Nominal Lawvere Theories

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Abstract. Lawvere theories provide a category theoretic view of equational logic, identifying equational theories with small categories equipped with finite products. This formulation allows equational theories to be investigated as first class mathematical entities. However, many formal systems, particularly in computer science, are described by equations modulated by side conditions asserting the "freshness of names"; these may be expressed as theories of Nominal Equational Logic (NEL). This paper develops a correspondence between NEL-theories and certain categories that we call nominal Lawvere theories.

Keywords: Lawvere theory, equational logic, nominal sets, Fraenkel-Mostowski set theory, fresh names.

1 Introduction

Many formal systems, particularly in computer science, may be expressed via equations modulated by side conditions asserting certain names are *fresh for* (not in the free names of) certain meta-variables:

First-order logic: $\Phi \supset (\forall a.\, \Psi) \;=\; \forall a.\, (\Phi \supset \Psi)$ if a is fresh for Φ;
λ-calculus: $\lambda\, a.\, f\, a \;=_\eta\; f$ if a is fresh for f;
π-calculus: $(\nu a\ x) \mid y \;=\; \nu a\, (x \mid y)$ if a is fresh for y.

We may express such modulated equations, and hence reason formally about the systems described by them, with *Nominal Equational Logic (NEL)* [5]. NEL-theories can also express the notions of binding and α-equivalence such systems exhibit [4]. NEL generalises standard equational logic by employing the *nominal sets* [17], and slightly more general *Fraenkel-Mostowski sets (FM-sets)* [9], models, where the manipulation of names is modelled by the action of *permutations.*

Lawvere's view of equational logic [15] identifies *equational theories* with certain *categories*; specifically, small categories with finite products, which are hence called *Lawvere theories.* This category theoretic formulation allows equational theories, as well as their models, to be treated as first class mathematical

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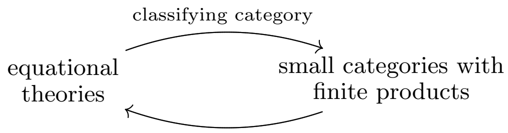
entities. This has a number of advantages for computer scientists: presentational details are abstracted away, theories may be considered in categories other than that of sets, and category theoretic tools, such as sums and monoidal tensor products, may be applied to theories in ways that are more natural than dealing directly with their presentations [11].

The chief development in Lawvere's result is the construction of a *classifying category* for each equational theory.



(1)

In the many-sorted case [6] this category's objects are tuples of sorts $(\mathsf{s}_1, \dots \mathsf{s}_n)$, which correspond to *sorting environments*, arrows are tuples of *terms* modulo provable equivalence, and composition is term *substitution*.

The elegance and usefulness of this perspective has led to a number of papers identifying generalisations of equational logic with the category theoretic structure beyond finite products needed to express them, e.g. [2, 20, 18]. This paper follows in this tradition by developing Lawvere-style correspondences for NEL.

In expressing theories of NEL as categories, let alone proving a correspondence, we encounter two major hurdles. The first is that variables within terms may have name permutations 'suspended' over them. These *suspensions* are necessary to establish many basic properties, such as $\alpha\beta$-equivalence in the λ-calculus (Ex. 3.3). Suspensions will be captured by a novel structure of arrows in our category, which we call the *internal* Perm-*action* (Def. 4.1).

The second hurdle is that while the objects of Lawvere's classifying categories are sorting environments, NEL has the richer notion of *freshness environments*, where variables are both assigned sorts and may have atoms asserted to be fresh for them. These will be captured by another novel structure called *fresh subobjects* (Def. 4.5).

These concepts, along with *equivariant* finite products (Def. 4.3), define what we will call *FM-categories*, and, where they are small, *nominal Lawvere theories*. An analogue of the correspondence (1) then follows (Sec. 6). The paradigmatic examples of FM-categories will be the category of FM-sets and the classifying category of a NEL-theory. The advantage of this generalised category theoretic view can be most clearly seen in this paper's Completeness Thm. 6.3, whose proof is more conceptually clear and less syntactic than that offered in [5].

The results of this paper are, with some minor differences in presentation, offered in full technical detail in Chap. 7 of the author's thesis [3].

2 Nominal sets and FM-sets

Fix a countably infinite set $\mathbb{A}$ of *atoms*, which we will use as names. The set Perm of (finite) *permutations* consists of all bijections $\pi : \mathbb{A} \to \mathbb{A}$ whose domain $supp(\pi) = \{a \mid \pi(a) \neq a\}$ is finite. Perm is generated by *transpositions* $(a\ b)$ that map a to b, b to a and leave all other atoms unchanged. We will make particular use of permutations known as *generalised transpositions* [5, Lem. 10.2]. Let

$$\mathbb{A}^{(n)} \triangleq \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid a_i \neq a_j \text{ for } 1 \leq i < j \leq n\} \tag{2}$$

and take $\vec{a} = (a_1, \ldots, a_n), \vec{a}' = (a'_1, \ldots, a'_n) \in \mathbb{A}^{(n)}$ with disjoint underlying sets. Then

$$(\vec{a}\ \vec{a}') \triangleq (a_1\ a'_1) \cdots (a_n\ a'_n)$$

Perm may be considered as a one-object category whose arrows are the finite permutations, identity is the trivial permutation ι, and composition is $(\pi' \circ \pi)(a) = \pi'\pi(a) = \pi'(\pi(a))$. A Perm-*set* is then a functor from Perm to the category of sets; that is, a set X equipped with a function, or Perm-action, $(\pi, x) \mapsto \pi \cdot x$ from $\mathsf{Perm} \times X$ to X such that $\iota \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = \pi'\pi \cdot x$.

We say that a set $\overline{a} \subseteq \mathbb{A}$ *supports* $x \in X$ if for all $\pi \in \mathsf{Perm}$, $\mathit{supp}(\pi) \cap \overline{a} = \emptyset$ implies that $\pi \cdot x = x$.

Definition 2.1. *A* nominal set *is a* Perm-*set* X *with the* finite support property*: for each $x \in X$ there exists some finite $\overline{a} \subseteq \mathbb{A}$ supporting x.*

If an element x is finitely supported then there is a unique least such support set [9, Prop. 3.4], which we write $\mathit{supp}(x)$ and call *the support of* x. This may be read as the *set of free names* of a term. If $\overline{a} \cap \mathit{supp}(x) = \emptyset$ for some $\overline{a} \subseteq \mathbb{A}$ we say that $\overline{a}$ *is fresh for* x and write $\overline{a} \# x$, capturing the *not free in* relation.

Example 2.2. (i) Any set becomes a nominal set under the trivial Perm-action $\pi \cdot x = x$, with finite support property $\mathit{supp}(x) = \emptyset$;
(ii) $\mathbb{A}$ is a nominal set with Perm-action $\pi \cdot a = \pi(a)$ and $\mathit{supp}(a) = \{a\}$;
(iii) $\mathbb{A}^{(n)}$ (2), and the set of finite sets of atoms $\mathcal{P}_{fin}(\mathbb{A})$, are nominal sets given the element-wise Perm-actions;
(iv) If X is a nominal set then the *finitely supported powerset* [9, Ex. 3.5]

$$\mathcal{P}_{fs}(X) \triangleq \{S \subseteq X \mid S \text{ is finitely supported in } \mathcal{P}(X)\}$$

is a nominal set given the element-wise Perm-action.

We can define a Perm-action on functions f between Perm-sets by

$$(\pi \cdot f)(x) = \pi \cdot (f(\pi^{-1} \cdot x)) \ . \tag{3}$$

If f has empty support, so that $\pi \cdot (f(x)) = f(\pi \cdot x)$, it is called *equivariant.* The nominal sets and equivariant functions between them form the category $\mathcal{N}om$.

Nominal sets are closed under their Perm-actions, but we can define a more general notion of sets that themselves have finite support. Consider the *Fraenkel-Mostowki hierarchy*:

$$\begin{aligned} \mathcal{FM}_0 &\triangleq \emptyset \\ \mathcal{FM}_{\alpha+1} &\triangleq \mathbb{A} + \mathcal{P}_{fs}(\mathcal{FM}_\alpha) \\ \mathcal{FM}_\lambda &\triangleq \textstyle\bigcup_{\alpha<\lambda} \mathcal{FM}_\alpha \qquad (\lambda \text{ a limit ordinal}) \end{aligned}$$

as α ranges over the ordinals. Each stage defines a nominal set, so the members of the hierarchy are finitely supported (but not necessarily closed) under the evident Perm-action.

