup we are interested in. As we will see in a moment the strategies we consider use substantially less space than \( O(\log T) \) with high probability (and \( O(\log T) \) space in the worst case).

**Relationship to data streaming** We find that our question is naturally related to algorithmic questions in data streaming. In data streaming one tries to estimate on-line various properties of a data stream while minimizing the amount of information stored about the stream. As we will see our solutions borrow ideas from data streaming in particular, we use sampling to estimate properties of the play so far. It is rather interesting that this is sufficient for a large class of games.

### 1.1 Our results

We provide two types of results. We show that there are \( \varepsilon \)-optimal strategies for repeated games with absorbing states, and we also show that there are limits on how small space such strategies could possibly use. Our strategies are first constructed for the Big Match. Then, following Kohlberg [11] these strategies are extended to general repeated games with absorbing states.

**Upper bounds on space usage.** Our first results concern the Big Match. We show that for all \( \varepsilon > 0 \), there exists an \( \varepsilon \)-optimal strategy that uses \( O(\log \log T) \) space with probability \( 1 - \delta \) for any \( \delta > 0 \). We note that the previous constructed strategies of Blackwell and Ferguson [2] and Kohlberg [11] uses space \( \Theta(\log T) \).

**Theorem 1.** For all \( \varepsilon > 0 \), there is an \( \varepsilon \)-optimal strategy \( \sigma_1 \) for Player 1 in the Big Match such that for any \( \delta > 0 \) with probability at least \( 1 - \delta \), the strategy \( \sigma_1 \) uses \( O(\log \log T) \) space in round \( T \).

**Remark.** We would like to stress the order of quantification above and their impact on the big-O notation used above for conciseness. The strategy we build depends on the choice of \( \varepsilon \), but only for the actions made – the memory updates are independent thereof, and thus likewise is the space usage. The dependence of \( \delta \) is also very benign. More precisely, there exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( \delta \), and an integer \( T_0 \) depending on \( \delta \), but independent of \( \varepsilon \), in such a way that with probability at least \( 1 - \delta \), the strategy \( \sigma_1 \) uses at most space \( C \log \log T \), for all \( T \geq T_0 \). The same remark holds elsewhere in our statements.

**Our results translated to the Turing based model.** After a slight modification our \( \varepsilon \)-optimal strategy can be implemented by a Turing machine so that (1) it processes \( T \) actions in time \( O(T) \); and (2) each time it processes an action, all randomness used comes from at most 1 unbiased coin flip; and (3) it, for all \( \delta > 0 \), uses \( O(\log \log T + \log \log \varepsilon^{-1}) \) space with probability \( 1 - \delta \), before round \( T \). See Corollary [19]

**Arbitrary small, but growing space for \( \limsup \).** For the case of \( \limsup \) evaluation of the average rewards we can design strategies that uses even less space, in fact arbitrarily small, but growing, space.
**Theorem 2.** For any non-decreasing unbounded function \( s \), there exists an \( \varepsilon \)-supremum-optimal strategy \( \sigma_1 \) for Player 1 in the Big Match such that for each \( \delta > 0 \), with probability at least \( 1 - \delta \), strategy \( \sigma_1 \) uses \( O(s(T)) \) space in round \( T \).

We may for instance think of \( s \) as the inverse of the Ackermann function. Although the strategy from this theorem has very uniform description it might not always be implementable by a Turing machine running in the same space since the machine needs to sample probabilistic events comparable to \( 1/T \) or smaller. That might not be achievable in small space using just a fair coin.

Our strategy that is \( \varepsilon \)-optimal is actually an instantiation of the \( \varepsilon \)-supremum-optimal strategy to the setting of \( O(\log \log T) \) space. We are unable to achieve \( \varepsilon \)-optimality in less space, and this seems to be inherent to our techniques.

**Generalization to repeated games with absorbing states.** We can generalize the above statements to the case of general repeated games with absorbing states.

**Theorem 3.** For all \( \varepsilon > 0 \) and any repeated game with absorbing states \( G \), there is an \( \varepsilon \)-optimal strategy \( \sigma_k \) for Player \( k \) in \( G \) such that, for each \( \delta > 0 \), with probability at least \( 1 - \delta \), the strategy \( \sigma_k \) uses \( O(\log \log T + \log 1/\varepsilon \cdot \text{poly}(|G|)) \) space in round \( T \).

**Theorem 4.** For all \( \varepsilon > 0 \), any repeated game with absorbing states \( G \), and any non-decreasing unbounded function \( s \), there exists an \( \varepsilon \)-supremum optimal strategy \( \sigma_k \) for Player \( k \) in \( G \) such that for each \( \delta > 0 \), with probability at least \( 1 - \delta \), the strategy \( \sigma_1 \) uses \( O(s(T) + \log 1/\varepsilon \cdot \text{poly}(|G|)) \) space in round \( T \).

These strategies are obtained by reducing to a special simple case of repeated games with absorbing states, generalized Big Match games, to which our Big Match strategies can be generalized. This reduction can furthermore be done effectively by a polynomial time algorithm.

**Lower bound on space usage.** We provide two lower bounds on space addressing different aspects of our strategies. One property of our strategies is that the smaller the space used is, the smaller the probabilities of actions employed are. The reciprocal of the smallest non-zero probability is the patience of a strategy. This is a parameter of interest for strategies. We show that the patience of our strategies is close to optimal. In particular, we show that the first \( f(T) \) memory states must use probabilities close to \( 1/T^{f(T)} \), where \( s(T) = \log f(T) \) is the space usage. We can almost match this bound by our strategies.

**Finite-memory deterministic-update Markov strategies are no good.** Beside the lower bound on patience we investigate the possibility of using a good strategy for Player 1 which would use only a constant number of states but where the actions could also depend on the round number. This is what we call a finite-memory Markov strategy. We show that such a strategy which also updates its memory state deterministically cannot exist. This answers a question posed by Abraham Neyman.

**Theorem 5.** For all \( \varepsilon < \frac{1}{2} \), there exists no finite-memory deterministic-update \( \varepsilon \)-optimal Markov strategy for Player 1 in the Big Match.

**1.2 Our techniques**

The previously given strategies for Player 1 in the Big Match [2, 11] use space \( \Theta(\log T) \) as they maintain the count of the number of different actions taken by the other player. There are two
principal ways how one could try to decrease the number of states for such randomized strategies: either to use approximate counters [13, 4], or to sub-sample the stream of actions of the other player and use a good strategy on the sparse sample. In this paper we use the latter approach.

**Overview over our strategy for the Big Match.** Our strategies for Player 1 proceed by observing the actions of Player 2 and collecting statistics on the payoff. Based on these statistics Player 1 adjusts his actions. The statistics is collected at random sample points and Player 1 plays according to a “safe” strategy on the points not sampled and plays according to a good (but space-inefficient) strategy on the sample points. If the space of Player 1 is at least log log T then Player 1 is able to collect sufficient statistics to accurately estimate properties of the actions of Player 2. Namely, substantial dips in the average reward given to Player 1 can be detected with high probability and Player 1 can react accordingly. Thus that during infinite play, the average reward will not be able drop for extended periods of time, and this will guarantee that lim inf evaluation of the average rewards is close to the value of the game.

**The bottle-neck in the lim inf case.** However, if our space is considerably less than log log T we do not know how to accurately estimate these properties of the actions of Player 2. Thus, long stretches of actions of Player 2 giving low average rewards might go undetected as long as they are accompanied by stretches of high average rewards. Thus one could design a strategy for Player 2 that has low lim inf value of the average rewards, but has large lim sup value. Against such a strategy, our space-efficient strategy for Player 1 is unlikely to stop. So during infinite play, while our strategy guarantees that the lim sup evaluation of the average rewards is close to the value of the game, it performs poorly under lim inf evaluation. It is not clear whether this is an intrinsic property of all very small space strategies for Player 1 or whether one could design a very small space strategy achieving that the lim inf evaluation of the average rewards is close to the value of the game. We leave this as an interesting open question.

**Generalizing to repeated games with absorbing states.** Our extension to general repeated games with absorbing states follow closely the work of Kohlberg [11]. He showed that all such games have a value and constructed ε-optimal strategies for them, building on the work of Blackwell and Ferguson [2]. His construction is in two steps: The question of value and of ε-optimal strategies are solved for a special case of repeated games with absorbing states, generalized Big Match games, that are sufficiently similar to the Big Match game that one of the strategies given by Blackwell and Ferguson [2] can be extended to this more general class of games. Having done this, Kohlberg shows how to reduce general repeated games with absorbing states to generalized Big Match games.

In a similar way we can extend our small-space strategies for the Big Match to the larger class of generalized Big Match games. These can then directly be used for Kohlberg’s reduction. This reduction is however only given as an existence statement. We show how the reduction can be made explicit and computed by a polynomial time algorithm. This is done using linear programming formulations and fundamental root bounds of univariate polynomials. This also provides explicit bounds on the bit size of the reduced generalized Big Match games. We also give a simple polynomial time algorithm for approximating the value of any repeated game with absorbing states based on bisection and linear programming.

2 Definitions

**Probability distributions.** A *probability distribution* over a finite set $S$, is a map $d : S \rightarrow [0, 1]$, such that $\sum_{s \in S} d(s) = 1$. Let $\Delta(S)$ denote the set of all probability distributions over $S$.  

Repeated games with absorbing states. The games we consider are special cases of two player, zero-sum concurrent mean-payoff games in which all states except at most one are absorbing, i.e. never left if entered (note also that an absorbing state can be assumed to have just a single action for each player). We restrict our definitions to this special case, introduced by Kohlberg \[11\] as repeated games with absorbing states. Such a game \( G \) is given by sets of actions \( A_1 \) and \( A_2 \) for each player together with maps \( \pi : A_1 \times A_2 \to \mathbb{R} \) (the stage payoffs) and \( \omega : A_1 \times A_2 \to [0,1] \) (the absorption probabilities).

The game \( G \) is played in rounds. In every round \( T = 1,2,3,\ldots \), each player \( k \in \{1,2\} \) independently picks an action \( a_T^k \in A_k \). Player 1 then receives the stage payoff \( \pi(a_T^1,a_T^2) \) from Player 2. Then, with probability \( \omega(a_T^1,a_T^2) \) the game stops and all payoffs of future rounds are fixed to be \( \omega(a_T^1,a_T^2) \) (we may think of this as the game proceeding to an absorbing state where the (unique) stage payoff for future rounds is \( \pi(a_T^1,a_T^2) \)). Otherwise, the game just proceeds to the next round.

The sequence \( (a_1^1,a_2^1),(a_2^2,a_2^2),(a_3^1,a_2^3),\ldots \) of actions taken by the two players is called a play. A finite play occurs when the game stops after the last pair of actions. Otherwise the play is infinite.

To a given play \( P \) we associate an infinite sequence of rewards \( (r_T)_{T \geq 1} \) received by Player 1. If \( P = (a_1^1,a_2^1),(a_2^2,a_2^2),\ldots,(a_1^T,a_1^T) \) is a finite play of length \( T \) we let \( r_T = \pi(a_1^T,a_2^T) \) for \( 1 \leq T \leq \ell \), and \( r_T = \pi(a_1^\ell,a_2^\ell) \) for \( T > \ell \). In this case we say that the game stops with outcome \( r_T \).

Otherwise, if \( P = (a_1^1,a_2^1),(a_2^2,a_2^2),\ldots \) is infinite we simply let \( r_T = \pi(a_1^T,a_2^T) \) for all \( T \geq 1 \).

To evaluate the sequence of the rewards we consider both the \( \liminf \) and \( \limsup \) value of the average reward \( \frac{1}{T} \sum_{t=1}^{T} r_t \). We thus define the limit-infimum payoff to Player 1 of the play as

\[
u_{\text{inf}}(P) = \liminf_{n \to \infty} \frac{1}{n} \sum_{T=1}^{n} r_T ,
\]

and similarly the limit-supremum payoff to Player 1 of the play as

\[
u_{\text{sup}}(P) = \limsup_{n \to \infty} \frac{1}{n} \sum_{T=1}^{n} r_T .
\]

Strategies. A strategy for Player \( k \) is a function \( \sigma_k : (A_1 \times A_2)^* \to \Delta(A_k) \) describing the probability distribution of the next chosen action after each finite play. We say that Player \( k \) follows a strategy \( \sigma_k \) if for every finite play \( P \) of length \( T-1 \), at round \( T \) Player \( k \) picks the next action according to \( \sigma_k(P) \). We say that a strategy \( \sigma_k \) is pure if for every finite play \( P \) the distribution \( \sigma_k(P) \) assigns probability 1 to one of the actions of \( A_k \) (i.e. the next action is uniquely determined).

Also, we say that a strategy \( \sigma_k \) is a Markov strategy if for every \( T \) and every play \( P \) of length \( T-1 \), the distribution \( \sigma_k(P) \) does not depend on the particular actions during the first \( T-1 \) rounds but is just a function of \( T \). Thus Markov strategy \( \sigma_k \) can be viewed as a map \( \mathbb{Z}_+ \to \Delta(A_k) \) or simply a sequence of distributions over \( A_k \).

A strategy profile \( \sigma \) is a pair of strategies \((\sigma_1, \sigma_2)\), one for each player. A strategy profile \( \sigma \) defines a probability measure on plays in the natural way. We define the expected limit-infimum payoff to Player 1 of the strategy profile \( \sigma = (\sigma_1, \sigma_2) \) as \( u_{\text{inf}}(\sigma) = u_{\text{inf}}(\sigma_1, \sigma_2) = E_{P \sim (\sigma_1, \sigma_2)}[u_{\text{inf}}(P)] \) and similarly the expected limit-supremum payoff to Player 1 of the strategy profile \( \sigma \) as \( u_{\text{sup}}(\sigma) = u_{\text{sup}}(\sigma_1, \sigma_2) = E_{P \sim (\sigma_1, \sigma_2)}[u_{\text{sup}}(P)] \).
Values and near-optimal strategies. We define the lower values of $G$ by $v_{\inf} = \inf_{\sigma_1} \sup_{\sigma_2} u_{\inf}(\sigma_1, \sigma_2)$ and $v_{\sup} = \sup_{\sigma_1} \inf_{\sigma_2} u_{\sup}(\sigma_1, \sigma_2)$, and we define the upper values of $G$ by $v_{\inf} = \inf_{\sigma_1} \sup_{\sigma_2} u_{\inf}(\sigma_1, \sigma_2)$ and $v_{\sup} = \sup_{\sigma_2} \sup_{\sigma_1} u_{\sup}(\sigma_1, \sigma_2)$. Clearly $v_{\inf} \leq v_{\sup} \leq v_{\sup}$ and $v_{\inf} \leq v_{\inf} \leq v_{\sup}$. Kohlberg showed that all these values coincide and we call this common number $v(G)$ the value $v$ of $G$.

**Theorem 6** (Kohlberg, Theorem 2.1). $v_{\inf} = v_{\sup}$.

Note that this also shows that for the purpose of defining the value of $G$ the choice of the limit of the average rewards does not matter. But a given strategy $\sigma_1$ for Player 1 could be close to guaranteeing the value with respect to lim sup evaluation of the average rewards, while being far from doing so with respect to the more restrictive lim inf evaluation. We shall hence distinguish between these different guarantees.

Let $\varepsilon > 0$ and let $\sigma_1$ be a strategy for Player 1. We say that $\sigma_1$ is $\varepsilon$-supremum-optimal, if

$$v(G) - \varepsilon \leq \inf_{\sigma_2} u_{\sup}(\sigma_1, \sigma_2)$$

and that $\sigma_1$ is $\varepsilon$-optimal, if

$$v(G) - \varepsilon \leq \inf_{\sigma_2} u_{\inf}(\sigma_1, \sigma_2).$$

**Observation 1.** Clearly it is sufficient to take the infimum over just pure strategies $\sigma_2$ for Player 2, and hence when showing that a particular strategy $\sigma_1$ is $\varepsilon$-supremum-optimal or $\varepsilon$-optimal we may restrict our attention to pure strategies $\sigma_2$ for Player 2.

One can naturally make similar definitions for Player 2, where the roles of lim inf and lim sup would then be interchanged, but we shall restrict ourselves here to the perspective of Player 1.

If the strategy $\sigma_1$ is 0-supremum-optimal (0-optimal) we simply say that $\sigma_1$ is supremum-optimal (optimal). The Big Match gives an example where Player 1 does not have a supremum-optimal strategy [2].

Memory and memory-based strategies. A memory configuration or state is simply a natural number. We will often think of memory configurations as representing discrete objects such as tuples of integers. In such a case we will always have a specific encoding of these objects in mind.

Let $\mathcal{M} \subseteq \mathbb{N}$ be a set of memory states. A memory-based strategy $\sigma_1$ for Player 1 consists of a starting state $m_s \in \mathcal{M}$ and two maps, the action map $\sigma^a_1 : \mathcal{M} \rightarrow \Delta(A_1)$ and the update map $\sigma^u_1 : A_1 \times A_2 \times \mathcal{M} \rightarrow \Delta(\mathcal{M})$. We say that Player 1 follows the memory-based strategy $\sigma_1$ if in every round $T$ when the game did not stop yet, he picks his next move $a^T_1$ at random according to $\sigma^a_1(m_T)$, where the sequence $m_1, m_2, \ldots$ is given by letting $m_1 = m_s$ and for $T = 1, 2, 3, \ldots$, choosing $m_{T+1}$ at random according to $\sigma^u_1(a^T_1, a^T_2, m_T)$, where $a^T_2$ is the action chosen by Player 2 at round $T$.

The strategies we construct in this paper have the property that their action maps do not depend on the action $a^T_2$ of Player 1. In these cases we simplify notation and write just $\sigma^a_1(a^T_2, m_T)$.

Since each finite play can be encoded by a binary string, and thus a natural number, we can view any strategy $\sigma_k$ for Player $k$ as a memory-based strategy. One can find similarly defined types of strategies in the literature, but typically, the function corresponding to the update function is deterministic.
Memory sequences and space usage of memory-based strategies. Let $\sigma_1$ be a memory-based strategy for Player 1 on memory states $M$ and $\sigma_2$ be a strategy for Player 2. Assume that Player 1 follows $\sigma_1$ and Player 2 follows $\sigma_2$. The strategy profile $(\sigma_1, \sigma_2)$ defines a probability measure on (finite and infinite) sequences over $M$ in the natural way. For a (finite) sequence $M \in M^*$, let $\omega_1(M)$ denote the probability that Player 1 follows this sequence of memory states during the first $|M|$ rounds of the game, while the game does not stop before round $|M|$.

Fix a non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a probability $p$. The strategy $\sigma_1$ uses $\log f(T)$ space with probability at least $p$ against $\sigma_2$, if for all $T$, the probability $\Pr_{(\sigma_1, \sigma_2)}[\forall i \leq T : M_i \leq f(T)] \geq p$ (i.e., with probability at least $p$, the current memory has stayed below that of $f(T)$ before round $T$, for all $T$). If $\sigma_1$ uses $\log f(T)$ space with probability at least $p$ against every strategy $\sigma'_2$, then we say that $\sigma_1$ uses $\log f(T)$ space with probability at least $p$.

The Big Match. The Big Match, introduced by Gillette [5] is a simply defined repeated game with absorbing states, where each player has only two actions. In each round Player 1 has the choice to stop the game (action $R$), or continue with the next round (action $L$). Player 2 has the choice to declare the round safe (action $L$) or unsafe (action $R$). If play continues in a round declared safe, or if play stops in a round declared unsafe, Player 2 must give Player 1 a reward 1. In the other two cases no reward is given.

More formally, let the action sets be $A_1 = A_2 = \{L, R\}$. The rewards are given by $\pi(a_1, a_2) = 1$ if $a_1 = a_2$ and $\pi(a_1, a_2) = 0$ if $a_1 \neq a_2$. The stopping probabilities are given by $\omega(R, a_2) = 1$ and $\omega(L, a_2) = 0$.

We can illustrate this game succinctly in a matrix form as shown in Figure 1, where rows are indexed by the actions of Player 1, columns are indexed by the actions of Player 2, entries give the rewards, and a star on the reward means that the game stops with probability 1 (See Section 8.1 for a general definition of the matrix form of a repeated game with absorbing states).

Density of pure Markov Strategies in the Big Match. When constructing strategies for Player 1 in the Big Match, not only is it sufficient to consider only pure strategies for Player 2 as noted in Observation [1] but we may restrict our consideration to pure Markov strategies, since Player 2 only ever observes the action $L$ of Player 1. An important property of a pure Markov strategy $\sigma$ for Player 2 in the Big Match is the density of $L$ actions of a prefix of $\sigma$.

Denote by $\sigma^T \in \{L, R\}^T$, the length $T$ prefix of $\sigma$. The density of $L$ in $\sigma^T$, denoted $\text{dens}(\sigma^T)$, is defined by

$$\text{dens}(\sigma^T) = \frac{|\{i \mid (\sigma^T)_i = L\}|}{T} .$$

Further, for $T' < T$ we define

$$\text{dens}(\sigma, T', T) = \frac{|\{i \geq T' \mid (\sigma^T)_i = L\}|}{T - T' + 1} .$$
Observation 2. Suppose Player 2 follows a pure Markov strategy $\sigma$. Then for any play $P$ and $T < |P|$ we have

$$\text{dens}(\sigma^T) = \frac{1}{T} \sum_{T'=1}^{T} r_{T'},$$

where $r_{T}$ is the reward given to Player 1 in round $T$. In particular, when $P$ is infinite, we have

$$u_{\text{inf}}(P) = \lim_{T \to \infty} \text{inf}_{T} \text{dens}(\sigma^T),$$

and

$$u_{\text{sup}}(P) = \lim_{T \to \infty} \text{sup}_{T} \text{dens}(\sigma^T).$$

3 Small space $\varepsilon$-supremum-optimal strategies in the Big Match

For given $\varepsilon > 0$, let $\xi = \varepsilon^2$. For any non-decreasing and unbounded function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, we will now give an $\varepsilon$-supremum optimal strategy $\sigma^i_1$ for Player 1 in the Big Match that for all $\delta > 0$ with probability $1 - \delta$ uses $O(\log f(T))$ space. Let $\overline{f}$ be a strictly increasing unbounded function from $\mathbb{Z}_+$ to $\mathbb{R}_+$, such that $\overline{f}(x) \leq f(x)$ for all $x \in \mathbb{Z}_+$, and let $F$ be the inverse of $\overline{f}$. For simplicity, and without loss of generality, we assume that $F(1) = 1$ and $F(T + 1) \geq 2 \cdot F(T)$. Note that in particular $\overline{f}(2 \cdot T) \leq 2 \cdot \overline{f}(T)$.

Intuitive description of the strategy and proof. The main idea for building the strategy is to partition the rounds of the game into epochs, such that epoch $i$ has expected length $F(i)$. The $i$’th epoch is further split into $i$ sub-epochs. In each sub-epoch $j$ of the $i$-th epoch we sample $i^2$ rounds uniformly at random. In every round not sampled we simply stay in the same memory state and play $L$ with probability 1. We view the $i^2$ samples as a stream of actions chosen by Player 2. We then follow a particular $\xi$-optimal base strategy $\sigma^i_1, \xi_1$ for the Big Match on the samples of sub-epoch $j$. This strategy $\sigma^i_1, \xi_1$ is a suitably modified version of a strategy by Blackwell and Ferguson [2] and Kohlberg [11].

More precisely, if $\sigma^i_1, \xi_1$ stops in its $k$-th round when run on the samples of sub-epoch $j$, the strategy $\sigma^i_1$ stops on the $k$-th sample in sub-epoch $j$. This will ensure that if $\sigma^i_1$ stops with probability at least $\sqrt{\xi}$, the outcome is at least $\frac{1}{2} - \xi$.

Also, for any $0 < \delta < \frac{1}{2}$ and for sufficiently large $i$, depending on $\delta$, if the samples have density of $L$ at most $\frac{1}{2} - \delta$ then $\sigma^i_1, \xi_1$ stops on the samples with a positive probability depending only on $\xi$, namely $\xi^4$. For $f(T) = \Theta(\log T)$, the division into sub-epochs ensures that if $\lim_{T \to \infty} \text{inf}_{T} \text{dens}(\sigma^T) < \frac{1}{2}$ then infinitely many sub-epochs have density of $L$ smaller than $1/2$, and thus the play stops with probability 1 in one of such epochs. This is not necessarily true for $f(T)$ smaller than $\log T$.

The base strategy. The important inner part of our strategy is a $\xi$-optimal strategy $\sigma^i_1, \xi_1$ parametrized by a non-negative integer $i$. These strategies are similar to $\xi$-optimal strategies given by Blackwell and Ferguson [2] and Kohlberg [11] (in fact, setting $i = 0$ and replacing $\xi^4$ by $\xi^2$ below one obtains the strategy used by Kohlberg).
The strategy $\sigma_{i,1}^{\xi}$ uses deterministic updates of memory, and uses integers as memory states (we think of the memory as an integer counter). The memory update function is given by

$$\sigma_{1,a}^{i,j}(a,j) = \begin{cases} j + 1 & \text{if } a = L \\ j - 1 & \text{if } a = R \end{cases}$$

and the action function is given by

$$\sigma_{1,a}^{i,j}(j)(R) = \begin{cases} \xi(1 - \xi)^{i+j} & \text{if } i + j > 0 \\ \xi^4 & \text{if } i + j \leq 0 \end{cases}$$

The complete strategy We are now ready to define $\sigma_{1}^{*}$. The memory states of this strategy are 5-tuples $(i,j,k,\ell,b) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{N} \times \{0,1\}$. Here $i$ denotes the current epoch and $j$ denotes the current sub-epoch of epoch $i$. The number of samples already made in the current sub-epoch is $k$. The memory state of the inner strategy is stored as $\ell$. Finally $b$ is 1 if and only if the strategy will sample to the inner strategy in the next step.

The memory update function $\sigma_{1}^{*}\ Texture continues on the next page...
Proof preliminaries. It will be useful to consider the strategy modified to never stop. Thus denote by $\tilde{\sigma}_1$ the strategy for Player 1, where $\tilde{\sigma}_1 = \sigma_1^{*\text{hi}}$ and $\tilde{\sigma}_1^q(m)(L) = 1$ for all memory states $m$.

We next define random variables indicating the locations of the sample steps. Fix some strategy $\sigma_1$ for Player 1. Let $M_{\sigma_1}$ be the memory sequence assigned to Player 1 when Player 1 follows $\tilde{\sigma}_1$ and Player 2 follows $\sigma_2$. For positive integers $i$, $j$, $k$ let $t(i, j, k)$ be the random variable indicating the round in which we sample the $k$’th time in sub-epoch $j$ of epoch $i$. For simplicity of notation we let $t(i, 0, i^2)$ denote $t(i - 1, i - 1, (i - 1)^2)$, and we let $t(i, j) = t(i, j, 0)$ denote $t(i, j - 1, i^2)$.

3.1 Space usage of the strategy

We will here consider the space usage of $\sigma_1^*$. First we will argue that with high probability, for all large enough $i$ and any $j$, the length of sub-epoch $j$ of epoch $i$, $t(i, j, i^2) - t(i, j - 1, i^2)$, is close to $F(i)$.

**Lemma 7.** For any $\gamma, \delta \in (0, 1/4)$, there is a constant $M$ such that with probability at least $1 - \gamma$, for all $i \geq M$ and all $j \in \{1, \ldots, i\}$, we have that

$$t(i, j, i^2) - t(i, j - 1, i^2) \in [(1 - \delta)F(i)/i, (1 + \delta)F(i)/i] .$$

**Proof.** The expected number of times we sample during $(1 - \delta)F(i)/i$ steps of the $i$’th epoch is $(1 - \delta)i^2$. If $t(i, j, i^2) - t(i, j - 1, i^2) < (1 - \delta)F(i)/i$ then we sampled at least $i^2$ times during these $(1 - \delta)F(i)/i$ steps of epoch $i$. This means that the actual number of samples is larger than its expectation by a factor $\delta/(1 - \delta)$. Thus by the multiplicative Chernoff bound, Theorem 45, we see that,

$$\Pr[t(i, j, i^2) - t(i, j - 1, i^2) < (1 - \delta)F(i)/i] < \exp \left( -c(1 - \delta)i^2 \right) ,$$

where $c = (\delta^2) / (2 + \frac{\delta}{1 - \delta})$. Similarly, if $t(i, j, i^2) - t(i, j - 1, i^2) > (1 + \delta)F(i)/i$ then we sampled less than $i^2$ times during $(1 + \delta)F(i)/i$ steps which is less than the expected by a factor $\delta/(1 + \delta)$ of its expectation. Again, by the multiplicative Chernoff bound,

$$\Pr[t(i, j, i^2) - t(i, j - 1, i^2) > (1 + \delta)F(i)/i] < \exp \left( -\frac{\delta^2}{2} (1 + \delta)i^2 \right) .$$

Thus for any $M$, we can bound from above the probability of any of the differences for $i \geq M$ being outside of the required range by:

$$\sum_{i=M}^{\infty} i \cdot \left( \exp \left( -c(1 - \delta)i^2 \right) + \exp \left( -\frac{\delta^2}{2} (1 + \delta)i^2 \right) \right) .$$

This sum is convergent so for sufficiently large $M$ it can be bounded by $\gamma$. The lemma follows. \qed

Now we bound the space usage of the strategy $\sigma_1^*$.

**Lemma 8.** For all constants $\gamma > 0$, with probability at least $1 - \gamma$, the space usage of $\sigma_1^*$ is $O(\log f(T))$. 

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**Proof.** Recall that there are at most $4 \cdot i^4$ distinct possible memory states reachable during the $j$th sub-epoch of epoch $i$, for all $i,j$. Thus, since there are $i$ sub-epochs in epoch $i$ there are at most $4 \cdot i^5$ distinct possible memory states reachable during the $i$th epoch. It is then clear that there are at most $\sum_{r=0}^{i} 4 \cdot r^5 \leq 4 \cdot i^6$ distinct possible memory states that can have been reached before the end of epoch $i$, for all $i$. Because memory states in earlier epochs are mapped to smaller numbers than latter epochs, we have that the strategy has not been in any state above that of $4 \cdot i^6$ before the end of epoch $i$.

Fix some $\gamma > 0$. By Lemma 7 with probability at least $1 - \gamma$, there is a $M$ such that for all $i \geq M$, the number $t(i,i,i^2) - t(i,1,i^2)$ is greater than $\frac{F(i)}{2}$. Thus also $t(i,i,i^2) \geq \frac{F(i)}{2}$. Consider any round $T$, for $T \geq t(M,M,M^2)$. Let $i$ be the epoch containing round $T$. By the preceding we know that, before time step $t(i+1,i+1,(i+1)^2)$ (which is greater than $T$), the strategy have only been in memory states below $4 \cdot (i+1)^6$. We also have that $T \geq t(i,i,i^2) \geq \frac{F(i)}{2}$ and hence $2 \bar{f}(T) \geq \bar{f}(2T) \geq i$, where the first inequality is by our assumption on $F$. Thus the strategy can only have been in states below that of $4 \cdot (2 \cdot \bar{f}(T)+1)^6$ before time step $T$. This is true for all sufficiently large $T$ and thus the strategy uses at most $O(\log f(T))$ space with probability $1 - \gamma$.  

### 3.2 Play stopping implies good outcome

We first establish some properties of the base strategy $\sigma_i^{i^2}$. The proof of these uses ideas similar to proofs by Blackwell and Ferguson [2] and Kohlberg [11], where they showed $\varepsilon$-optimality of their strategies.

**Lemma 9.** Let $T$, $i \geq 1$ be integers and $0 < \xi < 1$ be a real number. Let $\sigma \in \{L,R\}^T$ be a arbitrary prefix of a pure Markov strategy for Player 2. Consider the first $T$ rounds where the players play the Big Match following $\sigma_1^{i^2}$ and $\sigma$ respectively. Let $p_{\text{win}}$ be the probability that Player 1 stops the game (i.e. plays $R$) and wins. Let $p_{\text{loss}}$ be the probability that Player 1 stops the game and loses. Then we have:

1. 

$p_{\text{loss}} \leq (1 - \xi)^i \xi^3 + (1 - \xi)^{-1} p_{\text{win}}$.

2. For any $0 < \delta \leq \frac{1}{2}$ and any $T > i/(2\delta)$, if $\text{dens}(\sigma) \leq \frac{1}{2} - \delta$ then 

\[p_{\text{win}} + p_{\text{loss}} \geq \xi^4.

**Proof.** Define $d_\ell = |\{\ell' < \ell \mid \sigma_{\ell'} = L\}| - |\{\ell' < \ell \mid \sigma_{\ell'} = R\}|$. Note that $d_\ell$ is precisely the value of the counter used by $\sigma_1^{i^2}$ as memory in step $\ell$. For integer $d$, define

\[K_d = \{\ell \in \{1, \ldots, T\} \mid (d_\ell = d \& \sigma_\ell = L) \text{ or } (d_\ell = d+1 \& \sigma_\ell = R)\}.

There is an illustration of how $d_\ell$ could evolve through the steps in Figure 2. The set $K_d$ is then the times the counter moves between the pair of rows $d$ and $d+1$. Observe that the counter is alternately moving up and down in each $K_d$ (for instance, in the gray row, the arrows are wider and first moves up, then down and then up again).

Notice that $K_d$ partitions $\{1, \ldots, T\}$. Let $p_{\text{loss},d}$ be the probability that the game stops at some step $\ell \in K_d$ where $\sigma_\ell = L$, and let $p_{\text{win},d}$ be the probability that the game stops at some step $\ell \in K_d$ where $\sigma_\ell = R$. We see that $p_{\text{loss}}$ is the sum of $p_{\text{loss},d}$, and that $p_{\text{win}}$ is the sum of $p_{\text{win},d}$.
Figure 2: Possible movement of the counter $\sigma_1^j$ uses as memory through the steps.

$K_d = \{k_1 < k_2 < \cdots < k_m\}$, for some $m$. Observe that for any $j$, $\sigma_{kj} = \mathbf{L}$ if and only if $\sigma_{kj+1} = \mathbf{R}$, so $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{km}$ is an alternating sequence (as mentioned in relation to the illustration) starting with $\mathbf{L}$ when $d \geq 0$ and starting with $\mathbf{R}$ otherwise. For any $d \geq -i$, the probability that the game stops in round $k_j \in K_d$, conditioned on the event that it did not stop before round $k_j$, is $\xi^4(1-\xi)^{i+d}$ so, it is $\xi^4(1-\xi)^{i+d+1}$ when $\sigma_{kj+1} = \mathbf{L}$, and it is $\xi^4(1-\xi)^{i+d+1}$ when $\sigma_{kj+1} = \mathbf{R}$.

Hence, for $d \geq 0$, $p_{\text{loss},d} \leq \xi^4(1-\xi)^{i+d} + (1-\xi)^{-1}p_{\text{win},d}$. This is because we stop at $k_1$ with probability at most $\xi^4(1-\xi)^{i+d}$ (in which case Player 1 loses) and then for each even $j$, the probability of stopping at step $k_j$ (and winning) is at least $(1-\xi)$-times the probability of stopping at step $k_{j+1}$ (and losing). (Indeed, the probability of stopping at step $k_{j+1}$ might be even substantially smaller as the probability of stopping between $k_j$ and $k_{j+1}$ might be non-zero.)

For $d \in \{-i, -i+1, \ldots, -1\}$, $p_{\text{loss},d} \leq (1-\xi)^{-1}p_{\text{win},d}$ as for each odd $j$, the probability of stopping at step $k_j$ (and winning) is at least $(1-\xi)$-times the probability of stopping at step $k_{j+1}$ (and losing). The probability of stopping at any such $k_j$ conditioned on not stopping sooner is $\xi^4$ in this case.

Finally, for $d < -i$, $p_{\text{loss},d} \leq p_{\text{win},d}$, as for each odd $j$, the probability of stopping at step $k_j$ (and winning) is at least the probability of stopping at step $k_{j+1}$ (and losing). (The probability of stopping at any such $k_j$ conditioned on not stopping sooner is $\xi^4$ in this case.)

Hence,

$$p_{\text{loss}} = \sum_d p_{\text{loss},d} \leq \sum_{d \geq 0} \xi^4(1-\xi)^{i+d} + (1-\xi)^{-1} \sum_d p_{\text{win},d}$$

$$\leq \xi^3(1-\xi)^i + (1-\xi)^{-1}p_{\text{win}}.$$

For the second part, if the density of $\sigma_1, \ldots, \sigma_T$ is at most $\frac{1}{2} - \delta$ then it must contain at most $T/2 - \delta T < T/2 - i/2$ occurrences of the letter $\mathbf{L}$. Hence, it contains more than $t/2 + i/2$ occurrences of the letter $\mathbf{R}$. This implies that when the game reaches round $T$, we have that the memory state $j$ (recalling that the memory states are integers and in any round corresponds to the difference between the number of times Player 2 has played $\mathbf{R}$ minus the time he played $\mathbf{L}$ up till now) is such that $j \leq -i$ and hence Player 1 plays $\mathbf{R}$ at step $t$ with probability $\xi^4$ if the play did not stop, yet. \hfill \square
We can now prove the main statement of this subsection, that if the probability of stopping is not too small, Player 1 wins with probability close to 1/2 if play the stops.

**Lemma 10.** Let $\xi \in (0, 1)$. Let $\sigma$ be an pure Markov strategy for Player 2. If the probability that play stops is at least $\sqrt{\xi}$, then we have that

$$\Pr[\sigma_1, \sigma | \text{play stops}] \geq \frac{1}{2} - \sqrt{\xi}.$$

**Proof.** We will in this proof continue playing in a state $v$, even if Player 1 plays $R$. Note that $\sigma_1$ can still generate a choice in this case. Let $A_{i,j}$ be set of plays in which, between round 1 and round $t(i, j, 1) - 1$, Player 1 does not play $R$. Let $W_{i,j}$ be the set of plays, in which, between round $t(i, j, 1)$ and round $t(i, j, i^2)$, Player 1 plays $R$ and when he plays $R$ for the first time between these rounds, Player 2 plays $R$ as well. Similarly, let $L_{i,j} be the set of plays, in which, between round $t(i, j, 1)$ and round $t(i, j, i^2)$, Player 1 plays $R$ and when he does so for the first time Player 2 plays $L$. Let $S$ be the set of plays, in which Player 1 plays $R$ in some round. Let $W$ be the set of plays, in which Player 1 plays $R$ in some round and the first time he does so Player 2 also plays $R$. Let $L$ be the set of plays, in which Player 1 plays $R$ in some round and the first time he does not play $R$. We see that $S = W \cup L$. Clearly, $\Pr[\sigma_1, \sigma | W] = \sum_{i,j} \Pr[\sigma_1, \sigma | W_{i,j} & A_{i,j}]$ and $\Pr[\sigma_1, \sigma | L] = \sum_{i,j} \Pr[\sigma_1, \sigma | L_{i,j} & A_{i,j}]$.

Fix a possible value of all $(i,j,k)$'s and denote by $Y$ the event that these particular values actually occurs. Fix $i$ and $j$. Conditioned on $Y$ and $A_{i,j}$, between time $t(i, j, 1)$ and $t(i, j, i^2)$ Player 1 plays $\sigma_1^i, \xi$ against a fixed strategy $\sigma_{t(i,j,1)}, \sigma_{t(i,j,2)}, \ldots, \sigma_{t(i,j,i^2)}$ for Player 2. By Lemma 9 the probability of losing such a game for Player 1 is at most $\xi^3(1 - \xi)^i$ plus the probability of winning in this game divided by $1 - \xi$. Hence,

$$\Pr[L_{i,j} | Y, A_{i,j}] \leq \xi^3(1 - \xi)^i + (1 - \xi)^{-1} \Pr[W_{i,j} | Y, A_{i,j}].$$

Since the above inequality is true conditioned on arbitrary values of $t(i, j_1, j_2)$'s, it is true also without the conditioning:

$$\Pr[L_{i,j} | A_{i,j}] \leq \xi^3(1 - \xi)^i + (1 - \xi)^{-1} \Pr[W_{i,j} | A_{i,j}].$$

Thus,

$$\Pr[\sigma_1, \sigma | L] = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \Pr[L_{i,j} | A_{i,j}]$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \Pr[L_{i,j} | A_{i,j}] \cdot \Pr[A_{i,j}]$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} \left( \xi^3(1 - \xi)^i + (1 - \xi)^{-1} \Pr[\sigma_1, \sigma | W_{i,j} | A_{i,j}] \right) \cdot \Pr[A_{i,j}]$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} \xi^3(1 - \xi)^i + (1 - \xi)^{-1} \sum_{i=1}^{\infty} \sum_{j=1}^{i} \Pr[\sigma_1, \sigma | W_{i,j} & A_{i,j}]$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \xi^3(1 - \xi)^i + (1 - \xi)^{-1} \sum_{i=1}^{\infty} \sum_{j=1}^{i} \Pr[\sigma_1, \sigma | W_{i,j} & A_{i,j}]$$

$$\leq \xi + (1 - \xi)^{-1} \Pr[\sigma_1, \sigma | W].$$
Hence, \((1 - \xi) \Pr^{\sigma_1, \sigma}[L] \leq \xi - \xi^2 + \Pr^{\sigma_1, \sigma}[W]\), and so \(\Pr^{\sigma_1, \sigma}[L] - \Pr^{\sigma_1, \sigma}[W] \leq 2\xi\). By our assumption we also have that \(\Pr^{\sigma_1, \sigma}[L] + \Pr^{\sigma_1, \sigma}[W] = \Pr^{\sigma_1, \sigma}[S] \geq \sqrt[4]{\xi}\).