Definition 2.3. *The members of the Fraenkel-Mostowki hierarchy are divided by disjoint union into atoms and sets. Call these sets the* FM-sets*.*

Most of the usual set theoretic constructions (with the notable exception of choice) can be performed with FM-sets. In particular, *FM-functions* may be defined, which are finitely supported under the Perm-action (3). The FM-sets and FM-functions form the category $\mathcal{FM}$-Set, while small subcategories $\mathcal{FM}_\lambda$-Set may be defined by taking the hierarchy only up to some limit ordinal λ.

3 Nominal Equational Logic

This section summarises Nominal Equational Logic (NEL), mildly generalising [5] so that the collection of sorts forms a nominal set rather a set. This generalisation will be discussed in Rem. 6.6.

Definition 3.1. *A* NEL-signature Σ *is specified by*

(i) a nominal set Sort_Σ*, whose elements are called the* sorts of Σ*;*
(ii) a nominal set Op_Σ*, whose elements are called the* operation symbols of Σ*;*
(iii) an equivariant map from each operation symbol $op \in \mathsf{Op}_\Sigma$ *to a* type*, which we write* $op : (\mathsf{s}_1, \ldots, \mathsf{s}_n) \to \mathsf{s}$*. Where* $n = 0$ *we write* $op : \mathsf{s}$*.*

Fix a countably infinite set Var of *variables*. Then the *terms* over Σ are

$$t ::= \pi\, x \mid op\, t \cdots t$$

for all $\pi \in \mathsf{Perm}, x \in \mathsf{Var}$ and $op \in \mathsf{Op}_\Sigma$. We call $\pi\, x$ a *suspension* and write $\iota\, x$ simply as x. We call $op\, t_1 \cdots t_n$ a *constructed term*.

The *sorting environments* SE_Σ are partial functions $\Gamma : \mathsf{Var} \rightharpoonup \mathsf{Sort}_\Sigma$ with finite domain. We define the set $\Sigma_\mathsf{s}(\Gamma)$ of *terms of sort* s *in* Γ by

(i) if $\pi \in \mathsf{Perm}$ and $x \in dom(\Gamma)$ then $\pi\, x \in \Sigma_{\pi \cdot \Gamma(x)}(\Gamma)$;
(ii) given $op : (\mathsf{s}_1, \ldots, \mathsf{s}_n) \to \mathsf{s}$ and $t_i \in \Sigma_{\mathsf{s}_i}(\Gamma)$ for $1 \le i \le n$, $op\, t_1 \cdots t_n \in \Sigma_\mathsf{s}(\Gamma)$.

The *object-level* Perm*-action* on terms, $(\pi, t \in \Sigma_\mathsf{s}(\Gamma)) \mapsto \pi * t \in \Sigma_{\pi \cdot \mathsf{s}}(\Gamma)$, is

$$\begin{aligned} \pi * (\pi'\, x) &\triangleq \pi\pi'\, x \ ; \\ \pi * (op\, t_1 \cdots t_n) &\triangleq (\pi \cdot op)(\pi * t_1) \cdots (\pi * t_n) \ . \end{aligned} \tag{4}$$

This Perm-action is not in general finitely supported, but is used in the definition of substitution: given $\Gamma, \Gamma' \in \mathsf{SE}_\Sigma$, a *substitution* $\sigma : \Gamma \to \Gamma'$ is a map from each $x \in dom(\Gamma)$ to $\sigma(x) \in \Sigma_{\Gamma(x)}(\Gamma')$. Given a term $t \in \Sigma_\mathsf{s}(\Gamma)$, the term $t\{\sigma\} \in \Sigma_\mathsf{s}(\Gamma')$ is defined by

$$\begin{aligned} (\pi\, x)\{\sigma\} &\triangleq \pi * \sigma(x) \ ; \\ (op\, t_1 \cdots t_n)\{\sigma\} &\triangleq op\, t_1\{\sigma\} \cdots t_n\{\sigma\} \ . \end{aligned} \tag{5}$$

The *freshness environments* FE_Σ are partial functions ∇ with finite domain on Var, mapping each $x \in dom(\nabla)$ to a pair $(\overline{a}, \mathsf{s})$ where $\overline{a} \in \mathcal{P}_{fin}(\mathbb{A})$, $\mathsf{s} \in \mathsf{Sort}_\Sigma$ and $\overline{a} \# \mathsf{s}$. If $\nabla(x_i) = (\overline{a}_i, \mathsf{s}_i)$ we write ∇ as

$$(\overline{a}_1 \not\# x_1 : \mathsf{s}_1, \ldots, \overline{a}_n \not\# x_n : \mathsf{s}_n) \ . \tag{6}$$

The intended meaning is that $\overline{a}_i$ is fresh for x_i, which has sort s_i. Each $\nabla \in \mathsf{FE}_\Sigma$ gives rise to a sorting environment $\nabla^{:} \in \mathsf{SE}_\Sigma$ by taking the second projection. We will abbreviate $\{a\} \not\# x : \mathsf{s}$ as $a \not\# x : \mathsf{s}$ and $\emptyset \not\# x : \mathsf{s}$ as $x : \mathsf{s}$. Given a finite set of atoms $\overline{a} \# \nabla$ we can define a new freshness environment by

$$\nabla^{\not\#\overline{a}} \triangleq (\overline{a}_1 \cup \overline{a} \not\# x_1 : \mathsf{s}_1, \ldots, \overline{a}_n \cup \overline{a} \not\# x_n : \mathsf{s}_n) \ . \tag{7}$$

A *NEL-judgement* has the form

$$\nabla \vdash t \approx t' : \mathsf{s} \tag{8}$$

where $\nabla \in \mathsf{FE}_\Sigma$, $\mathsf{s} \in \mathsf{Sort}_\Sigma$ and $t, t' \in \Sigma_\mathsf{s}(\nabla^{:})$. A *NEL-theory* $\mathbb{T}$ is a collection of such judgements.

Fig. 1 present the *proof rules* of NEL, and uses the following new pieces of notation:

- In the rule (SUBST), if ∇ is (6) then $\nabla' \vdash \sigma \approx \sigma'$ stands for the hypotheses $\nabla' \vdash \sigma(x_i) \approx \sigma'(x_i) : \mathsf{s}_i$ for $1 \leq i \leq n$. The notation $\nabla' \vdash \sigma : \nabla$ checks the freshness assumptions of ∇, by $\nabla' \vdash \overline{a}_i \not\# \sigma(x_i) : \mathsf{s}_i$ (see Rem. 3.2).
- $\nabla \leq \nabla'$ if $\nabla(x) = (\mathsf{s}, \overline{a})$ implies $\nabla'(x) = (\mathsf{s}, \overline{a}')$ for $\overline{a} \subseteq \overline{a}'$.
- The *disagreement set* of permutations is $ds(\pi, \pi') \triangleq \{a \in \mathbb{A} \mid \pi(a) \neq \pi'(a)\}$.

If (8) follows from the axioms of $\mathbb{T}$ via these proof rules we write $\nabla \vdash_\mathbb{T} t \approx t' : \mathsf{s}$.

Remark 3.2. In [5] we also used freshness assertions on the right hand side of the turnstile $\vdash$. In fact such judgements do not add expressiveness [3, Sec. 5.5]:

$$\nabla \vdash_\mathbb{T} \overline{a} \not\# t : \mathsf{s} \ \Leftrightarrow \ \nabla^{\not\# supp(\vec{a}')} \vdash_\mathbb{T} t \approx (\vec{a}\ \vec{a}') * t : \mathsf{s} \tag{9}$$

where $\overline{a}$ has ordering $\vec{a} \in \mathbb{A}^{(n)}$ and $\vec{a}' \in \mathbb{A}^{(n)}$ is fresh, so $supp(\vec{a}') \# (\nabla, \overline{a}, t)$. Such freshness judgements will be used only as syntactic sugar in this paper, as equations have a more direct interpretation, as equalities between arrows.