We want to find the minimum probability that we might win, conditioned on us stopping

\[
\Pr^{\sigma_1, \sigma}[W | S] = \frac{\Pr^{\sigma_1, \sigma}[W]}{\Pr^{\sigma_1, \sigma}[W] + \Pr^{\sigma_1, \sigma}[L]}
\]

We see that it is greater than the solution of:

\[
\min \quad \frac{x}{x+y} \\
\text{s.t.} \quad x + y \geq \sqrt[4]{\xi} \\
y - x \leq 2\xi
\]

Solving the above, we see that \(\Pr^{\sigma_1, \sigma}[W | S] \geq \frac{1}{2} - \sqrt[4]{\xi}\) and the lemma follows. \(\square\)

### 3.3 Low density means play stops

In this subsection we will prove that play will stop with probability 1 if the density of prefixes of the pure Markov strategy used by Player 2 is not infinitely often at least 1/2. First, we will show that low density implies that some sequence of sub-epochs also have low density.

**Lemma 11.** Let \(\delta \in (0, 1)\). Let \(\sigma\) be an arbitrary pure Markov strategy for Player 2. Let \(a_{i,j}\) be some numbers. Consider the event \(Y\) where \(t(i, j) = a_{i,j}\) for all \(i, j\). If \(\limsup_{T \to \infty} \text{dens}(\sigma^T) \leq 1/2 - \delta\), then conditioned on \(Y\), there is an infinite sequence of sub-epochs and epochs \((i_n, j_n)\) such that \(\text{dens}(\sigma, a_{i_n,j_n} + 1, a_{i_n,(j_n+1)}) \leq 1/2 - \delta/4\).

**Proof.** Let \(M\) be such that for every \(T' \geq M\) we have that \(\text{dens}(\sigma^{T'}) \leq 1/2 - \delta/2\). Let \((T_n)\) be a sequence such that \(T_1 \geq M\) and for all \(n \geq 1\) we have that \(T_{n+1} \cdot \delta/4 \geq T_n\) and \(T_n = a_{i,j}\) for some \(i, j\). Let \((i'_n, j'_n)\) be the sequence such that \(T_n = a_{i'_n, j'_n}\). This means that even if \(\text{dens}(\sigma^{T_n}) = 0\), the density \(\text{dens}(\sigma, T_n + 1, T_{n+1})\) is at most \(1/2 - \delta/4\), because \(\text{dens}(\sigma^{T_n+1}) \leq 1/2 - \delta/2\). But, we then get that there exists some sub-epoch \(j_n\) in epoch \(i_n\), such that \(j'_n \leq j_n \leq j'_{n+1}\) and such that \(i'_n \leq i_n \leq i'_{n+1}\) for which the density of that sub-epoch \(\text{dens}(\sigma, a_{i_n,j_n} + 1, a_{i_n,(j_n+1)})\) is at most \(1/2 - \delta/4\), because not all sub-epochs can have density below that of the average sub-epoch. But then \((i_n, j_n)\) satisfies the lemma statement. \(\square\)

We are now ready to prove the main statement of this subsection.

**Lemma 12.** Let \(\sigma\) be an arbitrary pure Markov strategy for Player 2. If

\[
\limsup_{T \to \infty} \text{dens}(\sigma^T) < 1/2
\]

then when played against \(\sigma_1^*\) the play stops with probability 1.

**Proof.** Let \(\delta > 0\) be such that \(\limsup_{T \to \infty} \text{dens}(\sigma^T) \leq 1/2 - \delta\). Consider arbitrary numbers \(a_{i,j}\) and the event \(Y\) stating that \(t(i, j) = a_{i,j}\) for all \(i, j\). Let \((i_n, j_n)\) be the sequence of sub-epochs and epochs shown to exists by Lemma 11 with probability 1. That is, for each \((i_n, j_n)\) we have that sub-epoch \(j_n\) of epoch \(i_n\) has density at most \(1/2 - \delta/4\). We see that, conditioned on \(Y\) that each sample are sampled uniformly at random in each sub-epoch \(j\) of each epoch \(i\), except for the last sample. Now consider some fixed \(n\). By Hoeffding’s inequality, Theorem 46 (setting \(a_i = 0\) and...
Let \( b_i = 1 \), and let \( c_i \) be \( i \)'th payoff for Player 1), the probability that in sub-epoch \( j_n \) of epoch \( i_n \) that among our \( n^2 \) first samples we have \( \delta \cdot (i_n^2 - 1) \) additional L on top of the expectation (which is at most \( (1/2 - \delta/4) \cdot (i_n^2 - 1) \)) is bounded as

\[
\Pr \left[ \text{dens}(\sigma_{t(i_n,j_n,1)}, \sigma_{t(i_n,j_n,2)}), \ldots, \sigma_{t(i_n,j_n,(i_n^2 - 1))} \right] \geq \frac{1}{2} - \frac{\delta}{8}
\leq 2 \exp \left( -\frac{2(\delta /8)}{(i_n^2 - 1)} \right) = 2 \exp \left( -\frac{\delta^2}{32} (i_n^2 - 1) \right).
\]

For sufficiently large \( i_n \), this is less than \( \frac{1}{2} \). If on the other hand the number of L’s we sample is less than \( \left( \frac{1}{2} - \frac{\delta}{8} \right)(i_n^2 - 1) \), we see that \( (i_n^2 - 1) > i_n/(16\delta) \) for large enough \( i_n \) and in that case we have, by Lemma \ref{lem: bounded}, that we stop with probability at least \( \xi^4 \) in sub-epoch \( j_n \) of epoch \( i_n \). Thus, for each of the infinitely many \( n \)'s for which \( i_n \) is sufficiently high, we have a probability of at least \( \xi^4 \) of stopping. Thus play must stop with probability 1.

The argument was conditioned on some fixed assignment of endpoints of sup-epochs and epochs, but since there is such a assignment with probability 1 (since they are finite with probability 1), we conclude that the proof works without the condition.

\[\square\]

### 3.4 Proof of main result

**Theorem 13.** The strategy \( \sigma^\dagger \) is \( \sqrt{\xi} \)-supremum-optimal, and for all \( \delta > 0 \), with probability at least \( 1 - \delta \) it uses space \( O(\log f(T)) \).

**Proof.** The space usage follows from Lemma \ref{lem: space}. Let \( s \) be the probability that the play stops. We now consider three cases, either (i) \( s = 1 \); or (ii) \( \sqrt{\xi} < s < 1 \); or (iii) \( s \leq \sqrt{\xi} \). In case (i), if \( s = 1 \), Player 1 wins with probability \( \frac{1}{2} - \sqrt{\xi} \), by Lemma \ref{lem: win}. In case (ii), if \( \sqrt{\xi} < s < 1 \), then, by Lemma \ref{lem: win}, conditioned on the play stopping, Player 1 wins with probability \( W_s = \frac{1}{2} - \sqrt{\xi} \) and, since \( s < 1 \), by Lemma \ref{lem: win} \( \hat{W}_s = \limsup_{t \to \infty} \text{dens}(\sigma^T) \geq \frac{1}{2} \). Thus, Player 1 wins with probability

\[
s \cdot W_s + (1 - s) \cdot \hat{W}_s \geq s \cdot \left( \frac{1}{2} - \sqrt{\xi} \right) + (1 - s) \frac{1}{2} \geq \frac{1}{2} - \sqrt{\xi}
\]

In case (iii), if \( s \leq \sqrt{\xi} \), then by Lemma \ref{lem: win} \( \hat{W}_s = \limsup_{t \to \infty} \text{dens}(\sigma^T) \geq \frac{1}{2} \). The winning probability is then at least

\[
s \cdot 0 + (1 - s) \cdot \frac{1}{2} \geq \frac{1 - \sqrt{\xi}}{2} \geq \frac{1}{2} - \sqrt{\xi}
\]

\[\square\]

### 4 An \( \varepsilon \)-optimal strategy that uses \( \log \log T \) space for the Big Match

For given \( \varepsilon > 0 \), let \( \xi = \varepsilon^2 \). In this section we give a \( \varepsilon \)-optimal strategy \( \sigma^\dagger \) for Player 1 in the Big Match that for all \( \delta > 0 \) with probability \( 1 - \delta \) uses \( O(\log \log T) \) space. The strategy is simply an instantiation of the strategy \( \sigma^\dagger \) from Section \ref{sec: main} setting \( f(T) = \lceil \log T \rceil \). We can then let \( \bar{f} = \log T \) and \( F(T) = 2^T \).

The claim about the space usage of \( \sigma^\dagger \) is thus already established in Section \ref{sec: main}. To obtain the stronger property of \( \varepsilon \)-optimality rather than just \( \varepsilon \)-supremum-optimality, we just need to establish a \( \liminf \) version of Lemma \ref{lem: win}.
First we show a technical lemma. For a pure Markov strategy \( \sigma \) and a sequence of integers \( I = \{i_1, i_2, \ldots, i_m\} \), let \( \sigma_I \) be the sequence, \( \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m} \). Note that \( \sigma^k = \sigma_{\{1, \ldots, k\}} \).

**Lemma 14.** Let \( \sigma \) be a pure Markov strategy for Player 2, \( \delta < 1/4 \) be a positive real, and \( M \) be a positive integer. Let \( \lim \inf_{T \to \infty} \text{dens}(\sigma^T) \leq \frac{1}{2} - \delta \). Let \( \ell_1, \ell_2, \ldots \) be such that for all \( i \geq M \), we have that \( \ell_i \in [(1 - \delta) \cdot (2^{i+1} - 1), (1 + \delta) \cdot (2^{i+1} - 1)] \). Then there exists a sequence \( k_2, k_3, \ldots \) such that for infinitely many \( i > M \), we have that \( \ell_{i-1} + \delta 2^{i-2} \leq k_i \leq \ell_i \) and \( \text{dens}(\sigma_{\{i-1+1, \ldots, k_i\}}) \leq \frac{1}{2} - \frac{\delta}{4} \).

**Proof.** Let \( \ell_i \) be as required. If there are infinitely many \( i \) such that \( \text{dens}(\sigma_{\{i-1+1, \ldots, \ell_i\}}) \leq \frac{1}{2} - \frac{\delta}{4} \) then set \( k_i = \ell_{i+1} \) and the lemma follows by observing \( (k_i - \ell_{i-1}) \geq (1 - \delta)(2^{i+1} - 1) - (1 + \delta)(2^i - 1) = (1 - 3\delta)2^i \geq \delta 2^{i-2} \), for \( i > M \). So assume that only for finitely many \( i \), \( \text{dens}(\sigma_{\{i-1+1, \ldots, \ell_i\}}) \leq \frac{1}{2} - \frac{\delta}{4} \).

Thus the following claim can be applied for arbitrary large \( i_0 \).

**Claim 15.** Let \( i_0 \geq M \) be given. If for every \( i \geq i_0 \), \( \text{dens}(\sigma_{\{i-1+1, \ldots, \ell_i\}}) > \frac{1}{2} - \frac{\delta}{4} \) then there exist \( j > i_0 \) and \( k \) such that \( \ell_{j-1} + \delta 2^{j-2} \leq k \leq \ell_j \) and \( \text{dens}(\sigma_{\{j-1+1, \ldots, k\}}) \leq \frac{1}{2} - \delta \).

We can use the claim to find \( k_2, k_3, \ldots \) inductively. Start with large enough \( i_0 \geq M \) and set \( k_i = \ell_i \) for all \( i \leq i_0 \). Then provided that we already inductively determined \( k_2, k_3, \ldots, k_{i_0} \), we apply the above claim to obtain \( j \) and \( k \), and we set \( j = k = k_i = \ell_i \), for all \( i = i_0 + 1, \ldots, j - 1 \).

So it suffices to prove the claim. For any \( d \geq 1 \), \( \ell_{i_0} \cdot 2^{d-1} \leq \ell_{i_0+d} \) and

\[
\text{dens}(\sigma^d) \geq \frac{(1 + \frac{\delta}{4})(\ell_{i_0+d} - \ell_{i_0})}{\ell_{i_0+d}}.
\]

Furthermore, if \( d \geq 1 + \log(4/\delta) \) then \( \ell_{i_0} \leq \frac{\delta}{4} \ell_{i_0+d} \) and

\[
\text{dens}(\sigma^d) \geq \left( \frac{1}{2} - \frac{\delta}{4} \right) - \frac{\delta}{4} = \frac{1}{2} - \frac{\delta}{2}.
\]

Since \( \lim \inf_{k \to \infty} \text{dens}(\sigma^k) \leq \frac{1}{2} - \delta \), there must be \( k \) and \( d \geq 1 + \log(4/\delta) \) such that \( \ell_{i_0+d} \leq k \leq \ell_{i_0+d} \) and \( \text{dens}(\sigma^k) \leq \frac{1}{2} - \delta \). Set \( j = i_0 + d + 1 \). Also

\[
\text{dens}(\sigma^k) = \frac{\text{dens}(\sigma^j)}{\ell_{j-1} + (k - \ell_{j-1})},
\]

which means

\[
\left( \text{dens}(\sigma^k) - \text{dens}(\sigma_{\{j-1+1, \ldots, k\}}) \right) (k - \ell_{j-1}) = \left( \text{dens}(\sigma^j) - \text{dens}(\sigma^k) \right) \ell_{j-1}
\]

\[
ier \geq \left[ \left( \frac{1}{2} - \frac{\delta}{2} \right) - \left( \frac{1}{2} - \delta \right) \right] \ell_{j-1} = \frac{\delta}{2} \ell_{j-1}.
\]

Thus \( \text{dens}(\sigma_{\{j-1+1, \ldots, k\}}) \leq \text{dens}(\sigma^k) \) which in turn is less than \( \frac{1}{2} - \delta \). Furthermore, \( k - \ell_{j-1} \geq \frac{\delta}{2} \ell_{j-1} \geq \frac{\delta}{2} (1 - \delta)(2^j - 1) \geq \frac{\delta}{4} 2^j \), provided that \( j \geq 2 \). Hence, \( k \) and \( j \) have the desired properties.

We are now ready to prove the \( \lim \inf \) version of Lemma [12]

**Lemma 16.** Let \( \sigma \) be an arbitrary pure Markov strategy for Player 2. If

\[
\lim \inf_{T \to \infty} \text{dens}(\sigma) < \frac{1}{2},
\]

then when played against \( \sigma^*_1 \) the play stops with probability 1.
Proof. Let $0 < \delta < \frac{1}{4}$ be such that

$$\lim_{t \to \infty} \inf \text{dens}(\sigma_1, \ldots, \sigma_t) \leq 1/2 - \delta$$

Pick arbitrary $\gamma \in (0, 1)$. We will show that with probability at least $1 - \gamma$ the game stops, and this implies the statement. Let $M$ be given by Lemma 7 applied for $\gamma$ and $\delta/2$. Then we have that with probability at least $1 - \gamma$, for all $i \geq M$ and $j \in \{1, \ldots, i\},$

$$t(i, j, i^2) - t(i, j, i^2 - 1) \in [(1 - \frac{\delta}{2})2^i/i, (1 + \frac{\delta}{2})2^i/i] .$$

Pick $t_{i,j} \in \mathbb{N}$, for $i = 1, 2, \ldots$ and $j \in \{1, \ldots, i\}$, so that $t_{i,j-1} < t_{i,j}$ where $t_{i,0}$ stands for $t_{i-1,i-1}$. Let $t_{i,j} - t_{i,j-1} \in [(1 - \frac{\delta}{2})2^i/i, (1 + \frac{\delta}{2})2^i/i]$, for all $i \geq M$ and $j \in \{1, \ldots, i\}$. Pick $M'$ so that $\frac{\delta}{2}(2^{M'} - 1) \geq \max\{\ell_M, (1 - \frac{\delta}{2})2^M\}$. Define $\ell_i = t_{i,i}$ for all $i \geq 1$. Then for all $i \geq M'$, $\ell_i \in [(1 - \delta) \cdot (2^{i+1} - 1), (1 + \delta) \cdot (2^{i+1} - 1)]$ as

$$\ell_i = t_{i,i} = t_{M,0} + \sum_{M \leq \nu \leq i, 1 \leq j \leq \nu} t_{\nu,j} - t_{\nu,j-1} \leq \frac{\delta}{2}(2^{M'} - 1) + \sum_{M \leq \nu' \leq i} \nu' \cdot (1 + \frac{\delta}{2})2^\nu'/\nu' \leq \frac{\delta}{2}(2^{M'} - 1) + (1 + \frac{\delta}{2}) \cdot (2^{i+1} - 1) \leq (1 + \delta) \cdot (2^{i+1} - 1),$$

and similarly for the lower bound: $\ell_i \geq \sum_{\nu,j} t_{\nu,j} - t_{\nu,j-1} \geq (1 - \frac{\delta}{2})(2^{i+1} - 2^M) \geq (1 - \delta) \cdot (2^{i+1} - 1).$ Thus Lemma 14 is applicable on $\ell_i$ with $M$ set to $M'$, and we obtain a sequence $k_2, k_3, \ldots$ such that $\text{dens}(\sigma_{k_{i-1}+1}, \ldots, k_i) \leq \frac{1}{2} - \frac{\delta}{4}$ and $k_i - \ell_{i-1} \geq \delta 2^{i-2}$ for infinitely many $i$. Pick any of the infinitely many $i \geq \max\{M', 32(1 + \delta)/\delta\}$ for which $k_i - \ell_{i-1} \geq \delta 2^{i-2}$ and $\text{dens}(\sigma_{k_{i-1}+1}, \ldots, k_i) \leq \frac{1}{2} - \frac{\delta}{4}$. Since $\delta 2^{i-3} \geq (1 + \delta) 2^i/i$, there is some $j \in \{1, \ldots, i\}$ such that $\ell_{i-1} + \delta 2^{i-3} \leq k_i - (1 + \delta) 2^i/i \leq t_{i,j} \leq k_i$. Fix such $j$. Since $k_i \leq t_{i,j} + (1 + \delta) 2^i/i$, we have

$$\text{dens}(\sigma_{k_{i-1}+1}, \ldots, t_{i,j}) = \frac{\text{dens}(\sigma_{k_{i-1}+1}, \ldots, k_i)(k_i - \ell_{i-1})}{t_{i,j} + \ell_{i-1}} \leq \frac{\text{dens}(\sigma_{k_{i-1}+1}, \ldots, k_i)((1 + \delta)2^i/i + t_{i,j} - \ell_{i-1})}{t_{i,j} + \ell_{i-1}} \leq \frac{1}{2} - \frac{\delta}{4} + \frac{4(1 + \delta)}{i} \leq \frac{1}{2} - \frac{\delta}{8}.$$ 

Hence, $\text{dens}(\sigma_{k_{i-1}+1}, \ldots, t_{i,j}) \leq \frac{1}{2} - \frac{\delta}{8}$. So for some $j' \in \{1, \ldots, j\}$, $\text{dens}(\sigma_{t_{i,j'}-1}, \ldots, t_{i,j'}) \leq \frac{1}{2} - \frac{\delta}{8}$. We can state the following claim.

Claim 17. For $i$ large enough, conditioned on $t(a,b)d^2 = t_{a,b}$, for all $a \geq M$ and all $b$, and conditioned on that the game did not stop before the time $t_{i,j'-1} + 1$, the game stops during times $t_{i,j'-1} + 1, \ldots, t_{i,j'}$ with probability at least $\xi^4/2$. 

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Conditioned on \( t(a, b, a^2) = t_{a,b} \), for all \( a, b \), the claim implies that the game stops with probability 1. Note that the condition is true for some valid choice of \( t_{a,b} \) with probability \( 1 - \gamma \). This is because the claim can be invoked for infinitely many \( i \)'s and for each such \( i \) we will have \( \xi^4/2 \) chance of stopping.

It remains to prove the claim. Assume \( t(i, j', 1, i^2) = t_{i,j'-1} \) and \( t(i, j', i^2) = t_{i,j'} \). Clearly,

\[
dens(\sigma_{t_{i,j'-1}+1 : t_{i,j}'-1}) \leq dens(\sigma_{t_{i,j'-1}+1 : t_{i,j}'-1}) \cdot (2^{i-1}/(2^{i-1} - 1)) \leq \frac{1}{2} - \frac{\delta}{16},
\]

for \( i \) large enough. So if we sample \( i^2 - 1 \) times from \( \sigma_{t_{i,j'-1}+1 : t_{i,j}'-1} \) we expect at most \((\frac{1}{2} - \frac{\delta}{16})(i^2 - 1)\) of the letters to be \( L \). By Hoeffding’s inequality, Theorem 18 (again setting \( a_i = 0 \) and \( b_i = 1 \), and letting \( c_i \) be \( i \)'th payoff for Player 1), the probability that we get at least \( \frac{\delta}{32}(i^2 - 1) \) additional \( L \) items on top of the expectation is given by

\[
Pr \left[ dens(\sigma_{t(i,j',1)}, \sigma_{t(i,j',2)}, \ldots, \sigma_{t(i,j',i^2-1)}) \geq \frac{1}{2} - \frac{\delta}{32} \right] \leq 2 \exp \left( -\frac{\delta^2 (i^2 - 1)}{512} \right).
\]

The probability is taken over the possible choices of \( t(i, j', 1) < t(i, j', 2) < \cdots < t(i, j', i^2 - 1) \) assuming \( t(i, j' - 1, i^2) = t_{i,j'-1} \) and \( t(i, j', i^2) = t_{i,j'} \). For \( i \) sufficiently large, \( 2e^{-\delta^2/(512)} \leq 1/2 \). Whenever

\[
dens(\sigma_{t(i,j',1)}, \sigma_{t(i,j',2)}, \ldots, \sigma_{t(i,j',i^2-1)}) \leq \frac{1}{2} - \frac{\delta}{32}
\]

we have at least \( \xi^4 \) chance of stopping by Lemma 9 as Player 1 plays \( \sigma_{1+i,j' : \xi} \) against

\[
\sigma_{t(i,j',1)}, \sigma_{t(i,j',2)}, \ldots, \sigma_{t(i,j',i^2-1)}
\]

and \( i^2 - 1 > i/\delta \geq (i + j')/2\delta \) for sufficiently large \( i \). Hence, the game stops with probability at least \((1 - 1/2) \cdot \xi^4 = \xi^4/2 \). The claim, and thus the lemma, follows.

We can now conclude with the main result of this section.

**Theorem 18.** The strategy \( \sigma_{1}^1 \) is \( \sqrt{\xi} \)-optimal, and for all \( \delta > 0 \), with probability at least \( 1 - \delta \) it uses space \( O(\log \log T) \).

**Proof.** This is proved just like Theorem 13 except that Lemma 16 is used in place of Lemma 12.

We can also improve the strategy and get the following corollary.

**Corollary 19.** For any natural number \( k \), there is a strategy which is \( 2^{-k} \)-optimal, has patience 2 and can be implemented on a Turing machine, using at most 1 random bit and amortized constant time\(^1\) per round and with probability at least \( 1 - \delta \) does it use tape space \( O(\log \log (T) + \log k) \) upto round \( T \).

\(^1\)i.e. for all \( T \), let \( c(T) \) be the computation used for the first \( T \) rounds, then \( \frac{c(T)}{T} \) is some constant.
Proof. We will only sketch the necessary ingredients for the proof. We will modify the strategy $\sigma_1^*$ as defined in Section 4 to use less patience. First, it is easy to see that the probability that two sample points are ever within a polynomial in $i$ distance of each other is small, in any sufficiently high epoch $i$, and thus we can ignore this event. The idea is as follows: Each sub-epoch of epoch $i$ is split into blocks of length $B = i^{O(1)} \cdot k$ (where we simply keep track of the current location within the current block, but not the current block number). The purpose of each block is to provide all the randomness needed for the next block. Hence, it needs to (1) decide if there is a sample point in the next block; and (2) if so where; and (3) if we should stop on that sample point. Observe that we are ignoring the possibility that there are more than 1 sample point in a block. It is easy to see that the number of random bits needed in total to generate all those events is less than $i^{O(1)} \cdot k$ and thus it can be done with at most 1 random bit per round, implying the patience bound. To get the tape space bound, we utilize the three following simple ideas:

Idea 1: To get a probability like $\prod a_j$, for some sequence of probabilities $a_j$ of length $\ell$, one can test if sub-events that happens with probability $a_i$ for all $i$ all happens. This requires only space for a counter counting up to $\ell$ and space for the event that uses the most space (by reusing the space for the event).

Idea 2: To get an probability like $2^{-x}$ (or similarly $1 - 2^{-x}$) one can simply flip $x$ coins and if all comes up tails, then the event happens. This requires only space for a counter counting up to $x$.

Idea 3: To get an probability like $\frac{y}{x}$, for any natural number $x$ and $y$, one can simply use $x$ random bits. This uses $x$ many bits of space.

Direct application of these three ideas suffice to get all three events in $O(\log i + \log k)$ many bits. Observe that we can do this easily on a two tape Turing machine (one is used for events following idea 3 and one for the constant number of counters) and we only need amortized constant time in each round (to increment and/or reset some subset of the counters on the first tape and perhaps to add one more random bit or reset the second tape).

5 Lower bound on patience

When considering a strategy of a player one may want to look at how small or large the probabilities occurring in that strategy are. The parameter of interest is the patience of the strategy which is the reciprocal of the smallest non-zero probability occurring in the strategy. Patience is closely related to the expected length of finite plays as small probability events will not occur if the play is too short so they will have little influence on the overall outcome [11, 6]. Care has to be taken how to define patience for strategies with infinitely many possible events. One thing to note of our space efficient strategies is that the patience of the states in which we are with high probability during the first $T$ steps is approximately $T$, for rounds $T$ close to the end of an epoch. In this section we show that this is essentially necessary. So if the space used by the strategy with high probability is $\log f(T)$, then the first $f(T)$ states must have patience about $T$. Thus the smaller the space the strategy uses the larger the patience the states must have.

The main theorem of this section states that if the patience of the $f(T)$ states in which Player 1 is with high probability is less than about $T^{1/f(T)}$ then the strategy is bad for Player 1. It is easy to observe that events with probability substantially less than $1/T$ are unlikely to occur during the
there is a non-zero probability of playing

One may wonder whether the exponent $1/f(T)$ in $T^{1/f(T)}$ is necessary. It turns out that it is, so our lower bound is close to optimal using a technique like in Corollary [19].

We use the following definitions to deal with the fact that our strategies use infinitely many transitions so their overall patience is infinite.

For a memory based strategy $\sigma_1$ of Player 1 in a repeated zero-sum game with absorbing states, the patience of a set of memory states $M$ is defined as:

$$\text{pat}(M) = \max \left\{ \frac{1}{\sigma_1^a(m)(a_1)}, \frac{1}{\sigma_1^a(a_1, a_2, m)(m')}, m, m' \in M, a_1 \in A_1, a_2 \in A_2 \right\}.$$  

The patience of the player is defined similarly.

**Theorem 20.** Let $\delta, \varepsilon > 0$ be reals and $f : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function such that $f(T) \leq \frac{1}{T} \log_{1/\varepsilon} T$ for all large enough $T$. If a strategy $\sigma_1$ of Player 1 uses space $\log f(T)$ before time $T$ with probability at least $1 - \delta$, and the patience of the set of lexicographically first $f(T)$ memory states is at most $T^{1/(2f(T))}$ for all $T$ large enough, then there is a strategy $\sigma_2$ of Player 2 such that $u(\sup, (\sigma_1, \sigma_2)) \leq \delta + 2\varepsilon$.

**Proof.** Assume that $\varepsilon < 1/2$ and pick an integer $k$ sufficiently large. For $i > 0$, define $\ell_i = \varepsilon^{-i}$ and $T_i = \sum_{j=1}^i \ell_j$. The strategy $\sigma_2$ of Player 2 proceeds in phases, each phase $i$ is of length $\ell_i$. In the first $k$ phases, Player 2 plays $L$ with probability $1 - \varepsilon$ and $R$ with probability $\varepsilon$. In each phase $i > k$, Player 2 plays $R$ for the first $(1 - \varepsilon)\ell_i$ steps, and afterwards he plays $L$ with probability $1 - \varepsilon$ and $R$ with probability $\varepsilon$. Notice, if the game does not stop by Player 1 playing $R$ at some point then the expected lim sup payoff to Player 1 is at most $\varepsilon$.

Our goal is to show that if the game stops then the expected payoff to Player 1 is at most $\delta + 2\varepsilon$. If the game stops during the last $\varepsilon \ell_i$ steps of a phase $i$, then the expected payoff to Player 1 is $\varepsilon$ as the probability of Player 2 playing $R$ at that time is $\varepsilon$. If the game stops during the first $(1 - \varepsilon)\ell_i$ steps of a phase $i > k$, then the payoff of Player 1 is 1. Our goal is to argue that the overall probability that Player 1 stops during the first $(1 - \varepsilon)\ell_i$ steps of some phase $i > k$ is small.

For any $t > 0$, denote by $M(t)$ the set of the lexicographically first $f(t)$ memory states (i.e. those mapped to a number below $f(t)$). Let $C$ be the event that for all steps $t$, Player 1 is in one of the states in $M(t)$. For $t < t'$, let $S(t, t')$ be the event that Player 1 plays $R$ in one of the steps $[t, t')$. Let $A_i(t)$ be the event that at time $t$, Player 1 is in one of the states in $M(T_i)$ and there is some memory state in $M(T_i)$ that can be reached from the state current at time $t$ and in which there is a non-zero probability of playing $R$ (i.e., stopping).

The probability that $C$ does not occur is at most $\delta$ so for the rest of the proof we will assume that $C$ occurs. Let $k$ be large enough, and $i > k$. It is clear that if $S(T_{i-1}, T_{i-1} + (1 - \varepsilon)\ell_i)$ occurs then $A_i(T_{i-1})$ must have occurred as well so:

$$\Pr[S(T_{i-1}, T_{i-1} + (1 - \varepsilon)\ell_i)] \leq \Pr[A_i(T_{i-1})].$$

Furthermore, if $A_i(T_{i-1})$ occurs then $A_i(t)$ occurs for all $t < T_{i-1}$. For $t \in [T_{i-1} - \varepsilon\ell_{i-1}, T_{i-1} - f(T_i))$, if $A_i(t)$ occurs then within the next $f(T_i)$ steps the strategy of Player 1 might reach a state in which Player 1 chooses the stopping action $R$ with non-zero probability. Because of the patience of $M(T_i)$
and the fact that Player 2 plays each of his possible actions with probability at least \( \varepsilon \) during that time steps we have for any \( t \in [T_{i-1} - \varepsilon \ell_i - 1, T_{i-1} - f(T_i)] \),

\[
\Pr[S(t, t + f(T_i)) \mid A_i(t)] \geq \left( \frac{\varepsilon}{\text{pat}(M(T_i))} \right)^{f(T_i)}.
\]

Hence, the probability that \( A_i(T_{i-1}) \) occurs and the game did not stop yet is at most:

\[
\left( 1 - \left( \frac{\varepsilon}{\text{pat}(M(T_i))} \right)^{f(T_i)} \right)^{\varepsilon \ell_i - 1 / f(T_i)} \leq e^{-\left( \frac{\varepsilon}{\text{pat}(M(T_i))} \right)^{f(T_i)} \varepsilon \ell_i - 1 / f(T_i)}.
\]

For sufficiently large \( T \), \( f(T) \leq \frac{1}{4} \log_{1/\varepsilon} T \). Furthermore, \( T_i \leq 2\ell_i \) and \( \varepsilon T_i - 1 \leq T_{i-1} \). Since \( i \) is sufficiently large we have:

\[
\left( \frac{\varepsilon}{\text{pat}(M(T_i))} \right)^{f(T_i)} \cdot \varepsilon \cdot \frac{\ell_i - 1}{f(T_i)} \geq \left( \frac{\varepsilon}{T_{i-1}^2 f(T_i)} \right)^{f(T_i)} \cdot \varepsilon \cdot \frac{T_{i-1} - 1}{2 f(T_i)}
\]

\[
\geq \varepsilon f(T_i) \cdot \varepsilon^2 \cdot \frac{T_{i-1}^{1/2}}{2 f(T_i)} - 1
\]

\[
\geq \varepsilon^2 \cdot \frac{T_{i-1}^{1/4}}{2 f(T_i)} - 1 \geq T_i^{1/5}.
\]

Since \( T_i^{1/5} \gg i \) for \( i \) large enough, we get

\[
\Pr[S(T_{i-1}, T_{i-1} + (1 - \varepsilon)\ell_i)] \leq \varepsilon \cdot e^{-i}.
\]

We set \( k \) to be large enough so that the above analysis would work for \( i > k \). Thus except for probability at most \( \delta + \varepsilon \), Player 1 stops in a step when Player 2 plays \( R \) with probability only \( \varepsilon \). Thus the expected payoff to Player 1 is at most \( \delta + 2\varepsilon \).

\[\square\]

6 No finite-memory \( \varepsilon \)-optimal deterministic-update Markov strategy exists

A memory-based Markov strategy is an extension of a memory-based strategy that may also depend on the round number. More precisely, for Player 1, the action map \( \sigma^1_T \) for a memory-based Markov strategy \( \sigma_1 \) is a map from \( \mathbb{Z}_+ \times \mathcal{M} \) to \( \Delta(A_1) \) and the update map \( \sigma^u_1 \) for memory-based Markov strategies is a map from \( \mathbb{Z}_+ \times A_1 \times A_2 \times \mathcal{M} \) to \( \Delta(\mathcal{M}) \). We say that Player 1 follows the memory-based Markov strategy \( \sigma_1 \) if in every round \( T \) when the game did not stop yet, he picks his next move \( a_T^1 \) at random according to \( \sigma^1_T(T, m_T) \), where the sequence \( m_1, m_2, \ldots \) is given by letting \( m_1 = m_s \) and for \( T = 1, 2, 3, \ldots \) choosing \( m_{T+1} \) at random according to \( \sigma^u_T(a_T^1, a_T^2, m_T, T) \), where \( a_T^2 \) is the action chosen by Player 2 at round \( T \). The definition of memory-based Markov strategies is similar for Player 2. Note that memory-based Markov strategies are more general than memory-based strategies.

A memory-based (resp. Markov) strategy \( \sigma_1 \) for Player 1 has deterministic-update, if for all \( a_1 \in A_1 \), all \( a_2 \in A_2 \) and all \( m \in \mathcal{M} \) (resp. all \( T \in \mathbb{Z}_+ \)) the distribution \( \sigma^1_T(a_1, a_2, m) \) (resp. \( \sigma^u_1(T, a_1, a_2, m) \)) is deterministic.
In this section we will argue that no finite-memory $\varepsilon$-optimal deterministic-update Markov strategy for Player 1 in the Big Match exists, for $\varepsilon < \frac{1}{2}$. Let $n$ be some positive integer. Let $\sigma_1$ be some Markov strategy using at most $n$ memory for Player 1. We will show that for all $\delta > 0$, there exists a strategy $\sigma_2$ for Player 2 that ensures that $u(\inf, (\sigma_1, \sigma_2)) < \delta$. This shows that $\sigma_1$ can only ensure value 0.

**Construction of $\sigma_2$ and the sequence of strategies $\sigma^k_2$.** We will now describe the construction of $\sigma_2$. The strategy $\sigma_2$ will be the final strategy in a finite sequence of strategies $(\sigma^k_2)$. Each of the strategies $\sigma^k_2$ (and thus also $\sigma_2$) is a deterministic memory-based Markov strategy and will use the same set of memory states $\mathcal{M}$ of size $n$ and update map $\sigma^1_u$ as $\sigma_1$. Observe that the action map $\sigma^k_a$ for such a strategy can be thought of as a $(\infty, n)$-matrix $A$ over $\{L, R\}$, where $A^T_{m} = \sigma^k_a(T, m)$.

- Let $\sigma^{k,a}_2$ be the action map for $\sigma_2$.
- Let $S^k_R = \{(T, m) | \sigma^{k,a}_2(T, m) = R\}$ (i.e. the pairs under which $\sigma^k_2$ plays R).
- For all $T$, let $\mathcal{M}^T = \{(T', m) | T' \leq T\}$ (i.e. the memory states before round $T$).
- For all $T$, let $S^{k,T}_R = S^k_R \cap \mathcal{M}^T$ (i.e. the pairs under which $\sigma^k_2$ plays $R$ before round $T$).

**Properties of strategies in the sequence $\sigma^k_2$.** Besides ensuring that the last strategy $\sigma_2$ in the sequence is such that $u(\inf, (\sigma_1, \sigma_2)) < \delta$, our construction of $\sigma^k_2$ will ensure the following properties:

1. **Property 1.** The probability to stop (using union bound) while Player 2 plays $R$ is at most
   \[
   \sum_{(T, m) \in S^k_R} \sigma^q_1(T, m)(R) \leq (1 - 2^{-k})\delta.
   \]

2. **Property 2.** The infimum limit, for $T$ going to infinite, of the fraction of all pairs before round $T$ under which $\sigma^k_2$ plays $R$ is at least
   \[
   \liminf_{T \to \infty} \frac{|S^{k,T}_R|}{n \cdot T} \geq \frac{\delta \cdot k}{n}.
   \]

**The sequence has finite length.** Observe that Property 2 ensures that the strategy $\sigma^k_2$ cannot exists, for $k > \frac{n}{\delta}$, implying that the sequence has finite length. This is because $\sigma^k_2$, for such $k$, otherwise would require that there is some $T$, such that the number of pairs such that $\sigma^k_2$ plays $R$ before round $T$ is strictly more than the number of pairs before round $T$.

**The strategy $\sigma^0_2$.** The action map $\sigma^{0,a}_2$ is such that $\sigma^{0,a}_2(T, m) = L$ for all $T \in \mathbb{N}$ and $m \in \mathcal{M}$. The strategy has the wanted properties (because $S^0_R$ is the empty set).
Lemma 21. Let \( x < 1 \) be some real number. Let \((x_T)_T\) be an infinite sequence, in which \( x_T \in [0,1] \) and \( 1 - \prod_T(1 - x_T) = x \), then \( \sum_T x_T \leq -\log(1-x) < \infty \).
\[ \sigma_2^0 \text{ together with the sequence of memory states } (m_1^0)_t \]

\[ \sigma_2^1 \text{ together with the sequence of memory states } (m_1^1)_t \]

\[ \sigma_2^2 \text{ together with the sequence of memory states } (m_1^2)_t \]

\[ \sigma_2^3 \text{ together with the sequence of memory states } (m_1^3)_t \]

Figure 3: Possible sequence of strategies \((\sigma_2^k)_k\) and corresponding sequence of memory states \((m_1^k)_t\).
Proof. We have that all $x_T < 1$, since otherwise the product would be 0, and hence $x = 1$. Also, we get that

$$\prod_T (1 - x_T) = 1 - x \Rightarrow \sum_T \log(1 - x_T) = \log(1 - x)$$

$$\Rightarrow -\sum_T \log(1 - x_T) = -\log(1 - x) \Rightarrow \sum_T x_T \leq -\log(1 - x).$$

The last inequality comes from that $f(y) = -\log(1 - y) \geq y$ for $y \in [0,1)$. This follows from $f(0) = 0$ and $f'(y) = \frac{1}{1-y} \geq 1$, for $y \in [0,1)$. \qed

We have that $1 - \prod_T (1 - p_T) = p$ and by Lemma 21 we then get that $\sum_T p_T$ is finite. Hence, there exists some $M$ such that $\sum_{T=M}^{\infty} p_T \leq \frac{\delta}{2^{k+1}}$.

We can now define $\sigma_{k+1}^{a}$. Let

$$\sigma_{k+1,a}(T, m) = \begin{cases} R & \text{if } (T, m) \in S_{k,M}^{k} \\ \sigma_{k,a}(T, m) & \text{otherwise} \end{cases}$$

The strategy $\sigma_{k+1}^{a}$ satisfies the wanted properties. We will now show that $\sigma_{k+1}^{a}$ satisfies the wanted properties.