Example 3.3. A NEL-signature for the untyped λ-calculus can be defined by letting our sorts be the singleton $\{\mathsf{tm}\}$ and operation symbols be

$$\{var_a \mid a \in \mathbb{A}\} \cup \{lam_a \mid a \in \mathbb{A}\} \cup \{app\}$$

with the evident Perm-action (see Ex. 2.2(ii)). The typing function is

$$var_a : \mathsf{tm}, \quad lam_a : (\mathsf{tm}) \to \mathsf{tm}, \quad app : (\mathsf{tm}, \mathsf{tm}) \to \mathsf{tm} \ .$$

The binding structure of lam_a is captured, following [4], by the axiom

$$(\text{REFL})\ \frac{}{\nabla \vdash t \approx t : \mathsf{s}}\ \nabla \in \mathsf{FE}_{\Sigma}, t \in \Sigma_{\mathsf{s}}(\nabla^{:}) \qquad (\text{SYMM})\ \frac{\nabla \vdash t \approx t' : \mathsf{s}}{\nabla \vdash t' \approx t : \mathsf{s}}$$

$$(\text{TRANS})\ \frac{\nabla \vdash t \approx t' : \mathsf{s} \qquad \nabla \vdash t' \approx t'' : \mathsf{s}}{\nabla \vdash t \approx t'' : \mathsf{s}}$$

$$(\text{SUBST})\ \frac{\nabla' \vdash \sigma \approx \sigma' \qquad \nabla' \vdash \sigma : \nabla \qquad \nabla \vdash t \approx t' : \mathsf{s}}{\nabla' \vdash t\{\sigma\} \approx t'\{\sigma'\} : \mathsf{s}}\ \sigma, \sigma' : \nabla^{:} \to (\nabla')^{:}$$

$$(\text{WEAK})\ \frac{\nabla \vdash t \approx t' : \mathsf{s}}{\nabla' \vdash t \approx t' : \mathsf{s}}\ \nabla \leq \nabla' \in \mathsf{FE}_{\Sigma} \qquad (\text{ATM-ELIM})\ \frac{\nabla^{\#\bar{a}} \vdash t \approx t' : \mathsf{s}}{\nabla \vdash t \approx t' : \mathsf{s}}\ \bar{a} \# (\nabla, t, t')$$

$$(\text{PERM})\ \frac{}{\nabla^{\# ds(\pi,\pi')} \vdash \pi * t \approx \pi' * t : \pi \cdot \mathsf{s}}\ \nabla \in \mathsf{FE}_{\Sigma}, t \in \Sigma_{\mathsf{s}}(\nabla), ds(\pi, \pi') \# (\nabla, t)$$

Fig. 1. Proof rules of Nominal Equational Logic

(α) $(b \# x : \mathsf{tm}) \vdash lam_a\, x \approx lam_b\, (a\ b)\, x : \mathsf{tm}$

which by (9) may be sugared as $(x : \mathsf{tm}) \vdash a \# lam_a\, x : \mathsf{tm}$. The NEL-theory for $\alpha\beta\eta$-equivalence, adapting [10, Ex. 2.15], is then (α) plus

(β_1) $(a \# x : \mathsf{tm}, y : \mathsf{tm}) \vdash app\, (lam_a\, x)\, y \approx x : \mathsf{tm}$
(β_2) $(y : \mathsf{tm}) \vdash app\, (lam_a\, var_a)\, y \approx y : \mathsf{tm}$
(β_3) $(x : \mathsf{tm}, b \# y : \mathsf{tm}) \vdash app\, (lam_a\, (lam_b\, x))\, y \approx lam_b\, (app\, (lam_a\, x)\, y) : \mathsf{tm}$
(β_4) $(x_1 : \mathsf{tm}, x_2 : \mathsf{tm}, y : \mathsf{tm}) \vdash app\, (lam_a\, (app\, x_1\, x_2))\, y \approx$
$app\, (app\, (lam_a\, x_1)\, y)\, (app\, (lam_a\, x_2)\, y) : \mathsf{tm}$
(β_5) $(b \# x : \mathsf{tm}) \vdash app\, (lam_a\, x)\, var_b \approx (a\ b)\, x : \mathsf{tm}$
(η) $(a \# x : \mathsf{tm}) \vdash x \approx lam_a\, (app\, x\, var_a) : \mathsf{tm}$

Example 3.4. For the π-calculus [16] let our sorts be $\{\mathsf{tm}\}$ and our operation symbols include

$$\{res_a \mid a \in \mathbb{A}\} \cup \{in_{a,b} \mid (a, b) \in \mathbb{A} \times \mathbb{A}\} \cup \{par\}$$

representing restriction, input and parallel composition (and in addition to symbols for output and so forth). The evident Perm-actions make these into a nominal set. The typing function is

$$res_a : (\mathsf{tm}) \to \mathsf{tm}, \quad in_{a,b} : (\mathsf{tm}) \to \mathsf{tm}, \quad par : (\mathsf{tm}, \mathsf{tm}) \to \mathsf{tm}\ .$$

The axioms for binding are then

(α_1) $(b \# x : \mathsf{tm}) \vdash res_a\, x \approx res_b\, (a\ b)\, x : \mathsf{tm}$
(α_2) $(c \# x : \mathsf{tm}) \vdash in_{a,b}\, x \approx in_{a,c}\, (b\ c)\, x : \mathsf{tm}$
(α_3) $(c \# x : \mathsf{tm}) \vdash in_{a,a}\, x \approx in_{a,c}\, (a\ c)\, x : \mathsf{tm}$

The other axioms for structural congruence are simple to write down, such as

$(-)$ $(x : \mathsf{tm}, a \# y : \mathsf{tm}) \vdash res_a\, (par\, x\, y) \approx par\, (res_a\, x)\, y$

4 Nominal Lawvere theories

Definition 4.1. *A category $\mathcal{C}$ has an* internal Perm-action *if for each $\pi \in \mathsf{Perm}$ and $\mathcal{C}$-object C there is a $\mathcal{C}$-arrow π_C with domain C such that*

(i) ι_C is the identity id_C;
(ii) $(\pi'\pi)_C = \pi'_{\pi\cdot C} \circ \pi_C$, where $\pi \cdot C$ is defined to be the codomain of π_C.

(i) and (ii) specify that the action respects the category structure of Perm. In fact, it is equivalent to a *cofunctor*[1] $\mathsf{Perm} \to \mathcal{C}$, in the sense of [1, Sec. 4.2].

Now given $\pi \in \mathsf{Perm}$ and $\mathcal{C}$-arrow $f : C \to D$, we define the $\mathcal{C}$-arrow $\pi \cdot f : \pi \cdot C \to \pi \cdot D$ to be

$$\pi \cdot C \xrightarrow{(\pi^{-1})_{\pi\cdot C}} C \xrightarrow{f} D \xrightarrow{\pi_D} \pi \cdot D \ . \tag{10}$$

The maps $C \mapsto \pi \cdot C$ and $f \mapsto \pi \cdot f$ define an endofunctor $\pi \cdot (\,) : \mathcal{C} \to \mathcal{C}$. In fact, we have a Perm-*category*: a functor from Perm to the category of categories picking out a category $\mathcal{C}$ and defining Perm-actions on its objects $ob\,\mathcal{C}$ and arrows $ar\,\mathcal{C}$. Each π_C is then a component of the natural isomorphism $\pi : id_{\mathcal{C}} \dot{\to} \pi \cdot (\,)$.

Definition 4.2. *An internal* Perm*-action is* finitely supported *if all arrows (and hence objects) are finitely supported under the maps defined above.*

If $\mathcal{C}$ has a finitely supported internal Perm-action and is small, then it is an *internal category in $\mathcal{N}om$* in the sense of [12, Cha. B2]; its objects and arrows form nominal sets, and the domain, codomain, identity and composition maps are equivariant. Just as standard Lawvere theories have finite products, we will want our category to have finite products in $\mathcal{N}om$:

Definition 4.3. *$\mathcal{C}$ has* equivariant finite products *if it has all finite products so that the terminal object* 1 *has empty support and the maps from tuples of objects $(C_1, \ldots, C_n)$ to their projection arrows $pr_i : C_1 \times \cdots C_n \to C_i$ are equivariant.*

Example 4.4. Given $\pi \in \mathsf{Perm}$ and FM-set X we define $\pi \cdot X$ by the usual element-wise Perm-action, and the internal Perm-action $\pi_X : X \to \pi \cdot X$ by

$$\pi_X(x) \triangleq \pi \cdot x \ .$$

Defining $\pi \cdot f$ by (10) clearly agrees with (3) and so is finitely supported. $\mathcal{FM}$-Set has finite products defined as usual and it is easy to confirm they are equivariant.