1. That Property 1 is satisfied comes from that

$$\sum_{(T, m) \in S_{k}^{k+1}} \sigma_{1}^{a}(T, m)(R) \leq \sum_{(T, m) \in S_{k}^{k+1}} \sigma_{1}^{a}(T, m)(R) + \sum_{(T, m) \in S_{k-M}^{k}} \sigma_{1}^{a}(T, m)(R)$$

$$\leq (1 - 2^{-k})\delta + 2^{-k-1}\delta = (1 - 2^{-k-1})\delta.$$ 

2. That Property 2 is satisfied can be seen as follows. We have that

$$\liminf_{T \to \infty} \frac{|S_{k+1,T}^{k+1} \setminus S_{R}^{k+1}|}{n \cdot T} = \liminf_{T \to \infty} \frac{|S_{k,T}^{k} \setminus S_{R}^{k}| + |(S_{k,M}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{n \cdot T}$$

$$\geq \liminf_{T \to \infty} \frac{|S_{k,T}^{k}|}{n \cdot T} + \liminf_{T \to \infty} \frac{|(S_{k,M}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{n \cdot T}.$$ 

Since the properties are satisfied for $\sigma_{k}^{k}$, we get that $\liminf_{T \to \infty} \frac{|S_{k,T}^{k}|}{n \cdot T} \geq \frac{\delta}{n}$. We thus just need to argue that

$$\liminf_{T \to \infty} \frac{|(S_{k,M}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{n \cdot T} \geq \frac{\delta}{n}$$

and we are done. That statement can be seen as follows

$$\delta \leq \nu \Rightarrow \delta \leq \liminf_{T \to \infty} \frac{|(S_{k,0}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{T} \Rightarrow \frac{\delta }{n} \leq \liminf_{T \to \infty} \frac{|(S_{k,0}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{n \cdot T} \Rightarrow$$

$$\frac{\delta }{n} \leq \liminf_{T \to \infty} \frac{|(S_{k,M}^{k} \setminus S_{R}^{k}) \cap M^{T}|}{n \cdot T},$$

where the last inequality comes from that $S_{k}^{k,0}$ consists of the same pairs as $S_{k,M}^{k}$ and then $M$ more.
The above leads to the following theorem.

**Theorem 22.** For all $\varepsilon < \frac{1}{2}$, there exists no finite-memory deterministic-update $\varepsilon$-optimal Markov strategy for Player 1 in the Big Match.

### 7 Generalized Big Match Games

In order to generalize our results to arbitrary repeated games with absorbing states, we follow the approach of Kohlberg and consider first the subset of such games where Player 1, like in the Big Match, has just the choice whether to declare that the game should stop or continue. But unlike the Big Match, in case Player 1 declares the game should stop, the game will only stop with some non-zero probability (that depends on the action of Player 2). Furthermore, Player 2 can have any number of actions and the rewards can be arbitrary. We call such games *generalized* Big Match games.

More formally, a generalized Big Match game $G$ is specified as follows. Let $A_1 = \{L, R\}$ and let $A_2$ be any finite set. The rewards $\pi(a_1, a_2)$ are arbitrary, but the stop probabilities $\omega(a_1, a_2)$ must satisfy that $\omega(L, a_2) = 0$ and $\omega(R, a_2) > 0$ for all $a_2 \in A_2$.

Our strategies for generalized Big Match games will follow the same template as those given in Sections 3 and 4. The change required is a modification of the base strategies $\sigma_i^{1,\xi}$. The proof that these new base strategies $\tau_i^{1,\xi}$ have the desired properties follow those for $\sigma_i^{1,\xi}$, but uses also additional ideas similar to those of Kohlberg[11].

Given $G$ we define the *derived matrix game* $\tilde{G}$ by

$$
\tilde{G}_{a_1, a_2} = \begin{cases} 
\pi(a_1, a_2) & \text{if } a_1 = L \\
\omega(a_1, a_2) \cdot \pi(a_1, a_2) & \text{if } a_1 = R
\end{cases}
$$

**Assumption 1.** In the remainder of this section we make the following assumptions about the given generalized Big Match game $G$:

- The entries of $\tilde{G}$ are integer.
- The value $\tilde{G}$ is 0.
- Player 1 does not have a pure optimal strategy in $\tilde{G}$.

We observe that the last requirement means that in the matrix game $\tilde{G}$ Player 1 has a unique optimal strategy and it plays each action with non-zero probability.

Define $\omega = \min_{a_2} \omega(R, a_2)$ to be the minimum non-zero stop probability of $G$ and $K = \max_{a_1, a_2} |\tilde{G}_{a_1, a_2}|$ be the maximum magnitude of an entry of $\tilde{G}$.

**Remark.** As noted in Observation 1, it is sufficient to consider only pure strategies for Player 2. Unlike for the Big Match, here play may continue even when Player 1 chooses the action $R$, and hence it is not sufficient to consider only pure Markov strategies for Player 2. We shall however (as done also by Kohlberg) give only the proof for this special case and just note that the proof for general pure strategies of Player 2 is done along the same lines.
Generalized density of pure Markov strategies  In order to generalize our strategies given in Sections 3 and 4, we must first generalize the notion of density of a pure Markov strategy $\sigma$ for Player 2. We then define

$$gdens(\sigma_T) = \frac{\sum_{i=1}^{T} \pi(L, \sigma_i)}{T},$$

and further, for $T' < T$ we define

$$gdens(\sigma, T', T) = \frac{\sum_{i=T'}^{T} \pi(L, \sigma_i)}{T - T'}.$$

We can make a similar correspondence between the generalized density of a play and the average of the rewards received by Player 1. The main difference here is that the play may possibly continue whenever Player 1 plays the action $R$. However, the event that Player 1 plays action $R$ an infinite number of times happens with probability 0, since each time Player 1 does play action $R$, the game stops with probability at least $\omega > 0$ by definition.

**Observation 3.** Suppose Player 2 follows a pure Markov strategy $\sigma$. Consider a play $P$ in which Player 1 plays only the action $L$. Then for any $T < |P|$ we have

$$gdens(\sigma_T) = \frac{1}{T} \sum_{T'=1}^{T} r_T,$$

where $r_T$ is the reward given to Player 1 in round $T$. Consider now an infinite play $P$ in which Player 1 plays action $R$ only a finite number of times. Then we have

$$u_{\inf}(P) = \lim \inf_{T \to \infty} gdens(\sigma_T),$$

and

$$u_{\sup}(P) = \lim \sup_{T \to \infty} gdens(\sigma_T).$$

We need the following simple statement, which can be viewed as a quantified version of [11, Lemma 2.5].

**Lemma 23.** Let $\sigma$ be a pure Markov strategy for Player 2, and let $j$ be any integer. If $T \cdot gdens(\sigma_T) \leq -j \cdot 2K$ then

$$\sum_{i=1}^{T} \omega(R, \sigma_i^T) \cdot \pi(R, \sigma_i^T) \geq j.$$

**Proof.** Let $\sigma_1$ be the (unique) optimal strategy in $\tilde{G}$. By Assumption [4] we have that $\sigma_1(L) > 0$ and that the value of $\tilde{G}$ is 0. This immediately implies that $\sigma_1(L) > 1/2K$, since $p = \sigma_1(L)$ must satisfy an equation of the form

$$p \cdot \tilde{G}_{L,a_2} + (1 - p) \cdot \tilde{G}_{R,a_2} = 0,$$

for some $a_2 \in A_2$, where the entries $\tilde{G}_{L,a_2}$ and $\tilde{G}_{R,a_2}$ are non-zero and integers of magnitude at most $K$. 

Now, by optimality of $\sigma_1$ we have

$$\sigma_1(L) \cdot \pi(L, a_2) + \sigma_1(R) \cdot \omega(R, a_2) \cdot \pi(R, a_2) \geq 0,$$

for any $a_2 \in A_2$, and thus also

$$\sum_{i=1}^{T} \sigma_1(L) \cdot \pi(L, \sigma^T_i) + \sigma_1(R) \cdot \omega(R, \sigma^T_i) \cdot \pi(R, \sigma^T_i) \geq 0.$$ 

By assumption $T \cdot gdens(\sigma^T) = \sum_{i=1}^{T} \pi(L, \sigma^T_i) \leq -j \cdot 2K$. Hence

$$\sum_{i=1}^{T} \omega(R, \sigma^T_i) \cdot \pi(R, \sigma^T_i) \geq -\frac{\sigma_1(L)}{\sigma_1(R)} \sum_{i=1}^{T} \pi(L, \sigma^T_i) \geq \frac{1}{2K} \cdot j \cdot 2K = j.$$

The Big Match as a generalized Big Match game  The Big Match as defined in Section 2 does not immediately fit our definition of generalized Big Match games, since the value of the derived matrix game is $1/2$ rather than 0. To achieve this we may simply replace the rewards of value 0 with value $-1$. Doing this we see that

$$gdens(\sigma^T) = \frac{|\{i \mid (\sigma^T)_i = L\}| - |\{i \mid (\sigma^T)_i = R\}|}{T} = 2|\{i \mid (\sigma^T)_i = L\}| - T = 2 dens(\sigma^T) - 1$$

for any $\sigma^T$.

7.1 Small space $\varepsilon$-supremum-optimal strategies in generalized Big Match games

For given $\varepsilon$, let $\xi = \epsilon^2/(4K^4)$. Thus $\varepsilon = 2K^2\sqrt{\xi}$. The strategy $\sigma^*_1$ for Player 1 is obtained from the strategy of Section 3 by exchanging the base strategy $\sigma^{i\xi}_1$ with the new base strategy $\tau^{i\xi}_1$ defined next.

The base strategy. Similar to $\sigma^{i\xi}_1$, the strategy $\tau^{i\xi}_1$ uses deterministic updates of memory, and uses integers as memory states The memory update function is changed to

$$\tau^{i\xi, ut}_1(a, j) = j - \tilde{G}_{R,a} = j - \omega(R, a)\pi(R, a),$$
whereas the action function is unchanged as

$$\tau_{i,\xi,1}^a(j)(R) = \begin{cases} 
\xi^4(1 - \xi)^{i+j} & \text{if } i + j > 0 \\
\xi^4 & \text{if } i + j \leq 0
\end{cases}.$$ 

Note that when the Big Match is redefined as a generalized Big Match game, we have that $\tau_{i,\xi}^1 = \sigma_{i,\xi}^1$.

We will next generalize the statements of Section 3 in the following paragraphs.

7.1.1 Space usage of the strategy

We will here consider the space usage of $\sigma_1^*$. Like for the strategy $\sigma_1^*$ for the Big Match we are interested in the length of the epochs and can generalize Lemma 7 to generalized Big Match.

Lemma 24. For any $\gamma, \delta \in (0, 1/4)$, there is a constant $M$ such that with probability at least $1 - \gamma$, for all $i \geq M$ and all $j \in \{1, \ldots, i\}$, we have that

$$t(i,j,i^2) - t(i,j - 1,i^2) \in [(1 - \delta)F(i)/i, (1 + \delta)F(i)/i].$$

Proof. The proof is precisely the same as for Lemma 7, since it only concerns itself with the length of epochs and not with the base strategy, and only the base strategy has changed as compared to section Section 3. \hfill \Box

We next give a generalization of Lemma 8.

Lemma 25. For all constants $\gamma > 0$, with probability at least $1 - \gamma$, the space usage of $\sigma_1^*$ is $O(\log f(T) + \log K)$.

Proof. Observe that the base strategy $\tau_{i,\xi}^1$ can reach a factor of $2K$ more memory states up to round $T$ than $\sigma_{i,\xi}^1$, since $\sigma_{i,\xi}^1$ changes its counter by $\pm 1$ in each round while $\tau_{i,\xi}^1$ changes its counter by some number in $\{-K, \ldots, K\}$. Thus, $\sigma_1^*$ uses at most a factor $2K$ more memory states than using $\tau_{i,\xi}^1$ as the base strategy instead of $\sigma_{i,\xi}^1$ in round $T$ for any $T$. The statement then follows from a similar proof as the one for Lemma 8. \hfill \Box

7.1.2 Play stopping implies good outcome

Lemma 26. Let $T, i \geq 1$ be integers and $0 < \xi < 1$ be a real number. Let $\sigma \in (A_2)^T$ be an arbitrary prefix of a pure Markov strategy for Player 2. Consider the first $T$ rounds where the players play $G$ following $\tau_{i,\xi}^1$ and $\sigma$ respectively.

Let $S \in \{1, \ldots, T\} \cup \{\infty\}$ be a random variable in case the game stops in the first $T$ rounds denotes that round, and is $\infty$ otherwise. Let $U$ be the random variable that denotes the outcome in case the game stops in the first $T$ rounds, and is $0$ otherwise. Then

1. $$-\mathbb{E}[U \mid U < 0] \Pr[U < 0] \leq \xi^3(1 - \xi)^{i-K+1} + (1 - \xi)^{1-2K} \mathbb{E}[U \mid U > 0] \Pr[U > 0].$$

2. If $\text{gdens}(\sigma^T) \leq -i \cdot 2K/T$ then

$$\Pr[S < \infty] \geq \xi^4 \cdot \omega.$$
Proof. Define
\[ d_\ell = - \sum_{\ell' < \ell} \tilde{G}_{R,\sigma_{\ell'}}. \]
and note that \( d_\ell \) is the value of the counter used by \( \tau_1^{i,\xi} \) as memory in step \( \ell \). There is an illustration of how \( d_\ell \) could evolve through the steps in Figure 5. Let \( I = \{ \ell \in \{1, \ldots, T\} \mid \tilde{G}_{R,\sigma_\ell} < 0 \} \) and \( D = \{ \ell \in \{1, \ldots, T\} \mid \tilde{G}_{R,\sigma_\ell} > 0 \} \) be the sets of times where the counter is incremented and decremented, respectively. For integer \( d \), define sets \( K_d \) by

\[ K_d = \{ \ell \in \{1, \ldots, T\} \mid (\tilde{G}_{R,\sigma_\ell} < 0 \& d_\ell \leq d < d_\ell - \tilde{G}_{R,\sigma_\ell}) \text{ or } (\tilde{G}_{R,\sigma_\ell} > 0 \& d_\ell - \tilde{G}_{R,\sigma_\ell} \leq d < d_\ell) \}. \]

Intuitively, \( K_d \) is now the set of times where the counter is either incremented to pass through the value \( d \) and end above, or is decreased from above \( d \) passing through the value \( d \). For instance, the gray row of Figure 5 corresponds to a set \( K_d \), which consists of the edges that starts below or on the bottom of the gray row and ends over or on the top of the gray row. Each such edge are wider, in the figure, than the remaining edges. Note that horizontal edges are not in \( K_d \). Notice each \( \ell \in \{1, \ldots, T\} \) belongs to exactly \( |\tilde{G}_{R,\sigma_\ell}| \) many of the sets \( K_d \). Also if \( K_d = \{k_1 < k_2 < \cdots < k_m\} \), for some \( m \), then \( k_j \in I \) if and only if \( k_{j+1} \in k_{j+1} \in D \). Thus \( K_d \) is an sequence of elements alternately from \( I \) and \( D \), and starts with an element of \( I \) when \( d \geq 0 \) and starts with an element of \( D \) otherwise.

Finally let, \( A_\ell \in \{L, R\} \) be the random variable indicating the action of Player 1 at time step...
\[ \ell, \text{ and define } p_\ell = \Pr[S \geq \ell & A_\ell = \textbf{R}] \text{. We now have} \]
\[ E[U] = \sum_{\ell=1}^{T} \Pr[S = \ell] \cdot \pi(\textbf{R}, \sigma_\ell) = \sum_{\ell=1}^{T} p_\ell \cdot \omega(\textbf{R}, \sigma_\ell) \pi(\textbf{R}, \sigma_\ell) \]
\[ = \sum_{\ell=1}^{T} p_\ell \cdot \tilde{G}_{\textbf{R}, \sigma_\ell} = \sum_{\ell=1}^{T} \sum_{d \in K_d} \sum_{\ell \in K_d} \text{sgn}(\tilde{G}_{\textbf{R}, \sigma_\ell}) \cdot p_\ell \]
\[ = \sum_{d} \sum_{\ell \in K_d} \text{sgn}(\tilde{G}_{\textbf{R}, \sigma_\ell}) \cdot p_\ell \ . \]
Similarly,
\[ E[U \mid U < 0] \Pr[U < 0] = - \sum_{d \in K_d \cap I} p_\ell \ , \]
and
\[ E[U \mid U > 0] \Pr[U > 0] = \sum_{d \in K_d \cap D} p_\ell . \]
For integer \( d \), define \( E_{\text{loss}, d} = \sum_{\ell \in K_d \cap I} p_\ell \) and \( E_{\text{win}, d} = \sum_{\ell \in K_d \cap D} p_\ell \). Recall that by definition of \( \tau^i \xi \) we have \( \Pr[A_\ell = \textbf{R} \mid S \geq \ell] = \xi^4(1 - \xi^{\max(0,i+d)}) \).
Consider now \( d \geq 0 \). Then \( p_{k_1} \leq \Pr[A_\ell = \textbf{R} \mid S \geq k_1] \leq \xi^4(1 - \xi)^{i+d-K+1} \), since \( d_{k_1} > d - K \).
For even \( j \) we have
\[ p_{k_j} = \Pr[A_{k_j} = \textbf{R} \mid S \geq k_j] \Pr[S \geq k_j] \geq (1 - \xi)^{2K-1} \Pr[A_{k_j} = \textbf{R} \mid S \geq k_j+1] \Pr[S \geq k_j+1] = (1 - \xi)^{2K-1} p_{k_{j+1}} \ , \]
since \( d_{k_j} < d + K < d_{k_{j+1}} + 2K \). It follows that
\[ E_{\text{loss}, d} \leq \xi^4(1 - \xi)^{i+d-K+1} + (1 - \xi)^{1-2K} E_{\text{win}, d} . \]
For \( d < 0 \), \( E_{\text{loss}, d} \leq (1 - \xi)^{1-2K} E_{\text{win}, d} \), since for odd \( j \), \( p_{k_j} \geq (1 - \xi)^{2K-1} p_{k_{j+1}} \) as above. For \( d < -i \) we can give better estimates, like in the case of Lemma 8, but it is not needed.
Taking the summation over \( d \) then gives
\[ -E[U \mid U < 0] \Pr[U < 0] = \sum_{d} E_{\text{loss}, d} \leq \sum_{d \geq 0} \xi^4(1 - \xi)^{i+d-K+1} + (1 - \xi)^{1-2K} \sum_{d} E_{\text{win}, d} \]
\[ = \xi^3(1 - \xi)^{i-K+1} + (1 - \xi)^{1-2K} E[U \mid U > 0] \Pr[U > 0] . \]
For the second part, if \( \text{gden}(\sigma^T) \leq -i \cdot 2K/T \) then Lemma 23 gives \( \sum_{\ell=1}^{T} \tilde{G}_{\textbf{R}, \sigma_\ell} \geq i \), and hence \( d_T \leq -i \). This implies that if the game reaches round \( T \), Player 1 plays \( \textbf{R} \) with probability \( \xi^4 \), which means the game stops with probability at least \( \xi^4 \cdot \omega \).

**Lemma 27.** Let \( 0 < \xi < 0 \). Let \( \sigma \) be a pure Markov strategy for Player 2. Let \( S \) be the event that the game stops. Let \( U \) be the random variable that denotes the outcome in case the game stops, and is 0 otherwise. If \( \Pr[S \geq K\sqrt{\xi}] \) then \( E[U \mid S] \geq -2K\sqrt{\xi} \).
Proof. The proof follows along that of Lemma 27. Let $A_{i,j}$ be the set of plays in which between round 1 and round $t(i,j,1) - 1$, the game does not stop. Let $U_{i,j}$ be the random variable that denotes the outcome in case the game stops between round $t(i,j,1)$ and round $t(i,j,i^2)$, and is 0 otherwise. Note that $E[U] = \sum_{i,j} E[U_{i,j} | A_{i,j}] Pr[A_{i,j}]$.