Now given $\overline{a} \in \mathcal{P}_{fin}(\mathbb{A})$ fresh for X, we will define their *fresh subobject* in $\mathcal{FM}$-Set as the inclusion FM-function $i^{\overline{a}}_X : X^{\#\overline{a}} \hookrightarrow X$, where

$$X^{\#\overline{a}} \triangleq \{x \in X \mid \overline{a} \# x\} \ . \tag{11}$$

[1] The term *cofunctor* is also sometimes used to abbreviate *contravariant functor*.

We now turn to the arrow-theoretic analogue of (11). The definition below is the most involved of this paper, so first we will provide some motivation. If $\mathcal{P}_{fin}(\mathbb{A})$ is the one object category whose arrows are finite sets of atoms, identity is $\emptyset$ and composition is union, then conditions (i) and (ii) require that fresh subobjects respect that category's structure. $\mathcal{P}_{fin}(\mathbb{A})$ is internal to $\mathcal{N}om$ (though it has no *internal* Perm-action), so (iii) requires that Perm-actions are preserved. (iv) requires that finite products be preserved, while (v)-(vii) are conditions specific to freshness that will be motivated for $\mathcal{FM}$-Set by Ex. 4.6.

Definition 4.5. *A category $\mathcal{C}$ with internal* Perm*-action and finite products has* fresh subobjects *if for each finite set of atoms $\overline{a}$ and $\mathcal{C}$-object C such that $\overline{a} \# C$ there is a $\mathcal{C}$-arrow $i^{\overline{a}}_C$ with codomain C so that the following conditions hold:*

(i) $i^{\emptyset}_C$ *is the identity* id_C*;*
(ii) $i^{\overline{a}\cup\overline{a}'}_C = i^{\overline{a}}_C \circ i^{\overline{a}'}_{C^{\#\overline{a}}}$*, where* $C^{\#\overline{a}}$ *is defined to be the domain of* $i^{\overline{a}}_C$*.*
(iii) $\pi \cdot i^{\overline{a}}_C = i^{\pi\cdot\overline{a}}_{\pi\cdot C}$*.*
(iv) $i^{\overline{a}}_{C_1\times\cdots\times C_n} = i^{\overline{a}}_{C_1} \times \cdots \times i^{\overline{a}}_{C_n}$*.*
(v) (Fresh permutations): Given $\pi \in \mathsf{Perm}$*, if* $supp(\pi) \# C$ *then* $\pi_{C^{\#supp(\pi)}}$ *is the identity* $id_{C^{\#supp(\pi)}}$*;*
(vi) (Fresh epis): If we have a finite set of atoms $\overline{a}$ and parallel $\mathcal{C}$-arrows $f : C \rightrightarrows D$ such that $f \circ i^{\overline{a}}_C = g \circ i^{\overline{a}}_C$ and $\overline{a} \# (f, g)$, then $f = g$;
(vii) (Fresh arrows): Suppose we have sets of atoms $\overline{a}, \overline{a}'$ which may be ordered $\vec{a}, \vec{a}' \in \mathbb{A}^{(n)}$, and a $\mathcal{C}$-arrow $f : D \to C$. If $\overline{a} \# C$, $\overline{a}' \# (\overline{a}, f)$ and $(\vec{a}\ \vec{a}')_C \circ f \circ i^{\overline{a}'}_D = f \circ i^{\overline{a}'}_D$, then we have a unique $\hat{f} : D \to C^{\#\overline{a}}$ such that $i^{\overline{a}}_C \circ \hat{f} = f$:

$$\begin{array}{ccc} D^{\#\overline{a}'} & \xrightarrow{i^{\overline{a}'}_D} & D \\ & & \hat{f}\ \downarrow \quad \searrow f \\ & & C^{\#\overline{a}} \xrightarrow[i^{\overline{a}}_C]{} C \circlearrowleft (\vec{a}\ \vec{a}')_C \end{array} \tag{12}$$

Example 4.6. Conditions (v)-(vii) above applied to (11) in $\mathcal{FM}$-Set ask that

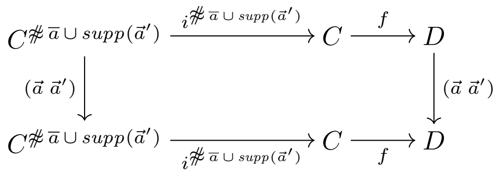
- If we have $x \in X$ and $supp(\pi) \# (x, X)$ then $\pi \cdot x = x$;
- If $\overline{a} \# (f, g)$, and $f(x) = g(x)$ whenever $\overline{a} \# x$, then $f = g$;
- Say $\overline{a} \# C$ and $\overline{a}' \# (\overline{a}, f)$. If $(\vec{a}\ \vec{a}') \cdot f(x) = f(x)$ whenever $\overline{a}' \# x$, then $\overline{a} \# f(x)$ for all x.

These are all readily verifiable properties of FM-sets and functions. In particular, (vii) encodes the freshness property (9) for arrows of our category. $\mathcal{FM}$-Set and the small $\mathcal{FM}_\lambda$-Set therefore fulfil the conditions of the following definition.

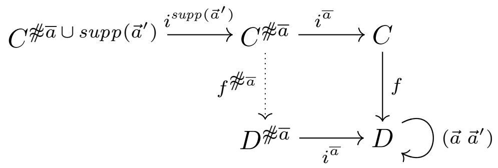
Definition 4.7. *An* FM-category *is a category with a finitely supported internal* Perm*-action, equivariant finite products and fresh subobjects. A small FM-category will be called a* nominal Lawvere theory*.*

From now on we will abbreviate the internal Perm-action π_C to π and fresh subobjects $i^{\overline{a}}_C$ to $i^{\overline{a}}$ where this is clear. The following definition and lemma demonstrate the utility of the Fresh Arrows condition above.

Definition 4.8. *Suppose that $\overline{a} \# (f : C \to D)$ in $\mathcal{C}$. Let $\vec{a} \in \mathbb{A}^{(n)}$ be an ordering of $\overline{a}$ and $\vec{a}' \in \mathbb{A}^{(n)}$ be a fresh tuple of the same size. $(\vec{a}\ \vec{a}')$ defines a natural transformation $id_{\mathcal{C}} \dot{\to} (\vec{a}\ \vec{a}') \cdot ()$, so*

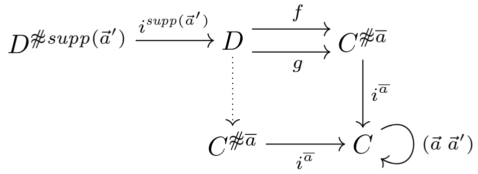


commutes. But $(\vec{a}\ \vec{a}')_{C^{\# \overline{a} \cup supp(\vec{a}')}}$ is the identity by Def. 4.5(v), so $(\vec{a}\ \vec{a}') \circ f \circ i^{\# \overline{a} \cup supp(\vec{a}')} = f \circ i^{\# \overline{a} \cup supp(\vec{a}')}$. Therefore by Def. 4.5(vii), the Fresh Arrows condition, we induce a unique arrow which we will call $f^{\# \overline{a}}$:



Lemma 4.9. *All fresh subobjects $i^{\overline{a}}_C : C^{\# \overline{a}} \to C$ are mono.*

Proof. Say $i^{\overline{a}} \circ f = i^{\overline{a}} \circ g$ for $f, g : D \rightrightarrows C^{\# \overline{a}}$, and take $\vec{a}$ as an ordering of $\overline{a}$ and $\vec{a}'$ as a fresh tuple of the same size. $(\vec{a}\ \vec{a}')_C \circ i^{\overline{a}}_C \circ f \circ i^{supp(\vec{a}')}_D = (\vec{a}\ \vec{a}')_C \circ i^{\overline{a} \cup supp(\vec{a}')}_C \circ f^{\# supp(\vec{a}')}$ by Def. 4.8. By the naturality of $(\vec{a}\ \vec{a}')$ and Def 4.5(v) this equals $i^{\overline{a}}_C \circ f \circ i^{supp(\vec{a}')}_D$. Therefore by Def. 4.5(vii) we have a unique arrow:



But $i^{\overline{a}} \circ f = i^{\overline{a}} \circ g$, so by uniqueness $f = g$.