Fix a possible value of all $t(i,j,k)$’s and denote by $Y$ the event that these particular values actually occur. Fix $i$ and $j$. Conditioned on $Y$ and $A_{i,j}$, between time $t(i,j,1)$ and $t(i,j,i^2)$ Player 1 plays $\tau_1^{i,j}$ against a fixed strategy $\sigma_{t(i,j,1)}, \sigma_{t(i,j,2)}, \ldots, \sigma_{t(i,j,i^2)}$ for Player 2. By Lemma 26 we thus have

$$-E[U_{i,j} | U_{i,j} < 0, Y, A_{i,j}] Pr[U_{i,j} < 0 | Y, A_{i,j}]$$

$$\leq \xi^3(1 - \xi)^{i-K+1} + (1 - \xi)^{1-2K} E[U_{i,j} | U_{i,j} > 0, Y, A_{i,j}] Pr[U_{i,j} > 0 | Y, A_{i,j}] .$$

Since the above inequality is true conditioned on arbitrary values of $t(i,j,1)$’s, it is true also without the conditioning:

$$-E[U_{i,j} | U_{i,j} < 0, A_{i,j}] Pr[U_{i,j} < 0 | A_{i,j}]$$

$$\leq \xi^3(1 - \xi)^{i-K+1} + (1 - \xi)^{1-2K} E[U_{i,j} | U_{i,j} > 0, A_{i,j}] Pr[U_{i,j} > 0 | A_{i,j}] .$$

Thus,

$$-E[U | U < 0] Pr[U < 0]$$

$$= -\sum_{i=1}^{\infty} \sum_{j=1}^{i} E[U_{i,j} | U_{i,j} < 0, A_{i,j}] Pr[U_{i,j} < 0 | A_{i,j}] Pr[A_{i,j}]$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} \left( \xi^3(1 - \xi)^{j-K+1} + (1 - \xi)^{1-2K} E[U_{i,j} | U_{i,j} > 0, A_{i,j}] Pr[U_{i,j} > 0 | A_{i,j}] \right) Pr[A_{i,j}]$$

$$\leq \xi + (1 - \xi)^{1-2K} \sum_{i=1}^{\infty} \sum_{j=1}^{i} E[U_{i,j} | U_{i,j} > 0, A_{i,j}] Pr[U_{i,j} > 0 | A_{i,j}] Pr[A_{i,j}]$$

$$= \xi + (1 - \xi)^{1-2K} E[U | U > 0] Pr[U > 0]$$

Hence

$$-(1 - \xi)^{2K-1} E[U | U < 0] Pr[U < 0] \leq \xi(1 - \xi)^{2K-1} + E[U | U > 0] Pr[U > 0] ,$$

and so

$$-E[U] = -E[U | U < 0] Pr[U < 0] - E[U | U > 0] Pr[U > 0]$$

$$\leq -(1 - (1 - \xi)^{2K-1}) E[U | U < 0] Pr[U < 0] + \xi(1 - \xi)^{2K-1}$$

$$\leq K(1 - (1 - \xi)^{2K-1}) + \xi(1 - \xi)^{2K-1}$$

$$\leq K(2K - 1)\xi + \xi \leq 2K^2 \cdot \xi ,$$

where we use the equation $1 - (1 - x)^k \leq kx$, which is valid for positive integer $k$ and $0 \leq x \leq 1$. Hence the expected outcome conditioned on the game stopping can be estimated by

$$E[U | S] = \frac{E[U]}{Pr[S]} \geq \frac{-2K^2\xi}{K\sqrt{\xi}} = -2K\sqrt{\xi} ,$$

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using the above estimate on $E[U]$ and the assumption $\Pr[S] \geq K\sqrt{\xi}$. 

7.1.3 Low density means play stops

**Lemma 28.** Let $\varepsilon, \delta \in (0,1)$. Let $\sigma$ be an arbitrary pure Markov strategy for Player 2. Let $a_{i,j}$ be some numbers. Consider the event $Y$ where $t(i,j) = a_{i,j}$ for all $i,j$. If $\limsup_{T \to \infty} \text{gdens}(\sigma^T) \leq -\delta$, then conditioned on $Y$, there is an infinite sequence of sub-epochs and epochs $(i_n, j_n)_n$ such that $\text{gdens}(\sigma, a_{i_n,j_n}, a_{i_n,j_n+1}) \leq -\delta/4$.

**Proof.** The proof follows that of Lemma 12 with small changes. Thus, let $M$ be such that for every $T' \geq M$ we have $\text{gdens}(\sigma^{T'}) \leq -\delta/2$. Let $(T_n)_n$ be a sequence such that $T_1 \geq M$ and for all $n \geq 1$ we have that $T_{n+1} \cdot \delta/4 \geq K \cdot T_n$ and $T_n = a_{i,j}$ for some $i,j$. Let $(i_n', j_n')_n$ be the sequence such that $T_n = a_{i_n', j_n'}$. This means that even if $\text{gdens}(\sigma^{T_n}) = -K$, the generalized density $\text{gdens}(\sigma, T_n + 1, T_n + 1)$ is at most $-\delta/4$, because $\text{gdens}(\sigma^{T_n+1}) \leq -\delta/2$. But, we then get that there exists some sub-epoch $j_n$ in epoch $i_n$, such that $i_n' \leq j_n \leq j_n' + 1$ and such that $i_n' \leq i_n \leq i_n' + 1$ for which the generalized density of that sub-epoch $\text{gdens}(\sigma, a_{i_n,j_n} + 1, a_{i_n,j_n+1})$ is at most $1/2 - \delta/4$, because not all sub-epochs can have generalized density below that of the average sub-epoch. But then $(i_n, j_n)_n$ satisfies the lemma statement. \hfill \Box

**Lemma 29.** Let $\sigma$ be an arbitrary pure Markov strategy for Player 2. If $\limsup_{T \to \infty} \text{gdens}(\sigma^T) < 0$ then when played against $\sigma^*_1$ the play stops with probability 1.

**Proof.** The proof follows that of Lemma 11 with small changes. Let $\delta > 0$ be such that $\limsup_{T \to \infty} \text{gdens}(\sigma^T) \leq -\delta$. Consider arbitrary numbers $a_{i,j}$ and the event stating that $t(i,j) = a_{i,j}$ for all $i,j$. Let $(i_n, j_n)_n$ be the sequence of sub-epochs and epochs shown to exists by Lemma 28 with probability 1. That is, for each $(i_n, j_n)$ we have that sub-epoch $j_n$ of epoch $i_n$ has generalized density at most $-\delta/4$. We see that, conditioned on $Y$ that each sample are sampled uniformly at random in each sub-epoch $j$ of each epoch $i$, except for the last sample.

Now consider some fixed $n$. By Hoeffding’s inequality, Theorem 46 (setting $a_i = -K$ and $b_i = K$, and letting $c_i = \pi(L, \sigma_{t(i_n,j_n,i)})$), the probability that the generalized density of the subsequence given by the first $(i_n)^2 - 1$ samples is more than $-\delta/8$ is bounded by $2 \exp \left( -\frac{2\delta^2((i_n)^2 - 1)}{128 \cdot K^2 (i_n)^2 - 1)} \right)$. For sufficiently large $i_n$, this is less than $\frac{1}{2}$. If on the other hand the generalized density of the subsequence is more than $-\delta/8$, we see that $(i_n)^2 - 1 \geq i_n \cdot 2K \cdot 8/\delta$ for large enough $i_n$, and in that case we have, by Lemma 26 that the game stops with probability at least $\xi^4 \cdot \omega$ in sub-epoch $j_n$ of epoch $i_n$. Thus, for each of the infinitely many $n$’s for which $i_n$ is sufficiently high, we have a probability of at least $\xi^4 \cdot \omega$ of stopping. Thus play must stop with probability 1.

The argument was conditioned on some fixed assignment of endpoints of sup-epochs and epochs, but since there is such a assignment with probability 1 (since they are finite with probability 1), we conclude that the proof works without the condition. \hfill \Box
7.1.4 Proof of main result

**Theorem 30.** The strategy $\sigma_1^*$ is $2K^2\sqrt{\xi}$-supremum-optimal, and for all $\delta > 0$, with probability at least $1 - \delta$ does it use space $O(\log f(T) + \log K)$.

**Proof.** The space usage follows from Lemma 25. Let $\sigma$ be a pure Markov strategy for Player 2. Let $S$ be the event that the game stops. Let $U$ be the random variable that denotes the outcome in case the game stops, and is equal to $\limsup_{T \to \infty} \text{gden}(\sigma^T)$ otherwise. By Observation 8 we have $u_{\sup}(\sigma_1^*, \sigma) = E[U]$. Let $s = \Pr[S]$ the the probability that the game stops. We now consider three cases, either (i) $s = 1$; or (ii) $K\sqrt{\xi} < s < 1$; or (iii) $s \leq K\sqrt{\xi}$. In case (i), by Lemma 27 we have $E[U] = E[U | S] \geq -2K\sqrt{\xi}$.

In case (ii), by Lemma 27 $E[U | S] \geq -2K\sqrt{\xi}$, since $K\sqrt{\xi} < s$. And by Lemma 29 we have $E[U | \overline{S}] \geq 0$, since $s < 1$. Thus

$$E[U] \geq s \cdot (-2K\sqrt{\xi}) + (1 - s) \cdot 0 \geq -2K\sqrt{\xi}.$$ 

In case (iii), again by Lemma 29 we have $E[U | \overline{S}] \geq 0$, since $s < 1$. Thus

$$E[U] \geq s \cdot (-K) + (1 - s) \cdot 0 \geq -K^2\sqrt{\xi}.$$ 

\[\square\]

7.2 An $\varepsilon$-optimal strategy that uses $\log \log T$ space for generalized Big Match games

In this section we give a $\varepsilon$-optimal strategy $\sigma_1^*$ for Player 1 in any generalized Big Match game that for all $\delta > 0$ with probability $1 - \delta$ uses $O(\log \log T)$ space. Similarly to how Section 4 showed that the strategy $\sigma_1^*$, if initialized correctly, from Section 3 was $\varepsilon$-optimal for Player 1 in the Big Match, we here show that the strategy $\sigma_1^*$ from Section 7.1 if initialized correctly, is $\varepsilon$-optimal. Similarly to Section 4 the $\varepsilon$-optimal strategy for generalized Big Match games is simply an instantiation of the strategy $\sigma_1^*$ from Section 7.1 setting $f(T) = \lceil \log T \rceil$. We can then let $\overline{T} = \log T$ and $F(T) = 2^\overline{T}$. The proofs of the statements in this section is nearly identical to those of Section 4 but there are minor changes and they are thus given here in full.

The claim about the space usage of $\sigma_1^*$ is already established in Section 7.1. To obtain the stronger property of $\varepsilon$-optimality rather than just $\varepsilon$-supremum-optimality, we just needs to establish a lim inf version of Lemma 29 like how we in 4 gave a lim inf version of Lemma 12.

First we show a technical lemma similarly to Lemma 14. Recall that for a pure Markov strategy $\sigma$ and a sequence of integers $I = \{i_1, i_2, \ldots, i_m\}$, we have that $\sigma_I$ is the sequence, $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m}$. Again, note that $\sigma^{k} = \sigma_{\{1, \ldots, k\}}$.

**Lemma 31.** Let $\sigma$ be a pure Markov strategy for Player 2, $\delta < 1/4$ be a positive real, and $M$ be a positive integer. Let $\liminf_{T \to \infty} \text{gden}(\sigma^T) \leq -\delta$. Let $\ell_1, \ell_2, \ldots$ be such that for all $i \geq M$, we have that $\ell_i \in [(1 - \delta) \cdot (2^{2i+1} - 1), (1 + \delta) \cdot (2^{2i+1} - 1)]$. Then there exists a sequence $k_2, k_3, \ldots$ such that for infinitely many $i > M$, we have that $\ell_{i-1} + \delta / K^2 \leq k_i \leq \ell_i$ and that $\liminf_{\ell \to \infty} \text{gden}(\sigma_{\{\ell_{i-1}+1, \ldots, \ell_i\}}) \leq -\delta^2 / 4$.

**Proof.** The proof follows Lemma 14 with small changes. Let $\ell_i$ be as required. If there are infinitely many $i$ such that $\text{gden}(\sigma_{\{\ell_{i-1}+1, \ldots, \ell_i\}}) \leq -\delta^2 / 4$ then set $k_i = \ell_{i+1}$ and the lemma follows by observing...
\((k_i - \ell_{i-1}) \geq (1 - \delta)(2^{i+1} - 1) - (1 + \delta)(2^i - 1) = (1 - 3\delta)2^i \geq \delta/K2^{i-2}\), for \(i > M\). So assume that only for finitely many \(i\), \(\text{gdens}(\sigma_{\ell_{i-1}+1,\ldots,\ell_i}) \leq -\frac{\delta}{4}\). Thus the following claim can be applied for arbitrary large \(i_0\).

**Claim 32.** Let \(i_0 \geq M\) be given. If for every \(i \geq i_0\), \(\text{gdens}(\sigma_{\ell_{i-1}+1,\ldots,\ell_i}) > -\frac{\delta}{4}\) then there exist \(j > i_0\) and \(k\) such that \(\ell_{j-1} + (\delta/K)2^{j-2} \leq k \leq \ell_j\) and \(\text{gdens}(\sigma_{\ell_{j-1}+1,\ldots,k}) \leq -\delta\).

We can use the claim to find \(k_2, k_3, \ldots\) inductively. Start with large enough \(i_0 \geq M\) and set \(k_i = \ell_i\) for all \(i \leq i_0\). Then provided that we already inductively determined \(k_2, k_3, \ldots, k_{i_0}\), we apply the above claim to obtain \(j\) and \(k\), and we set \(k_j = k\) and \(k_i = \ell_i\), for all \(i = i_0 + 1, \ldots, j - 1\).

So it suffices to prove the claim. For any \(d \geq 1\), \(\ell_{i_0} \cdot 2^{d-1} \leq \ell_{i_0+d}\) and

\[
\text{gdens}(\sigma^{\ell_{i_0+d}}) \geq \frac{-\frac{\delta}{4}(\ell_{i_0+d} - \ell_{i_0}) - K \cdot \ell_{i_0}}{\ell_{i_0+d}}.
\]

Furthermore, if \(d \geq 1 + \log (4K/\delta)\) then \(\ell_{i_0} \leq \frac{\delta}{4K} \ell_{i_0+d}\) and

\[
\text{gdens}(\sigma^{\ell_{i_0+d}}) \geq \frac{-\frac{\delta}{4} - \frac{\delta}{4}}{\ell_{i_0+d}} = -\frac{\delta}{2}.
\]

Since \(\liminf_{k \to \infty} \text{gdens}(\sigma^k) \leq -\delta\), there must be \(k\) and \(d \geq 1 + \log (4/\delta)\) such that \(\ell_{i_0+d-1} \leq k \leq \ell_{i_0+d}\) and \(\text{gdens}(\sigma^k) \leq -\delta\). Set \(j = i_0 + d\). Also

\[
\text{gdens}(\sigma^k) = \frac{\text{gdens}(\sigma^{\ell_{j-1}})\ell_{j-1} + \text{gdens}(\sigma_{\ell_{j-1}+1,\ldots,k})(k - \ell_{j-1})}{\ell_{j-1} + (k - \ell_{j-1})},
\]

which means

\[
\left(\text{gdens}(\sigma^k) - \text{gdens}(\sigma_{\ell_{j-1}+1,\ldots,k})\right)(k - \ell_{j-1}) = \left(\text{gdens}(\sigma^{\ell_{j-1}}) - \text{gdens}(\sigma^k)\right)\ell_{j-1}
\]

\[
\geq \left[-\frac{\delta}{2} - \delta\right]\ell_{j-1} = \frac{\delta}{2}\ell_{j-1}.
\]

Thus \(\text{gdens}(\sigma_{\ell_{j-1}+1,\ldots,k}) \leq \text{gdens}(\sigma^k)\) which in turn is less than \(-\delta\). Furthermore, \(k - \ell_{j-1} \geq \frac{\delta}{2K}\ell_{j-1} \geq \frac{\delta}{2K}(1 - \delta)(2^j - 1) \geq \frac{\delta}{4K}2^j\), provided that \(j \geq 2\). Hence, \(k\) and \(j\) have the desired properties.

We are now ready to prove the \(\lim\inf\) version of Lemma \[16\].

**Lemma 33.** Let \(\sigma\) be an arbitrary pure Markov strategy for Player 2. If

\[
\liminf_{t \to \infty} \text{gdens}(\sigma_1, \ldots, \sigma_t) < 0,
\]

then when played against \(\sigma_1^*\) the play stops with probability 1.

**Proof.** The proof is similarly to Lemma \[16\] with minor modifications. Let \(0 < \delta < \frac{1}{4}\) be such that

\[
\liminf_{t \to \infty} \text{gdens}(\sigma_1, \ldots, \sigma_t) \leq -\delta
\]

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Pick arbitrary $\gamma \in (0,1)$. We will show that with probability at least $1 - \gamma$ the game stops, and this implies the statement. Let $M$ be given by Lemma 24 applied for $\gamma$ and $\delta/2$. Then we have that with probability at least $1 - \gamma$, for all $i \geq M$ and $j \in \{1, \ldots, i\}$,

$$t(i, j, i^2) - t(i, j - 1, i^2) \in \left[\left(1 - \frac{\delta}{2}\right)2^i/i, (1 + \frac{\delta}{2})2^i/i\right].$$

Pick $t_{i,j} \in \mathbb{N}$, for $i = 1, 2, \ldots$ and $j \in \{1, \ldots, i\}$, so that $t_{i,j-1} < t_{i,j}$ where $t_{i,0}$ stands for $t_{i-1,i-1}$. Let $t_{i,j} - t_{i,j-1} \in \left[\left(1 - \frac{\delta}{2}\right)2^i/i, (1 + \frac{\delta}{2})2^i/i\right]$, for all $i \geq M$ and $j \in \{1, \ldots, i\}$. Pick $M'$ so that $\frac{\delta}{2}(2^{M'} - 1) \geq \max\{t_{M,0}, (1 - \frac{\delta}{2})2^{M}\}$. Define $\ell_i = t_{i,i}$ for all $i \geq 1$. Then for all $i \geq M'$, $\ell_i \in [\left(1 - \delta\right) \cdot (2^{i+1} - 1), (1 + \delta) \cdot (2^{i+1} - 1)]$ as

$$
\ell_i = t_{i,i} = t_{M,0} + \sum_{M \leq i' \leq i \leq i'} t_{i',j} - t_{i',j-1} \\
\leq \frac{\delta}{2}(2^{M'} - 1) + \sum_{M \leq i' \leq i} i' \cdot \left(1 + \frac{\delta}{2}\right)2^{i'/i'} \\
\leq \frac{\delta}{2}(2^{M'} - 1) + \left(1 + \frac{\delta}{2}\right) \cdot (2^{i+1} - 1) \\
\leq (1 + \delta) \cdot (2^{i+1} - 1),
$$

and similarly for the lower bound: $\ell_i \geq \sum_{M \leq j' \leq j \leq j'} t_{i',j'} - t_{i',j'-1} \geq (1 - \frac{\delta}{2}) \cdot (2^{i+1} - 2M) \geq (1 - \delta) \cdot (2^{i+1} - 1)$.

Thus Lemma 31 is applicable on $\ell_i$ with $M$ set to $M'$, and we obtain a sequence $k_2, k_3, \ldots$ such that $\text{gdens}(\sigma_{\ell_i+1+\ldots,k_i}) \leq -\frac{\delta}{4}$ and $k_i - \ell_i \geq \delta2^{i-2}/K$ for infinitely many $i$. Pick any of the infinitely many $i \geq \max\{M', 3(1 + \delta)K/\delta\}$ for which $k_i - \ell_i \geq \delta2^{i-2}/K$ and $\text{gdens}(\sigma_{\ell_i+1+\ldots,k_i}) \leq -\frac{\delta}{4}$. Since $\delta2^{i-3}/K \geq (1 + \delta)2^i/i$, there is some $j \in \{1, \ldots, i\}$ such that $\ell_i - 1 + \delta/K2^{i-3} \leq k_i - (1 + \delta)2^i/i \leq t_{i,j} \leq k_i$. Fix such $j$. Since $k_i \leq t_{i,j} + (1 + \delta)2^i/i$, we have

$$
\text{gdens}(\sigma_{\ell_i+1+\ldots,t_{i,j}}) = \frac{\text{gdens}(\sigma_{\ell_i+1+\ldots,k_i})(k_i - \ell_i)}{t_{i,j} + \ell_i - 1} \\
\leq \frac{\text{gdens}(\sigma_{\ell_i+1+\ldots,k_i})((1 + \delta)2^i/i + t_{i,j} - \ell_i - 1)}{t_{i,j} + \ell_i - 1} \\
\leq -\frac{\delta}{4} \cdot \left(1 + \frac{8(1 + \delta)}{i}\right) \\
\leq -\frac{\delta}{4} + \frac{4(1 + \delta)}{i} \leq -\frac{\delta}{8}.
$$

Hence, $\text{gdens}(\sigma_{\ell_i+1+\ldots,t_{i,j}}) \leq -\frac{\delta}{8}$. So for some $j' \in \{1, \ldots, j\}$, $\text{gdens}(\sigma_{t_{i,j'+1}+\ldots,t_{i,j'}}) \leq -\frac{\delta}{8}$. We can state the following claim.