Definition 4.10. *Given FM-categories $\mathcal{C}, \mathcal{C}'$, an* FM-functor $\mathcal{C} \to \mathcal{C}'$ *is a functor that strictly preserves*

(i) the internal Perm*-action: $F(\pi_C) = \pi_{FC}$;*
(ii) finite products: $F(pr_i) = pr_i : FC_1 \times \cdots \times FC_n \to FC_i$;
(iii) fresh subobjects: $F(i^{\overline{a}}_C) = i^{\overline{a}}_{FC}$.

$FM(\mathcal{C}, \mathcal{C}')$ is the category of FM-functors $\mathcal{C} \to \mathcal{C}'$ and natural transformations.

5 Algebra in FM-categories

Definition 5.1. *Given a NEL-signature Σ and FM-category $\mathcal{C}$, a Σ-*structure *M in $\mathcal{C}$ is defined by*

(i) An equivariant function $M[\![-]\!] : \mathsf{Sort}_\Sigma \to ob\,\mathcal{C}$;
(ii) An equivariant function $M[\![-]\!] : \mathsf{Op}_\Sigma \to ar\,\mathcal{C}$ where

$$M[\![op]\!] : M[\![\mathsf{s}_1]\!] \times \cdots \times M[\![\mathsf{s}_n]\!] \to M[\![\mathsf{s}]\!] \tag{13}$$

if op has type $(\mathsf{s}_1, \ldots, \mathsf{s}_n) \to \mathsf{s}$.

Where the structure in question is clear we will write $M[\![\mathsf{s}]\!]$ as $[\![\mathsf{s}]\!]$ and so forth.

Definition 5.2. *Given a freshness environment ∇ as (6) and a Σ-structure in an FM-category $\mathcal{C}$ we define the $\mathcal{C}$-object*

$$[\![\nabla]\!] \;\triangleq\; [\![\mathsf{s}_1]\!]^{\#\overline{a}_1} \times \cdots \times [\![\mathsf{s}_n]\!]^{\#\overline{a}_n} \quad .$$

Given a term $t \in \Sigma_{\mathsf{s}}(\nabla^{\cdot})$ the value arrow *$[\![\nabla \vdash t : \mathsf{s}]\!]$ is a $\mathcal{C}$-arrow $[\![\nabla]\!] \to [\![\mathsf{s}]\!]$:*

$$\begin{aligned} [\![\nabla \vdash \pi\, x_i : \mathsf{s}_i]\!] \;&\triangleq\; \pi_{[\![\mathsf{s}_i]\!]} \circ i^{\overline{a}_i}_{[\![\mathsf{s}_i]\!]} \circ pr_i \;\; ; \\ [\![\nabla \vdash op\, t_1 \cdots t_n : \mathsf{s}]\!] \;&\triangleq\; [\![op]\!] \circ \langle [\![\nabla \vdash t_1 : \mathsf{s}_1]\!], \ldots, [\![\nabla \vdash t_n : \mathsf{s}_n]\!] \rangle \;\; . \end{aligned}$$

Definition 5.3. *A structure M in an FM-category $\mathcal{C}$* satisfies *the judgement $\nabla \vdash t \approx t' : \mathsf{s}$ if $M[\![\nabla \vdash t : \mathsf{s}]\!] = M[\![\nabla \vdash t' : \mathsf{s}]\!]$. If M satisfies all axioms of a theory $\mathbb{T}$ then it is a $\mathbb{T}$-*algebra in *$\mathcal{C}$.*

Theorem 5.4 (Soundness). *If M is a $\mathbb{T}$-algebra in $\mathcal{C}$ and $\nabla \vdash_{\mathbb{T}} t \approx t' : \mathsf{s}$ then M satisfies that judgement.*

Proof. We need to show closure under the proof rules of Fig. 1; see App. A.1.

Definition 5.5. *A $\mathbb{T}$-*homomorphism *$M \to M'$ in $\mathcal{C}$ is an equivariant function h from Sort_Σ to $\mathcal{C}$-arrows, with $h(\mathsf{s}) = h_{\mathsf{s}} : M[\![\mathsf{s}]\!] \to M'[\![\mathsf{s}]\!]$, such that*

$$h_{\mathsf{s}} \circ M[\![op]\!] = M'[\![op]\!] \circ (h_{\mathsf{s}_1} \times \cdots \times h_{\mathsf{s}_n}) \tag{14}$$

for all op. The $\mathbb{T}$-algebras in $\mathcal{C}$ and $\mathbb{T}$-homomorphisms form the category $\mathcal{C}^{\mathbb{T}}$.

6 Category-theory correspondence

Given a NEL-theory $\mathbb{T}$, this section defines a nominal Lawvere theory called the *classifying category*, $Cl(\mathbb{T})$. This construction gives rise to a simple completeness proof, along with the key correspondences of this paper.

Lemma 6.1. *Fix an ordering $v_1, v_2, \ldots$ on the set of variables Var and let $\mathbb{T}$ be a theory over a signature Σ. Then the following constructions define a nominal Lawvere theory, which we will call the* classifying category *and write $Cl(\mathbb{T})$.*

Objects: *ob* $Cl(\mathbb{T})$ *is the set of freshness environments whose domain is an initial sublist of* Var, $\{v_1,\ldots,v_n\}$*, so the typical object is*

$$\nabla \;=\; (\overline{a}_1 \# v_1 : \mathsf{s}_1, \ldots, \overline{a}_n \# v_n : \mathsf{s}_n) \tag{15}$$

Arrows: *Taking* ∇ *as* (15)*,* $Cl(\mathbb{T})$*-arrows* $f : \nabla' \to \nabla$ *are defined by*

$$\nabla' \vdash (\overline{a}_1 \# [t_1] : \mathsf{s}_1, \ldots, \overline{a}_n \# [t_n] : \mathsf{s}_n) \tag{16}$$

where, for $1 \le i \le n$*,*
(i) $t_i \in \Sigma_{\mathsf{s}_i}((\nabla')^{:})$*;*
(ii) $\nabla' \vdash_{\mathbb{T}} \overline{a}_i \# t_i : \mathsf{s}_i$ *(see* (9)*);*
(iii) $[t_i]$ *is the equivalence class of terms* u *such that* $\nabla' \vdash_{\mathbb{T}} t_i \approx u : \mathsf{s}_i$*.*

Identity: *The identity on* ∇ (15) *is*

$$id_{\nabla} \;\triangleq\; \nabla \vdash (\overline{a}_1 \# [v_1] : \mathsf{s}_1, \ldots, \overline{a}_n \# [v_n] : \mathsf{s}_n) \;\; .$$

Composition: *Given* f (16)*,* $g = \nabla \vdash (\overline{a}'_1 \# [t'_1] : \mathsf{s}'_1, \ldots, \overline{a}'_m \# [t'_m] : \mathsf{s}'_m)$*,*

$$g \circ f \;\triangleq\; \nabla' \vdash (\overline{a}'_1 \# [t'_1\{\sigma\}] : \mathsf{s}'_1, \ldots, \overline{a}'_m \# [t'_m\{\sigma\}] : \mathsf{s}'_m)$$

where σ *is the substitution* (5) $\sigma(v_i) = t_i$ *for* $1 \le i \le n$*.*

Finitely supported internal Perm**-action:** *Given* ∇ (15)*,*

$$\pi_{\nabla} \;\triangleq\; \nabla \vdash (\pi \cdot \overline{a}_1 \# [\pi\, v_1] : \pi \cdot \mathsf{s}_1, \ldots, \pi \cdot \overline{a}_n \# [\pi\, v_n] : \pi \cdot \mathsf{s}_n) \;\; .$$

Equivariant finite products: *The terminal object of* $Cl(\mathbb{T})$ *is the empty freshness environment. Given* ∇ (15) *and* $\nabla' = (\overline{a}'_1 \# v_1 : \mathsf{s}'_1, \ldots, \overline{a}'_m \# v_m : \mathsf{s}_m)$*, their binary product* $\nabla \times \nabla'$ *is*

$$(\overline{a}_1 \# v_1 : \mathsf{s}_1, \ldots, \overline{a}_n \# v_n : \mathsf{s}_n, \overline{a}'_1 \# v_{n+1} : \mathsf{s}'_1, \ldots, \overline{a}'_m \# v_{n+m} : \mathsf{s}'_m)$$

with projections

$$\begin{aligned} pr_1 &\triangleq \nabla \times \nabla' \vdash (\overline{a}_1 \# [v_1] : \mathsf{s}_1, \ldots, \overline{a}_n \# [v_n] : \mathsf{s}_n) \;\; ; \\ pr_2 &\triangleq \nabla \times \nabla' \vdash (\overline{a}'_1 \# [v_{n+1}] : \mathsf{s}'_1, \ldots, \overline{a}'_m \# [v_{n+m}] : \mathsf{s}'_m) \;\; . \end{aligned}$$

Fresh subobjects: *Define* $\nabla^{\#\overline{a}}$ *by applying* (7) *to* (15)*. Then*

$$i^{\overline{a}}_{\nabla} \;\triangleq\; \nabla^{\#\overline{a}} \vdash (\overline{a}_1 \# [v_1] : \mathsf{s}_1, \ldots, \overline{a}_n \# [v_n] : \mathsf{s}_n) \;\; .$$

Proof. App. A.2.

Definition 6.2. *Define the* generic algebra G *by*

$$\begin{aligned} G[\![\mathsf{s}]\!] &\triangleq (v_1 : \mathsf{s}) \\ G[\![op]\!] &\triangleq (v_1 : \mathsf{s}_1, \ldots, v_n : \mathsf{s}_n) \vdash (\emptyset \# [op\, v_1 \cdots v_n] : \mathsf{s}) \end{aligned}$$

for $op : (\mathsf{s}_1, \ldots, \mathsf{s}_n) \to \mathsf{s}$*. It is easy to prove that* G *is a* $\mathbb{T}$*-algebra in* $Cl(\mathbb{T})$*.*

Theorem 6.3 (Completeness). *Given a NEL-theory* $\mathbb{T}$*, if* $\nabla \vdash t \approx t' : \mathsf{s}$ *is satisfied by all* $\mathbb{T}$*-algebras in all nominal Lawvere theories then* $\nabla \vdash_{\mathbb{T}} t \approx t' : \mathsf{s}$.

Proof. If $\nabla \vdash t \approx t' : \mathsf{s}$ is satisfied by the generic algebra G in $Cl(\mathbb{T})$ then $\nabla \vdash_{\mathbb{T}} t \approx t' : \mathsf{s}$ by the definition of $Cl(\mathbb{T})$-arrows.

Theorem 6.4. *Given any NEL-theory* $\mathbb{T}$ *and nominal Lawvere theory* $\mathcal{C}$*, there is an isomorphism*

$$FM(Cl(\mathbb{T}), \mathcal{C}) \cong \mathcal{C}^{\mathbb{T}}$$

between the category of FM-functors $Cl(\mathbb{T}) \to \mathcal{C}$ *(Def. 4.10) and the category of* $\mathbb{T}$*-algebras in* $\mathcal{C}$ *(Def. 5.5).*

Proof. App. A.2.

Definition 6.5. *Given a nominal Lawvere theory* $\mathcal{C}$*, define the signature* $Sg(\mathcal{C})$ *by setting* $\mathsf{Sort}_{Sg(\mathcal{C})} = ob\,\mathcal{C}$ *and*

$$\mathsf{Op}_{Sg(\mathcal{C})} \triangleq \{f : (C_1, \ldots, C_n) \to C \mid f : C_1 \times \cdots \times C_n \to C \in ar\,\mathcal{C}\}$$

with Perm*-actions defined via the internal* Perm*-action on* $\mathcal{C}$*. We use smallness here as our sorts and operation symbols must form nominal* sets. *Note that one arrow can give rise to multiple operation symbols; for example,* $f : C_1 \times C_2 \to C$ *induces operation symbols* $(C_1 \times C_2) \to C$ *and* $(C_1, C_2) \to C$*.*

Let $M(\mathcal{C})$ *be the* $Sg(\mathcal{C})$*-structure in* $\mathcal{C}$ *which we define by* $M(\mathcal{C})[\![C]\!] = C$ *and* $M(\mathcal{C})[\![f]\!] = f$*, then let* $Th(\mathcal{C})$ *be the* $Sg(\mathcal{C})$*-theory whose axioms are all the judgements that are satisfied by* $M(\mathcal{C})$*, so* $M(\mathcal{C})$ *is trivially an algebra of* $\mathcal{C}^{Th(\mathcal{C})}$*.*

Remark 6.6. It is clear that nominal Lawvere theories require a Perm-action on their objects; without this we could not adequately represent freshness environments in the classifying category. The translation from nominal Lawvere theories back to NEL-theories, if it is not to lose information, then requires that our sorts may form any nominal set. Note that this is only a mild generalisation of previous presentations of NEL [5], as by Ex. 2.2(i) any set may be considered a nominal set under the trivial Perm-action.

Theorem 6.7. *For any small FM-category* $\mathcal{C}$ *there is an equivalence*

$$\mathcal{C} \simeq Cl(Th(\mathcal{C})) \ .$$

Proof. App. A.2.

Proving the converse of this theorem, that $\mathbb{T} \simeq Th(Cl(\mathbb{T}))$, requires a notion of morphism between theories. The most natural definition of such a translation $\mathbb{T} \to \mathbb{T}'$ is an FM-functor $Cl(\mathbb{T}) \to Cl(\mathbb{T}')$, so the converse of Thm. 6.7 is actually a corollary, and we have a correspondence between NEL-theories and nominal Lawvere theories.

7 Related and further work

- This work opens the way for consideration of NEL-theories in FM-categories other than $\mathcal{FM}$-Set, such as the FM-cppos of [21].
- The most similar system to NEL is the independently produced Nominal Algebra (NA) [10], which also addresses languages with binding, and equations modulo freshness, via the nominal sets model. A number of different design choices were made in these logics' constructions: NA employs sets, rather than nominal sets, of sorts and operation symbols; NA uses 'nominal signatures' with explicit binding sorts; and freshness in NA is sound, but not complete, for freshness in the underlying nominal sets interpretation. Nonetheless it is not too difficult to translate from one logic to the other, as is done to some extent in [10, Sec. 5] and [4, Sec. 7].
  The motivation for many of the design choices made for NEL was to cleave as close as possible to the standard account of equational logic, so that established equational logic techniques may be transferred to the nominal setting. This paper is one of the fruits of this philosophy. While nominal Lawvere theories obviously apply to NA-theories if we translate them to NEL-theories as our first step, a more interesting open question is whether a compelling Lawvere theoretic account can be developed directly for some of the design choices of NA, most notably explicit binding sorts.
- The results of this paper are unlikely to be a straightforward application of existing work on generalised Lawvere theories [19, 14], concerned as it is with expressing the particular syntax of NEL. Nonetheless, connecting nominal Lawvere theories with the more general picture would be valuable.
- The most well known alternative to Lawvere theories for the category theoretic expression of equational logic are algebras for a monad. This was investigated for NEL in [3, Chap. 6], and more extensively in [13]. Unlike Lawvere theories, the monadic view offers no explicit category theoretic description of a logic's theories.
- Fiore-Hur equational systems are a newer category theoretic approach to equational logic, and have an established application to NEL [8, Sec. 7.3]. There is as yet no general relationship established between this approach and Lawvere theories, but such results have been produced in the special case of second-order equational logic [7].

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A Technical Appendices

A.1 Algebra in FM-categories

Lemma A.1. *(i)* $[\![\nabla \vdash \pi * t : \pi \cdot \mathsf{s}]\!] = \pi_{[\![\mathsf{s}]\!]} \circ [\![\nabla \vdash t : \mathsf{s}]\!]$*;*

(ii) Given $\nabla \leq \nabla'$ *there exists an arrow* $\mathsf{weak} : [\![\nabla']\!] \to [\![\nabla]\!]$ *such that for any* $t \in \Sigma_{\mathsf{s}}(\nabla^{\cdot})$*,* $[\![\nabla' \vdash t : \mathsf{s}]\!] = [\![\nabla \vdash t : \mathsf{s}]\!] \circ \mathsf{weak}$*;*

(iii) $[\![\nabla^{\#\overline{a}} \vdash t : \mathsf{s}]\!] = [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{\overline{a}}_{[\![\nabla]\!]}$*;*

(iv) If ∇ *is as* (6) *then* $[\![\nabla \vdash t : \mathsf{s}]\!] = [\![x_1 : \mathsf{s}_1, \ldots, x_n : \mathsf{s}_n \vdash t : \mathsf{s}]\!] \circ (i^{\overline{a}_1}_{[\![\mathsf{s}_1]\!]} \times \cdots \times i^{\overline{a}_n}_{[\![\mathsf{s}_n]\!]})$*;*

(v) Given $\Gamma \in \mathsf{SE}_{\Sigma}$ *with domain* $\{x_1, \ldots, x_n\}$*, a substitution* $\sigma : \Gamma \to \nabla^{\cdot}$ *and a term* $t \in \Sigma_{\mathsf{s}}(\Gamma)$*,*

$$[\![\nabla \vdash t\{\sigma\} : \mathsf{s}]\!] = [\![\Gamma \vdash t : \mathsf{s}]\!] \circ \langle \ldots, [\![\nabla \vdash \sigma(x_i) : \Gamma(x_i)]\!], \ldots \rangle$$

where Γ *is defined to be a freshness environment in* FE_{Σ} *by* $\Gamma(x) = (\Gamma(x), \emptyset)$*.*

Proof. (i) follows by induction on the structure of t, using the naturality of π in the constructed term chase.