**Claim 34.** For $i$ large enough, conditioned on $t(a,b,a^2) = t_{a,b}$, for all $a \geq M$ and all $b$, and conditioned on that the game did not stop before the time $t_{i,j'-1} + 1$, the game stops during times $t_{i,j'-1} + 1, \ldots, t_{i,j'}$ with probability at least $\xi^4\omega/2$.

Conditioned on $t(a,b,a^2) = t_{a,b}$, for all $a, b$, the claim implies that the game stops with probability 1. Note that the condition is true for some valid choice of $t_{a,b}$ with probability $1 - \gamma$. This is because the claim can be invoked for infinitely many $i$'s and for each such $i$ we will have $\xi^4\omega/2$ chance of stopping.
It remains to prove the claim. Assume \( t(i, j' - 1, i^2) = t_{i,j'-1} \) and \( t(i, j', i^2) = t_{i,j'} \). Clearly,

\[
gdens(\sigma_{t, i,j'-1+1}, \ldots, t_{i,j'-1}) \leq gdens(\sigma_{t, i,j'-1+1}, \ldots, t_{i,j'}) \cdot (2^{i-1}/(2^{i-1} - 1)) \leq -\frac{\delta}{16},
\]

for \( i \) large enough. So if we sample \( i^2 - 1 \) times from \( \sigma_{t, i,j'-1+1}, \ldots, \sigma_{t, i,j'-1} \) the generalized density is at most \(-\frac{\delta}{16}\) in expectation. By Hoeffding’s inequality, Theorem \( 46 \) (setting \( a_i = -K \) and \( b_i = K \), and letting \( c_i = \pi(L, \sigma_t(i_n, j_n, i)) \)), the probability that the generalized density of the subsequence given by the first \( i^2 - 1 \) samples is more than \(-\delta/32\) (i.e. \( \delta/32 \) greater than the expectation) is bounded by

\[
2 \exp \left( -\frac{2(\delta/32(i^2 - 1))^2}{(i^2 - 1)(2K)^2} \right) = 2 \exp \left( -\frac{\delta^2}{2048 \cdot K^2 (i^2 - 1)} \right).
\]

The probability is taken over the possible choices of \( t(i, j', 1) < t(i, j', 2) < \cdots < t(i, j', i^2 - 1) \) assuming \( t(i, j' - 1, i^2) = t_{i,j'-1} \) and \( t(i, j', i^2) = t_{i,j'} \). For \( i \) sufficiently large, \( 2 \exp \left( -\frac{\delta^2}{2048 \cdot K^2 (i^2 - 1)} \right) \leq 1/2. \) Also, whenever

\[
gdens(\sigma_{t, i,j'-1,1}, \sigma_{t, i,j',2}, \ldots, \sigma_{t, i,j',i^2 - 1}) \leq -\frac{\delta}{32},
\]

we have at least \( \xi^4 \cdot \omega \) chance of stopping by Lemma \( 26 \) as Player 1 plays \( \tau_1^{i, \delta} \) against

\[
\sigma_{t, i,j'-1,1}, \sigma_{t, i,j',2}, \ldots, \sigma_{t, i,j',i^2 - 1}
\]

and \( -\frac{\delta}{32} \leq -i \cdot 2K/(i^2 - 1) \) for sufficiently large \( i \).

Hence, the game stops with probability at least \((1 - 1/2) \cdot \xi^4 \cdot \omega = \xi^4 \cdot \omega / 2. \) The claim, and thus the lemma, follows.

We can now conclude with the main result of this section.

**Theorem 35.** The strategy \( \sigma_1^* \) is \( 2K^2 \sqrt{\xi} \)-optimal, and for all \( \delta > 0 \), with probability at least \( 1 - \delta \) does it use space \( O(\log \log T + \log K) \).

**Proof.** This is proved just like Theorem \( 30 \) except that Lemma \( 33 \) is used in place of Lemma \( 29 \).

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### 8 Reduction of repeated games with absorbing states to generalized Big Match games

As explained in Section \( 7 \) for defining strategies for repeated games with absorbing states, Kohlberg reduced such games in general form to the special case of generalized Big Match games. The actual terminology, “generalized Big-Match games”, is due to Coulomb \( 3 \).

Performing the reduction of Kohlberg requires two things. The first thing is to determine the value of the repeated game. Kohlberg showed that the value is the same as the limit of the value of the associated \( n \)-stage game as \( n \) goes to infinity. The other thing is finding two optimal strategies in an associated parametrized matrix game with certain closeness properties. Here Kohlberg appealed just to semi-continuity of the mapping from the parameter to an optimal strategy of the matrix game. In this section we show how to make these two ingredients efficient, namely by describing polynomial time algorithms for them.
Hansen et al. [7] recently showed the existence of a polynomial time algorithm for computing the value of any undiscounted stochastic game with a constant number of non-absorbing states. We present below a much simpler algorithm for the case of repeated games with absorbing states based on a characterization of the value of those given by Kohlberg [11]. This algorithm is based only on bisection together with solving linear programs.

The algorithm of Hansen et al. is in fact similar in spirit, based on bisection and linear programming as well, but is applied to a discounted version of the game with a discount factor for which no explicit expression is readily available.

Additional definitions The bit-size of an integer \( n \) is the smallest non-zero integer \( \tau \) such that \(|n| < 2^\tau \). Thus \( \tau = \lfloor \log_2 |n| \rfloor + 1 \) for non-zero \( n \). For a polynomial \( p \in \mathbb{Z}[x] \), we denote by \( \|p\|_\infty \) the maximum magnitude of a coefficient of \( p \).

8.1 Marginal value of matrix games and value of repeated games

A matrix game is given by a \( m \times n \) real matrix \( A = (a_{ij}) \). The game is played by the two players simultaneously choosing a pure strategy, where Player 1 chooses action \( i \) among the \( m \) rows and Player 2 chooses action \( j \) among the \( n \) columns. Hereafter Player 1 receives payoff \( a_{ij} \). A strategy of a player is a probability distribution over the player’s actions. Let \( \Delta^m \) denote the strategies of Player 1 and \( \Delta^n \) denote the strategies of Player 2. Given \( x \in \Delta^m \) and \( y \in \Delta^n \), the expected payoff to Player 1 when Player 1 uses strategy \( x \) and Player 2 uses strategy \( y \) is then \( x^T Ay \). As shown by von Neumann [17] every matrix game \( A \) has a value \( v(A) \) in mixed strategies, namely

\[
v(A) = \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T Ay = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T Ay.
\]

Let \( O_1(A) \subseteq \Delta^m \) and \( O_2(A) \subseteq \Delta^n \) denote the set of optimal strategies for Player 1 and Player 2, respectively. That is, \( O_1(A) = \{x \in \Delta^m \mid \forall y \in \Delta^n : x^T Ay \geq v(A)\} \) and \( O_2(A) = \{y \in \Delta^n \mid \forall x \in \Delta^m : x^T Ay \geq v(A)\} \).

Let \( B \) be another \( m \times n \) real matrix. Mills [12] showed that the limit

\[
\frac{\partial v(A)}{\partial B} := \lim_{\alpha \to 0^+} \frac{v(A + \alpha B) - v(A)}{\alpha}
\]

exists and characterized the limit as the value of the game \( B \) when the strategies of Player 1 and Player 2 are restricted to be optimal in \( A \).

Theorem 36 (Mills).

\[
\frac{\partial v(A)}{\partial B} = \max_{x \in O_1(A)} \min_{y \in O_2(A)} x^T By
\] (1)

The limit \( \frac{\partial v(A)}{\partial B} \) is called the marginal value of \( A \) with respect to \( B \). It is not hard to see that Equation (1) implies that \( \frac{\partial v(A)}{\partial B} \) may be computed using linear programming. Indeed, we may express that simultaneously \( x \in O_1(A) \) and \( y \in O_2(A) \) by linear equalities and inequalities with
auxiliary variable $v$ as

$$f_n v - A^T x \leq 0 \quad f_m v - A y \geq 0$$

$$x \geq 0 \quad y \geq 0$$

$$f_m^T x = 1 \quad f_n^T y = 1$$

Thus for fixed $x$, the quantity $\min_{y \in O_2(A)} x^T B y$ may be computed by the linear program

$$\text{min } x^T B y$$
$$\text{s.t. } A^T x' - f_n v \geq 0$$
$$f_m v - A y \geq 0$$
$$f_m^T x' = 1$$
$$f_n^T y = 1$$
$$x', y \geq 0$$

with auxiliary variables $x'$ and $v$. Taking the dual we obtain

$$\text{max } r + s$$
$$\text{s.t. } A p + f_m r \leq 0$$
$$f_n s - A^T q \leq x^T B$$
$$f_m^T q - f_n^T p = 0$$
$$p, q \geq 0$$

in variables $p, q, r, s$, and then by reintroducing $x \in O_1(A)$ as variables we obtain the following linear program for computing $\frac{\partial v(A)}{\partial B}$.

$$\text{max } r + s$$
$$\text{s.t. } A p + f_m r \leq 0$$
$$f_n s - A^T q - x^T B \leq 0$$
$$f_m^T q - f_n^T p = 0$$
$$f_n v - A^T x \leq 0$$
$$A y' - f_m v \leq 0$$
$$f_m^T x' = 1$$
$$f_n^T y' = 1$$
$$p, q, x, y' \geq 0$$

Appealing to the existence of polynomial time algorithms for linear programming we get:

**Corollary 37.** The marginal value $\frac{\partial v(A)}{\partial B}$ can be computed in polynomial time in the bit-size of $A$ and $B$.

In this section it will be useful to introduce an alternative notation for repeated games with absorbing states, their matrix form. Consider such a game given by action sets $A_1$ and $A_2$, the stage payoff function $\pi : A_1 \times A_2 \to \mathbb{R}$, and the absorption probability function $\omega : A_1 \times A_2 \to \mathbb{R}$.

We shall now assume $A_1 = \{1, 2, \ldots, m\}$ and $A_2 = \{1, 2, \ldots, n\}$. We then let $b_{ij} = \pi(i, j)$ and $\omega_{ij} = \omega(i, j)$. The game will now be identified by a $m \times n$ matrix $A = (a_{ij})$, populated by the symbolic entries $a_{ij}$ defined by letting $a_{ij} = \omega_{ij} b_{ij}^*$ if $\omega_{ij} > 0$ and $a_{ij} = b_{ij}$ if $\omega_{ij} = 0$.

---

2By $f_n$ we mean the vector $(1, \ldots, 1)^T$ of dimension $n$. 

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Let $A = (a_{ij})$ be a $m \times n$ repeated game with absorbing states. The notion of the derived matrix game $\tilde{A} = (\tilde{a}_{ij})$ obtained from $A$ is generalized from the definition of Section 7 to be given by

$$\tilde{a}_{ij} = \begin{cases} 
\omega_{ij}b_{ij} & \text{if } a_{ij} = \omega_jb_{ij}^* \\
b_{ij} & \text{if } a_{ij} = b_{ij}
\end{cases}$$

Also, given reals $u$ and $t$, we define an associated $m \times n$ matrix game denoted $A(u,t)$, letting entry $(i,j)$ be

$$A(u,t)_{ij} = \omega_{ij}b_{ij} + (1 - \omega_{ij})u + t((1 - \omega_{ij})(b_{ij} - u))$$

Let $A_1$ be the $m \times n$ matrix with entries $\omega_{ij}b_{ij} + (1 - \omega_{ij})u$, and $A_2$ be the $m \times n$ matrix with entries $(1 - \omega_{ij})(b_{ij} - u)$. In other words, we write $A(u,t) = A_1 + tA_2$. Then the limit

$$\frac{\partial v(A(u,t))}{\partial t^+} := \lim_{t \to 0^+} \frac{v(A(u,t)) - v(A(u,0))}{t} = \frac{\partial v(A_1)}{\partial A_2}$$

exists and may be computed in polynomial time by Theorem 36 and Corollary 37. Define the extended real number $\Delta_A(u)$ by the limit

$$\Delta_A(u) := \lim_{t \to 0^+} \frac{v(A(u,t)) - u}{t}.$$ 

Clearly, $\Delta_A(u) = \frac{\partial v(A(u,t))}{\partial t^+}$ when $v(A(u,0)) = u$, and otherwise $\Delta_A(u)$ is $\infty$ or $-\infty$ depending on whether $v(A(u,0)) > u$ or $v(A(u,0)) < u$. Kohlberg showed that the value of $A$ can be characterized by $\Delta_A(u)$.

**Theorem 38 (Kohlberg).** Let $A$ be a repeated game with absorbing states. The value of $A$ is the unique point $u_0$ for which

$$u < u_0 \Rightarrow \Delta_A(u) > 0$$

and

$$u > u_0 \Rightarrow \Delta_A(u) < 0.$$ 

Using this characterization and bisection together with Corollary 37 yields a very simple algorithm for approximating the value of a repeated game with absorbing states.

**Proposition 39.** There is an algorithm that given a repeated game with absorbing states $A$ and $\varepsilon > 0$ computes the value of $A$ to within an additive error $\varepsilon$ in polynomial time in the bit-size of $A$ and $\log(1/\varepsilon)$.

### 8.2 Parametrized Matrix Games

The value of a $m \times n$ matrix game $A$ as well as an optimal strategy for Player 1 may be computed by the following linear program in variables $(x,v)$.

$$\begin{align*}
\max \quad & v \\
\text{s.t.} \quad & f_nv - A^Tx \leq 0 \\
& x \geq 0 \\
& f_m^Tx = 1
\end{align*}$$
A basic solution to LP (2) is obtained by selecting \( m + 1 \) constraints indexed by \( B \), that includes the equality constraint. This gives the \((m + 1) \times (m + 1)\) matrix \( M^A_B \), consisting of the coefficients of these constraints, appropriately ordered (we shall assume the equality constraint is ordered last). A basic solution is determined by \( B \) if \( M^A_B \) is non-singular, and in that case it is \((x, v)^T = (M^A_B)^{-1}e_{m+1}\). By Cramer’s rule \( x_i = \det((M^A_B)_{ij})/\det(M^A_B) and \( v = \det((M^A_B)_{m+1})/\det(M^A_B) \), where \((M^A_B)_{ij}\) is the matrix obtained from \( M^A_B \) by replacing column \( i \) with \( e_{m+1} \). The basic solution \((x, v)^T\) is a basic feasible solution (bfs) if also \( x \geq 0 \) and \( f_nv - A^Tx \leq 0 \).

We consider the setting where each entry of \( A \) is a linear function in a variable \( t \). Let \( A(t) \) denote this matrix game. When \( t_0 > 0 \) is sufficiently small, then if \( B \) defines an optimal basic feasible solution for \( A(t_0) \), then \( B \) also defines an optimal basic feasible solution for any \( 0 < t \leq t_0 \). We give an explicit bound for this using the following fundamental root bound.

**Lemma 40.** Let \( f \in \mathbb{Z}[x] \) be a non-zero integer polynomial. Then for any non-zero root \( \gamma \) of \( f \) it holds \((2\|f\|_\infty)^{-1} < |\gamma| < 2\|f\|_\infty\).

Using this we have the following precise statement.

**Proposition 41.** Let \( A(t) \) be a \( m \times n \) matrix game parametrized by \( t \), where each entry is a linear function in \( t \) with integer coefficients of bit-size at most \( \tau \). Let \( t_0 = 4((m + 1)2^{\tau + 1})^{2(m + 1)}^{-1} \). If \( B \) defines an optimal bfs for \( A(t_0) \) then \( B \) also defines an optimal bfs for \( A(t) \) for all \( 0 < t \leq t_0 \).

**Proof.** Let \( P^B_i(t) = \det((M^A_B)_{ij}) \) and \( Q^B(t) = \det(M^A_B) \). These are polynomials of degree at most \((m + 1)\) having coefficients of magnitude at most \((m + 1)!2^{m+1}(2\tau)^{m+1} \leq ((m + 1)2^{\tau + 1})^{m+1} \). By Lemma 40 we then have that \( \text{sgn}(Q^B(t)) = \text{sgn}(Q(t_0)) \) and \( \text{sgn}(P^B_i(t)) = \text{sgn}(P^B_i(t_0)) \) for all \( 0 < t \leq t_0 \). This means that if \( B \) defines a bfs for \( t_0 \), then \( B \) also defines a bfs for all \( 0 < t < t_0 \). To ensure that the bfs defined by \( B \) is optimal we shall compare it with any other bfs defined by a different set \( B' \). We then need to ensure that \( \frac{p_{m+1}^B(t)}{Q^B(t)} \geq \frac{p_{m+1}^{B'}(t)}{Q^{B'}(t)} \). For this we consider the polynomial \( H(t) = P_{m+1}^B(t)Q^{B'}(t) - P_{m+1}^{B'}Q^B(t) \). Note that \( H \) is a polynomial of degree at most \( 2(m + 1) \) having coefficients of magnitude at most \( 2((m + 1)2^{\tau + 1})^{2(m + 1)} \). Then by Lemma 40 again, also \( \text{sgn}(H(t)) = \text{sgn}(H(t_0)) \) for all \( 0 < t < t_0 \), which means that if \( B \) defines a bfs that is also optimal for \( t_0 \) the bfs it defines for all \( 0 < t \leq t_0 \) is optimal as well. \( \square \)

### 8.3 Reduction to generalized Big Match games

We give here an effective version of the reduction of Kohlberg of repeated games with absorbing states to the special case of generalized big match games [11, Lemma 2.8 and Theorem 2.1]. We additionally make the (rather simple) extension to repeated games with generalized absorbing states.

We shall need the following lemma.

**Lemma 42.** Let \( P \) and \( Q \) be integer polynomials such that \( \lim_{t \to 0^+} \frac{P(t)}{Q(t)} \) exists. Let \( \eta = 1/k \) for a positive integer \( k \) and suppose \( \|P\|_\infty \leq M \) as well as \( \|Q\|_\infty \leq M \). Then

\[
\left| \frac{P(t)}{Q(t)} - \lim_{t \to 0^+} \frac{P(t)}{Q(t)} \right| < \eta
\]

whenever \( 0 < t \leq t_0 = (6kM^2)^{-1} \).

\footnote{By \( e_n \) we mean the standard \( n \)th unit vector of appropriate dimension.}
Proof. Note that \( \lim_{t \to 0^+} \frac{P(t)}{Q(t)} = 0 \), where \( b \) is the non-zero coefficient of \( Q \) of lowest degree, and \( a \) is the coefficient of \( P \) of the same degree. Let \( H_1(t) = k(bP(t) - aQ(t)) - bQ(t) \) and \( H_2(t) = k(aQ(t) - bP(t)) - bQ(t) \). Then Equation (3) holds if and only if \( H_1(t) < 0 \) and \( H_2(t) < 0 \). Noting that \( \|H_1\|_\infty \leq 3kM^2 \) as well as \( \|H_2\|_\infty \leq 3kM^2 \) the conclusion follows from Lemma 40.

Let \( A' \) be a \( m \times n \) repeated game with absorbing states with stage payoffs \( b'_{ij} \) and stopping probabilities \( \omega_{ij} \). For an optimal strategy \( x \) in \( A'(0,0) \), define:

\[
\omega_j = \sum_{i=1}^{m} x_i \omega_{ij}
\]

\[
b_j = \begin{cases} 
\frac{1}{\omega_j} \sum_{i=1}^{m} x_i \omega_{ij} b'_{ij} & \text{if } \omega_j > 0 \\
0 & \text{if } \omega_j = 0 
\end{cases}
\]

\[
e_j = \begin{cases} 
\frac{1}{1-\omega_j} \sum_{i=1}^{m} x_i (1 - \omega_{ij}) b'_{ij} & \text{if } \omega_j < 1 \\
0 & \text{if } \omega_j = 1
\end{cases}
\]

and similarly for an optimal strategy \( x(t) \) in \( A'(0,t) \) for some given \( t > 0 \), define:

\[
\omega(t)_j = \sum_{i=1}^{m} x(t)_i \omega_{ij}
\]

\[
b(t)_j = \begin{cases} 
\frac{1}{\omega(t)_j} \sum_{i=1}^{m} x(t)_i \omega_{ij} b'_{ij} & \text{if } \omega(t)_j > 0 \\
0 & \text{if } \omega(t)_j = 0 
\end{cases}
\]

\[
e(t)_j = \begin{cases} 
\frac{1}{1-\omega(t)_j} \sum_{i=1}^{m} x(t)_i (1 - \omega_{ij}) b'_{ij} & \text{if } \omega(t)_j < 1 \\
0 & \text{if } \omega(t)_j = 1
\end{cases}
\]

Suppose that the \( \omega_{ij} \)'s are rational numbers with common denominator \( \beta_1 \) and the nominators and \( \beta_1 \) are of bit-size \( \tau_1 \). Similarly suppose that the \( b'_{ij} \)'s are rational numbers with common denominator \( \beta_2 \) and the nominators and \( \beta_2 \) are of bit-size \( \tau_2 \). By definition \( A'(0,t)_{ij} = \omega_{ij} b'_{ij} + t(1 - \omega_{ij}) b'_{ij} \). Thus the entries of \( A'(0,t) \) are linear functions in \( t \) where the coefficients are rational numbers with common denominator \( \beta = \beta_1 \beta_2 \) and the bit-sizes of the nominators and denominators are at most \( \tau = \tau_1 + \tau_2 \). Multiplying each entry of \( A'(0,t) \) by \( \beta \) only scales every bfs, so setting \( t_0 = 4((m+1)2^{\tau+1})^{2(m+1)}^{-1} \), whenever \( B \) defines an optimal bfs in \( \beta A'(0,t_0) \) it also defines an optimal bfs for \( A'(0,t) \) for all \( 0 < t \leq t_0 \), by Proposition 11. So let \( B \) define an optimal bfs for \( A'(0,t_0) \). Let now \( P_i(t) = \det((M_B^{\beta A'(0,t)})_i) \) and \( Q(t) = \det(M_B^{\beta A'(0,t)}) \), and define \( x(t) = P_i(t)/Q(t) \). \( P_i(t) \) and \( Q(t) \) are polynomials of degree at most \( m + 1 \) having integer coefficients of magnitude at most \( ((m+1)2^{\tau+1})^{m+1} \). Furthermore is \( x(t) \) an optimal strategy in \( A'(0,t) \) for all \( 0 < t \leq t_0 \). Let \( x = \lim_{t \to 0^+} x(t) \). Then \( x \) is an optimal strategy in \( A'(0,0) \). Each coordinate \( x_i \) is the ratio between two coefficients from \( P_i(t) \) and \( Q(t) \) and is therefore a rational number with nominator and denominator of magnitude at most \( ((m+1)2^{\tau+1})^{m+1} \).

We have always

\[
\omega(t)_j = \sum_{i=1}^{m} \frac{P_i(t)}{Q(t)} \omega_{ij} = \frac{\sum_{i=1}^{m} P_i(t) \omega_{ij}}{Q(t)}
\]
and
\[ 1 - \omega(t)_j = \sum_{i=1}^{m} x(t)_i(1 - \omega_{ij}) = \frac{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})}{Q(t)} \]

Consider now the case of \( \omega(t)_j > 0 \). Then
\[ b(t)_j = \frac{Q(t)}{\sum_{i=1}^{m} P_i(t)\omega_{ij}} \sum_{i=1}^{m} P_i(t)\omega_{ij}b'_{ij} = \frac{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})b'_{ij}}{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})} \]

Consider now the case of \( \omega(t)_j < 1 \). Then
\[ e(t)_j = \frac{Q(t)}{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})} \sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})b'_{ij} = \frac{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})b'_{ij}}{\sum_{i=1}^{m} P_i(t)(1 - \omega_{ij})} \]

Since \( x = \lim_{t \to 0^+} x(t) \) we also have \( \lim_{t \to 0^+} \omega_j(t) = \omega_j \). Furthermore, in case that \( \omega_j > 0 \), we have \( \lim_{t \to 0^+} b_j(t) = b_j \), and in case \( \omega_j = 0 \), we have \( \lim_{t \to 0^+} e_j(t) = e_j \).

Let \( \eta = 1/k \) for a positive integer \( k \). Define \( t_1 = (2k((m + 1)2^{r+1})^{m+2})^{-1} \) and \( t_2 = (6k((m + 1)2^{r+1})^{2(m+1)})^{-1} \). Note that \( t_2 \leq \min(t_0, t_1) \).

Suppose that \( \omega_j > 0 \). Then \( \omega(t)_j > 0 \) if \( \sum_{i=1}^{m} P_i(t)\omega_{ij} > 0 \). Since \( \omega_j > 0 \) there exists \( i \) such that \( P_i(t) \) is a non-zero polynomial and \( \omega_{ij} > 0 \). Then by Lemma 40 it follows that \( \omega(t)_j > 0 \) for all \( 0 < t \leq t_0 \). Also from Lemma 42 we have \( |b_j - b(t)_j| < \eta \) whenever \( 0 < t \leq t_1 \). Suppose now that \( \omega_j = 0 \). Consider the polynomial \( H(t) = \frac{1}{2} \sum_{i=1}^{m} P_i(t)\omega_{ij} - \frac{1}{2} P(t)Q(t) \). We then have \( \omega(t)_j < \eta \) if and only if \( H(t) < 0 \), for \( 0 < t \leq t_0 \). The polynomial \( H \) is of degree at most \( m+1 \) and has integer coefficients of magnitude at most \( k((m + 1)2^{r+1})^{m+2} \). By Lemma 40 we have \( \omega(t)_j < \eta \) whenever \( 0 < t \leq \min(t_0, t_2) \). Also from Lemma 42 we have \( |e_j - e(t)_j| < \eta \), whenever \( 0 < t \leq t_1 \).

Putting all these observations together gives us the following effective version of [11, Lemma 2.8].

**Lemma 43.** There is an algorithm that given a \( m \times n \) repeated game \( A' \) with absorbing states as above and \( \eta = 1/k \) computes in polynomial time strategies \( x \) and \( x(t) \) that are optimal for Player 1 in \( A(0,0) \) and \( A(0, t_1) \), respectively, such that

1. If \( \omega_j > 0 \) then \( \omega(t)_2 > 0 \) and \( |b_j - b(t)_2| < \eta \).
2. If \( \omega_j = 0 \) then \( \omega(t)_2 < \eta \) and \( |e_j - e(t)_2| < \eta \).

where \( t_2 = (6k((m + 1)2^{r+1})^{2(m+1)})^{-1} \).
We will now show how to reduce an arbitrary repeated games with absorbing states to generalized Big Match games. By reduction we mean that a generalized Big Match game \( D \) is computed from a repeated game with absorbing states \( A \), such that a strategy \( \sigma_1 \) for \( D \) can be extended to a strategy \( \tau_1 \) for \( A \). In case \( \sigma_1 \) is \( \varepsilon' \)-optimal for \( D \) then \( \tau_1 \) is \( \varepsilon \)-optimal for \( A \), and likewise, in case \( \sigma_1 \) is \( \varepsilon' \)-supremum-optimal for \( D \) then \( \tau_1 \) is \( \varepsilon \)-supremum-optimal for \( A \), where \( \varepsilon' \) depends on \( \varepsilon \) and \( A \).

**Theorem 44.** Let \( A \) be a \( m \times n \) repeated game with absorbing states and let \( \varepsilon = 2^{-t} \). Assume the stage payoff \( b_{ij} \) are rational numbers such that \( |b_{ij}| \leq 1 \). Assume the stopping probabilities \( \omega_{ij} \) are rational numbers with common denominator \( \beta_1 \) and the nominator and \( \beta_1 \) are of bit-size at most \( \tau_1 \). Then \( A \) can be reduced in polynomial time to a generalized Big Match game satisfying Assumption \( \beta \) with integer entries of magnitude at most \( (24(m + 2)^{2\ell + \tau_1 + 1})20(m + 2)^2(2n + 1) \).

**Proof.** We will have 4 sources of error: In approximating the value of \( A \), in rounding the entries of \( A \), from the strategy for the generalized Big Match to which we reduce, and finally from additional strategies of Player 2 that are not part of this. We shall allow \( \varepsilon/4 \) to all these.

First we use the Algorithm of Proposition [39] to compute \( u \) such that

\[
 u + \frac{\varepsilon}{2} \leq v(A) < u + \frac{3\varepsilon}{4}.
\]

Using \( u \) we translate and round the entries of \( A \) to obtain another repeated game with absorbing states \( A' \) with the same stopping probabilities but with stage payoff \( b'_{ij} \) given by

\[
 b'_{ij} = \left\lfloor \frac{b_{ij} - u}{\varepsilon/4} \right\rfloor \frac{\varepsilon}{4}.
\]

Using Equation \((6)\) we have that \( \varepsilon/4 \leq v(A') < 3\varepsilon/4 \). Also the rounded and translated stage payoffs \( b'_{ij} \) satisfy \(-2 \leq b'_{ij} \leq 2 \) are rational numbers with common denominator \( \beta_2 = 4/\varepsilon = 2^{\ell+2} \) and nominators of bit-size at most \( \tau_2 = \ell + 3 \).

Since \( v(A') > 0 \) from Theorem [38] we have that \( \Delta_{A'}(0) > 0 \), and this means that \( v(A'(0,t)) \) can be bounded below by a linear function in an interval to the right of 0. We shall make this explicit below, providing constants \( \delta \) and \( t_1 \) such that

\[
 v(A'(0,t)) \geq \delta t
\]

whenever \( 0 \leq t \leq t_1 \).

So \( \Delta_{A'}(0) = \lim_{t \to 0^+} \frac{v(A'(0,t))}{t} > 0 \). We first fix \( \delta > 0 \) and then determine a corresponding \( t_1 \). If \( v(A'(0,0)) > 0 \) we may choose any \( \delta > 0 \). If \( v(A'(0,0)) = 0 \) we should choose \( \delta \) such that \( \delta < \Delta_{A'}(0) \).

Let \( \tau = \tau_1 + \tau_2 \) and \( \beta = \beta_1\beta_2 \). Scaling the entries of \( A'(0,t) \) by \( \beta \) and setting \( t_0 = (4((m + 1)2^{\ell+1})2(m + 1)^{-1} \) whenever \( B \) defines an optimal bfs for \( \beta A'(0,0) \) it also defines an optimal bfs for \( A'(0,t) \) for all \( 0 < t \leq t_0 \) by Proposition [41]. So let \( B \) define an optimal bfs for \( A'(0,0) \). Let now \( P(t) = \det((M_{B}^{\beta A'(0,t)})_{m+1}) \) and \( Q(t) = \det((M_{B}^{\beta A'(0,t)})_{m+1}). \) Then when \( 0 < t \leq t_0 \) we have \( v(A'(0,t)) = \frac{P(t)}{\beta Q(t)} \). The polynomials \( P \) and \( Q \) are of degree at most \( m + 1 \) and having integer coefficients of magnitude at most \( ((m + 1)2^{\ell+1})m+1 \).

Suppose that \( v(A'(0,0)) = 0 \). Then

\[
 \Delta_{A'}(0) = \frac{\partial v(A'(0,t))}{\partial t^+} = \lim_{t \to 0^+} \frac{d}{d^t} \frac{P(t)Q(t) - P(t)\frac{d}{d^t} Q(t)}{\beta Q(t)^2}.
\]
Thus $\Delta_A'(0)$ is the ratio between the coefficients of integers polynomial where the denominator has coefficients of maximum magnitude $2^r((m+1)2^{r+1})^{2(m+1)}$. It follows that

$$\frac{1}{2} \Delta_A'(0) \geq ((m+1)2^{r+1})^{2m+3}$$

so we let $\delta = ((m+1)2^{r+1})^{2m+3}$. To determine $t_1$ we need to ensure that $v(A'(0, t)) \geq \delta t$. To this end, define the polynomial $H(t) = P(t)/\delta - \beta tQ(t)$. This is an integer polynomial of degree at most $m+2$ and $||H||_{\infty} \leq 2((m+1)2^{r+1})^{3m+4}$. By Lemma [40] letting $t_1 = 4((m+1)2^{r+1})^{3m+4}$ we obtain the desired Equation (7).

Let $\eta = \delta/4$ and use the algorithm from Lemma [43] to compute strategies $x(t_2)$ and $x$, where $t_2 = (24((m+1)2^{r+1})^{4(m+1)})$. Note that $t_2 \leq t_1$. Also note for later that $\delta \leq 2^{-m} = e/8$.

We may now proceed as in [11, Theorem 2.1]. Player 1 will commit to at every stage playing either the strategy $x$ or the strategy $x(t_2)$. In this way Player 1 becomes restricted to the $2 \times n$ repeated game with absorbing states $C = (c_{ij})$, where

$$c_{1j} = \begin{cases} \omega_jb_{j*} & \text{if } \omega_j > 0 \\ e_j & \text{if } \omega_j = 0 \end{cases}$$

and similarly

$$c_{2j} = \begin{cases} \omega(t_2)lb_{j*} & \text{if } \omega(t_2) > 0 \\ e(t_2)j & \text{if } \omega(t_2) = 0 \end{cases}$$

Since $x$ is optimal in $A'(0, 0)$ Equation (7) gives for all $j$,

$$\omega_jb_j = \sum_{i=1}^{m} x_i\omega_{ij}b_{ij}^* \geq 0$$

and similarly since $x(t_2)$ is optimal in $A'(0, t_2)$ Equation (7) gives for all $j$,

$$\omega(t_2)lb_{j*} + (1 - \omega(t_2))t_2e(t_2)j = \sum_{i=1}^{m} x_i(t_2)_i(\omega_{ij}b_{ij}^* + t_2(1 - \omega_{ij}b_{ij}^*)) \geq \delta t_2$$

Let $J = \{ j \in \{1, \ldots, n\} \mid \omega_j = 0 \text{ and } \omega(t_2)_j > 0 \}$, and consider any $j \in J$. Since $\omega_j = 0$, Lemma [43] gives $\omega(t_2)_j < \eta = \delta/4$. Since $|b_{ij}^*| \leq 2$, we then get $\omega(t_2)_je(t_2)_j \leq \delta/2$, and Equation (11) gives

$$\omega(t_2)lb_{j*} + t_2e(t_2)j \geq \delta t_2/2 .$$

Also from Lemma [43] we have $|e_j - e(t_2)_j| \leq \eta \leq \delta/2$, which means $e(t_2)_j \geq e_j - \delta/2$, which in turn means we have

$$\omega(t_2)lb_{j*} + t_2e(t_2)j \geq 0 .$$

Let $\tilde{C} = (\tilde{c}_{ij})$ be the derived matrix game from $C$. For $j \in J$, $\tilde{c}_{1j} = e_j$ and $\tilde{c}_{2j} = \omega(t_2)_j b(t_2)_j$. Thus, dividing by $1 + t_2$ we get

$$\frac{1}{1 + t_2} \tilde{c}_{2j} + \frac{t_2}{1 + t_2} \tilde{c}_{1j} \geq 0 ,$$

which means that the value of the matrix game $\tilde{C}$ restricted to the columns of $J$ is at least 0. We define a $2 \times |J|$ repeated game with absorbing states $C' = (c'_{ij})$ by restricting $C$ to the columns $J$.
and subtracting a value from each entry such the value of the derived matrix game $\tilde{C}'$ is 0. More precisely, let $v$ be the value of $\tilde{C}$ restricted to the columns of $J$, and for $j \in J$ we let $c'_{1j} = e_j - v$ and $c'_{2j} = \omega(t_2)_j(b(t_2) - v/\omega(t_2)_j)^*$.

Using the expressions previously obtained for $\omega(t)_j$, $b(t)_j$, and $e(t)_j$, we have that for $0 < t \leq t_0$, each of $\omega(t)_j$, $b(t)_j$, $e(t)_j$, and $\omega(t)_j b(t)_j$ can be expressed as rational functions of integer polynomials of degree at most $m + 1$ and integer coefficients of magnitude at most $(m + 1)2^{r+1}m$. This means that $e_j = \lim_{t \to 0^+} e(t)_j$ is a rational number with nominator and denominator of magnitude at most $(m + 1)2^{r+1}m$ as well. We can bound the nominator and denominator of the numbers $\omega(t_2)_j$, $b(t_2)_j$, and $\omega(t_2)_j b(t_2)_j$ by estimating the magnitudes after substitution of $t_2$ in the corresponding rational functions. This yield that they are rational numbers with nominator and denominator of magnitude at most $(m + 1)2^{r+1}m = 24((m + 1)2^{r+1}m)^{m+2} \leq (24(m + 1)2^{r+1}m)^{m+2}$.

Now the value $v$ is given by the value of a $2 \times 2$ sub-game of the matrix game $\tilde{C}$ restricted to the columns of $J$. This in turn means that $v$ has nominator and denominator of magnitude at most $4(24(m + 1)2^{r+1}m)^{16(m+2)}$.

We can now estimate the entries of $\tilde{C}' = (c'_{ij})$. These are just the entries from $\tilde{C}$ subtracted $v$. Hence they all have nominator and denominators of magnitude at most $8(24(m + 1)2^{r+1}m)^{20(m+2)^2} \leq (24(m + 2)2^{r+1}m)^{20(m+2)^2}$.

We now scale the entries of $C'$ obtaining another repeated game with absorbing states $D$ such that the entries of $D$ are integers. We simply do this by multiplying by least common multiple $M$ of all the denominators of the entries of $C'$. Note that $M \leq (24(m + 2)2^{r+1}m)^{40(m+2)^2}$, which makes the entries of $D$ integers of magnitude at most $K = (24(m + 2)2^{r+1}m)^{20(m+2)^2}$. In case $D$ does not have a pure optimal strategy, then $D$ satisfies Assumption 1 and we let $\sigma_1$ be a memory based strategy for Player 1 for $D$ with action map $\sigma_1^a$ and update map $\sigma_1^u$ that is either $\epsilon/(4M)$-optimal or $\epsilon/(4M)$-supremum optimal. In case that $D$ has a pure optimal strategy we simple take $\sigma_1$ to be the strategy that plays this pure action always.

From $\sigma_1$ we now construct a strategy $\tau_1$ for $A$. The action map $\tau_1^a$ will sample an action from $\sigma_1^a$. In case of a $L$ sample, $\tau_1^a$ will sample the final action from $x$ and in case of a $R$ sample, $\tau_1^a$ will sample the final action from $x(t_2)$. The update map will be a simple filtering map $\tau_1^u$ given as follows. Let $(m, j)$ be a pair of a memory state $m$ and an action $j$ of Player 2. In case $j \notin J$ we let $\tau_1^u(m, j) = \sigma_1^u(m, j)$. But if $j \notin J$ we let $\tau_1^u(m, j)$ stay in the memory state $m$, that is we let the next state be $m$ with probability 1.

Looking at the rounds where $j \in J$, the strategy $\tau_1$ inherits the performance of $\sigma_1$. Consider now $j \notin J$. Then we have either (a) $\omega_j = 0$ and $\omega(t_2)_j = 0$ or (b) $\omega_j > 0$. In case (a) Equation 11 gives $e_j \geq \delta$ and from Lemma 43 follows $e_j \geq 0$. In case (b) Equation 10 gives $b_j \geq 0$. From Lemma 43 follows $\omega(t_2)_j \geq 0$ as well as $b(t_2)_j \geq -\eta = -\delta/4 \geq -\epsilon/32$. Thus in each case the expected stage payoff is at least $u - \epsilon/32 \geq v(A) - \epsilon$.

Note that $\log K = O(m^2n(\tau + \log m)) = O(m^2n(\log 1/\epsilon + \tau_1 + \log m))$, which means that for each $\delta > 0$ with probability at least $1 - \delta$ the resulting strategy $\epsilon$-supremum optimal strategy will use space $O(f(T) + m^2n(\log 1/\epsilon + \tau_1 + \log m))$ and the resulting $\epsilon$-optimal strategy will use space $O(\log \log T + m^2n(\log 1/\epsilon + \tau_1 + \log m))$. 

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References


A Tail inequalities

**Theorem 45** (Multiplicative Chernoff bound). Let $X = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are random variables independently distributed in $[0,1]$. Then for any $\varepsilon > 0$

$$
\Pr[X \geq (1 + \varepsilon)E[X]] \leq \left(\frac{e^\varepsilon}{(1 + \varepsilon)(1 + \varepsilon)}\right)^{E[X]} \leq \exp\left(-\frac{\varepsilon^2}{2 + \varepsilon}E[X]\right),
$$

and

$$
\Pr[X \leq (1 - \varepsilon)E[X]] \leq \left(\frac{e^{-\varepsilon}}{(1 - \varepsilon)(1 - \varepsilon)}\right)^{E[X]} \leq \exp\left(-\frac{\varepsilon^2}{2}E[X]\right).
$$

Hoeffding [8] gave the following bound for sampling without replacement.

**Theorem 46** (Hoeffding). Let a population $C$ consist of $N$ values $c_1, \ldots, c_N$, where $a_i \leq c_i \leq b_i$. Let $X_1, \ldots, X_n$ denote a random sample without replacement from $C$ and $X = \sum_{i=1}^{n} X_i$. Then

$$
\Pr[|X - E[X]| \geq t] \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).
$$