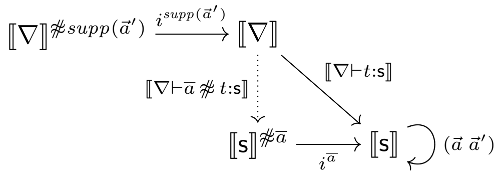
(ii): take ∇ as (6) and say $\nabla'(x_i) = (\overline{a}_i \cup \overline{a}'_i, \mathsf{s}_i)$ for $1 \le i \le n$. Then set

$$\mathsf{weak} \;\triangleq\; \langle i^{\overline{a}'_1} \circ pr_1, \ldots, i^{\overline{a}'_n} \circ pr_n \rangle \ .$$

The result follows by another induction on t, and (iii) is an immediate corollary.

(iv) is another induction on t and (v) likewise, using (i) in the suspension case.

Definition A.2. *Suppose $\nabla \vdash \overline{a} \;\#\; t : \mathsf{s}$ is satisfied, so by (9) and Def. 5.3, $[\![\nabla^{\#supp(\vec{a}')} \vdash t : \mathsf{s}]\!] = [\![\nabla^{\#supp(\vec{a}')} \vdash (\vec{a}\;\vec{a}') * t : \mathsf{s}]\!]$. Then by Lem. A.1(i) and (iii), $[\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{supp(\vec{a}')} = (\vec{a}\;\vec{a}') \circ [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{supp(\vec{a}')}$. By Def. 4.5(vii) a unique arrow is induced which we will call $[\![\nabla \vdash \overline{a} \;\#\; t : \mathsf{s}]\!]$:*



Lemma A.3. *Take ∇ as (6).*

(i) Given a term $t \in \Sigma_{\mathsf{s}}(\nabla^{:})$ and substitution $\sigma : \nabla^{:} \to (\nabla')^{:}$, if the arrows $[\![\nabla' \vdash \overline{a}_i \;\#\; \sigma(x_i) : \mathsf{s}_i]\!]$ are defined for $1 \le i \le n$ then $[\![\nabla' \vdash t\{\sigma\} : \mathsf{s}]\!]$ equals

$$[\![\nabla \vdash t : \mathsf{s}]\!] \circ \langle [\![\nabla' \vdash \overline{a}_1 \;\#\; \sigma(x_1) : \mathsf{s}_1]\!], \ldots, [\![\nabla' \vdash \overline{a}_n \;\#\; \sigma(x_n) : \mathsf{s}_n]\!] \rangle \ ;$$

(ii) $id_{[\![\nabla]\!]} = \langle [\![\nabla \vdash \overline{a}_1 \;\#\; x_1 : \mathsf{s}_1]\!], \ldots, [\![\nabla \vdash \overline{a}_n \;\#\; x_n : \mathsf{s}_n]\!] \rangle$;
(iii) $i^{\overline{a}}_{[\![\nabla]\!]} = \langle [\![\nabla^{\#\overline{a}} \vdash \overline{a}_1 \;\#\; x_1 : \mathsf{s}_1]\!], \ldots, [\![\nabla^{\#\overline{a}} \vdash \overline{a}_n \;\#\; x_n : \mathsf{s}_n]\!] \rangle$;
(iv) $\pi_{[\![\nabla]\!]} = \langle [\![\nabla \vdash \pi \cdot \overline{a}_1 \;\#\; \pi\, x_1 : \pi \cdot \mathsf{s}_1]\!], \ldots, [\![\nabla \vdash \pi \cdot \overline{a}_n \;\#\; \pi\, x_n : \pi \cdot \mathsf{s}_n]\!] \rangle$;
(v) Take ∇_1, ∇_2 with disjoint domain. Then $pr_j([\![\nabla_1]\!], [\![\nabla_2]\!]) = \langle [\![\nabla_1 \cup \nabla_2 \vdash \overline{a}_1 \;\#\; x_1 : \mathsf{s}_1]\!], \ldots, [\![\nabla_1 \cup \nabla_2 \vdash \overline{a}_n \;\#\; x_n : \mathsf{s}_n]\!] \rangle$ if ∇_j is (6) for $i = 1$ or 2.

Proof. (i) follows by Lem. A.1(iv) and (v). For (ii)-(v) we first need to confirm that the arrows in question exist, following Def A.2. For (ii), we see that

$$i^{\overline{a}_i} \circ pr_i \circ i^{supp(\vec{a}'_i)} \;=\; (\vec{a}_i\;\vec{a}'_i) \circ i^{\overline{a}_i} \circ pr_i \circ i^{supp(\vec{a}'_i)}$$

by the naturality of $(\vec{a}_i\;\vec{a}'_i)$ and Def. 4.5(v). (iii)-(v) follow similarly. The equalities (ii), (iii) and (v) then follow by applying $i^{\overline{a}_i} \circ pr_i$ to each side, as $i^{\overline{a}_i}$ is mono and projections jointly mono. (iv) follows by applying $i^{\pi \cdot \overline{a}_i} \circ pr_i$ to each side.

*Proof (**Thm. 5.4**).* Closure under (REFL), (SYMM) and (TRANS) is trivial, and (WEAK) follows from Lem A.1(ii).

(SUBST): Given ∇ as (6) the arrows $[\![\nabla' \vdash \overline{a}_1 \;\#\; \sigma(x_1) : \mathsf{s}_1]\!]$ (Def. A.2) are defined for $1 \le i \le n$, so by Lem. A.3(i)

$$[\![\nabla' \vdash t\{\sigma\} : \mathsf{s}]\!] \;=\; [\![\nabla \vdash t : \mathsf{s}]\!] \circ \langle [\![\nabla' \vdash \overline{a}_1 \;\#\; \sigma(x_1) : \mathsf{s}_1]\!], \ldots \rangle$$

and similarly for $t'\{\sigma'\}$. We have $[\![\nabla \vdash t : \mathsf{s}]\!] = [\![\nabla \vdash t' : \mathsf{s}]\!]$, while $i^{\overline{a}_i} \circ [\![\nabla' \vdash \overline{a}_i \;\#\; \sigma(x_i) : \mathsf{s}_i]\!] = [\![\nabla' \vdash \sigma(x_i) : \mathsf{s}_i]\!] = [\![\nabla' \vdash \sigma'(x_i) : \mathsf{s}_i]\!] = i^{\overline{a}_i} \circ [\![\nabla' \vdash \overline{a}_i \;\#\; \sigma'(x_i) : \mathsf{s}_i]\!]$. But $i^{\overline{a}_i}$ is mono by Lem. 4.9, so we are done.

(ATM-ELIM): $[\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{\overline{a}} = [\![\nabla \vdash t' : \mathsf{s}]\!] \circ i^{\overline{a}}$ by Lem. A.1(iii), so $[\![\nabla \vdash t : \mathsf{s}]\!] = [\![\nabla \vdash t' : \mathsf{s}]\!]$ by Def. 4.5(vi).

(PERM): By Lemma A.1(i) and (iii) we need to prove that $\pi \circ [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')} = \pi' \circ [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')}$. Now $\pi^{-1} \circ \pi' \circ [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')} = [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')} \circ \pi^{-1}\pi'_{[\![\nabla]\!]\#^{ds(\pi,\pi')}}$ by naturality, but $ds(\pi,\pi') = supp(\pi^{-1}\pi')$, so we can apply Def. 4.5(v) to make this equal $[\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')}$. Applying the identity to the front gives us $\pi^{-1} \circ \pi \circ [\![\nabla \vdash t : \mathsf{s}]\!] \circ i^{ds(\pi,\pi')}$. π^{-1} is iso, and therefore mono, so we are done.

A.2 Category-theory correspondence

*Proof (**Lem. 6.1**).* The various properties of FM-categories are easily verifiable corollaries of the proof rules, along with standard properties of NEL-terms from [5, Sec. 4 and 5]. For example, take f as (16) and apply (10):

$$\pi \cdot f \;=\; \pi \cdot \nabla' \vdash [\pi \cdot \overline{a}_1 \# [\pi \cdot t_1] : \pi \cdot \mathsf{s}_1, \ldots, \pi \cdot \overline{a}_n \# [\pi \cdot t_n] : \pi \cdot \mathsf{s}_n] \;.$$

$\pi \cdot t$ is the *meta-level* Perm-*action* of [5, Sec. 5] (not to be confused with the *object-level* Perm-*action* (4)) under which terms are indeed finitely supported.

Likewise, if we take f as (16) in (12) then

$$\hat{f} \;=\; \nabla' \vdash [\overline{a}_1 \cup \overline{a} \# [t_1] : \mathsf{s}_1, \ldots, \overline{a}_n \cup \overline{a} \# [t_n] : \mathsf{s}_n] \;.$$

Lemma A.4. *Given FM-categories $\mathcal{C}, \mathcal{C}'$ and an algebra $M \in ob\,\mathcal{C}^{\mathbb{T}}$, we can define a functor, called the* modelling functor*, $M(-) : FM(\mathcal{C}, \mathcal{C}') \to \mathcal{C}^{\mathbb{T}}$ by*

- $M(F)[\![\mathsf{s}]\!] = F(M[\![\mathsf{s}]\!])$ *and* $M(F)[\![op]\!] = F(M[\![op]\!])$*;*
- $M(\phi)_{\mathsf{s}} = \phi_{M[\![\mathsf{s}]\!]}$.

Proof. FM-functors preserve the internal Perm-action, so $\mathsf{s} \mapsto M(F)[\![\mathsf{s}]\!]$ and $op \mapsto M(F)[\![op]\!]$ are equivariant and $M(F)$ is a Σ-structure. Given $t \in \Sigma_{\mathsf{s}}(\nabla^{\cdot})$,

$$M(F)[\![\nabla \vdash t : \mathsf{s}]\!] \;=\; F(M[\![\nabla \vdash t : \mathsf{s}]\!]) \tag{17}$$

by induction on the structure of t, so if $M \in ob\,\mathcal{C}^{\mathbb{T}}$ then $M(F) \in ob\,\mathcal{C}'^{\mathbb{T}}$. $\mathsf{s} \mapsto M(\phi)_{\mathsf{s}}$ is evidently equivariant, and (14) holds because natural transformations between finite product preserving functors commute with those products.

Lemma A.5. *Given ∇ as* (15)*, a $\mathbb{T}$-homomorphism $h : M \to M'$, and a term $t \in \Sigma_{\mathsf{s}}(\nabla^{\cdot})$,*

$$h_{\mathsf{s}} \circ M[\![\nabla \vdash t : \mathsf{s}]\!] = M'[\![\nabla \vdash t : \mathsf{s}]\!] \circ (h_{\mathsf{s}_1}^{\#\overline{a}_1} \times \cdots \times h_{\mathsf{s}_n}^{\#\overline{a}_n})$$

Proof. An easy induction on the structure of t.

*Proof (**Thm. 6.4**).* We will show that the modelling functor (Lem. A.4) for the generic algebra (Def. 6.2) is an isomorphism $G(-) : FM(Cl(\mathbb{T}), \mathcal{C}) \to \mathcal{C}^{\mathbb{T}}$. Let $G^{-1}(-) : \mathcal{C}^{\mathbb{T}} \to FM(Cl(\mathbb{T}), \mathcal{C})$ be

$$\begin{aligned} G^{-1}(M)(\nabla) &= M[\![\nabla]\!] \\ G^{-1}(M)(f) &= \langle M[\![\nabla' \vdash \overline{a}_1 \,\#\!\!\!\!/\, t_1 : \mathsf{s}_1]\!], \ldots, M[\![\nabla' \vdash \overline{a}_n \,\#\!\!\!\!/\, t_n : \mathsf{s}_n]\!]\rangle \\ G^{-1}(h)_\nabla &= h_{\mathsf{s}_1}^{\#\!\!\!\!/\,\overline{a}_1} \times \cdots \times h_{\mathsf{s}_n}^{\#\!\!\!\!/\,\overline{a}_n} \end{aligned} \tag{18}$$

where ∇ is (15), $f : \nabla' \to \nabla$ is (16) and $h_{\mathsf{s}_i}^{\#\!\!\!\!/\,\overline{a}_i}$ is defined by Def. 4.8. For any $\mathbb{T}$-algebra M in $\mathcal{C}$, $G^{-1}(M)$ is an FM-functor $Cl(\mathbb{T}) \to \mathcal{C}$ by Lem. A.3.

Given a $\mathbb{T}$-homomorphism $h : M \to M'$, we can show that $i^{\overline{a}_i} \circ pr_i \circ G^{-1}(M')(f) \circ G^{-1}(h)_{\nabla'} = i^{\overline{a}_i} \circ pr_i \circ G^{-1}(h)_\nabla \circ G^{-1}(M)(f)$ by Lem. A.5 and Defs. 4.8 and A.2. But $i^{\overline{a}_i} \circ pr_i$ are jointly mono, so $G^{-1}(h)$ is a natural transformation $G^{-1}(M) \dot{\to} G^{-1}(M')$.

That $G(G^{-1}(-))$ is the identity on $\mathcal{C}^{\mathbb{T}}$ follows easily; the converse holds as follows. Given an FM-functor $F : Cl(\mathbb{T}) \to \mathcal{C}$ and $Cl(\mathbb{T})$-object ∇ we have $G^{-1}(G(F))(\nabla) = G(F)[\![\nabla]\!] = F(G[\![\nabla]\!]) = F\nabla$ because F preserves finite products and $G[\![\nabla]\!] = \nabla$. $i^{\overline{a}_i} \circ pr_i \circ G^{-1}(G(F))(f) = F(G[\![\nabla' \vdash t_i : \mathsf{s}_i]\!])$ by (17), which equals $F(\nabla \vdash ([t_i] : \mathsf{s}_i)) = i^{\overline{a}_i} \circ pr_i \circ f$. But $i^{\overline{a}_i} \circ pr_i$ are jointly mono so we have equality on objects. Finally, given a natural transformation $\phi : F \dot{\to} F'$ we must show that $G^{-1}(G(\phi))_\nabla = \phi_\nabla$. This follows by applying $i^{\overline{a}_i} \circ pr_i$ as above.

*Proof (**Thm. 6.7**).* The equivalence functor is $G^{-1}(M(\mathcal{C})) : Cl(Th(\mathcal{C})) \to \mathcal{C}$, defined by Def. 6.5 and (18).

Full: Take the $\mathcal{C}$-arrow $f : G^{-1}(M(\mathcal{C}))(\nabla') \to G^{-1}(M(\mathcal{C}))(\nabla)$. Then $(v_1 : G^{-1}(M(\mathcal{C}))(\nabla')) \vdash ([f\, v_1] : G^{-1}(M(\mathcal{C}))(\nabla))$ is a $Cl(Th(\mathcal{C}))$-arrow. Applying $G^{-1}(M(\mathcal{C}))$ to this arrow gives us $M(\mathcal{C})[\![f]\!] = f$.

Faithful: Say we have $Cl(Th(\mathcal{C}))$-arrows $f = \nabla' \vdash (\overline{a}_1 \,\#\!\!\!\!/\, [t_1] : C_1, \ldots)$ and $f' = \nabla' \vdash (\overline{a}_1 \,\#\!\!\!\!/\, [t_1'] : C_1, \ldots)$ so that $G^{-1}(M(\mathcal{C}))(f) = G^{-1}(M(\mathcal{C}))(f')$. Applying the jointly mono $i^{\overline{a}_i} \circ pr_i$ to each side gives us $M(\mathcal{C})[\![\nabla' \vdash t_i : C_i]\!] = M(\mathcal{C})[\![\nabla' \vdash t_i' : C_i]\!]$, so $\nabla' \vdash_{Th(\mathcal{C})} t_i \approx t_i' : C_i$ by definition and $f = f'$.

Surjective: For any $\mathcal{C}$-object C, $G^{-1}(M(\mathcal{C}))(v_1 : C) = C$.