Fairness in allocation problems

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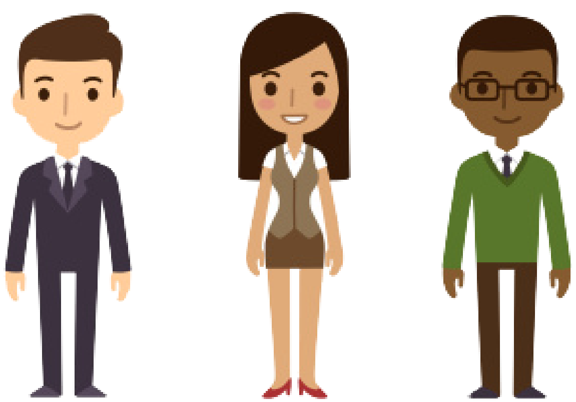
Allocations of goods

- **Indivisible** goods

- **Agents** with additive **valuations** for goods



- Goal: **divide** the goods **fairly**

Formally …

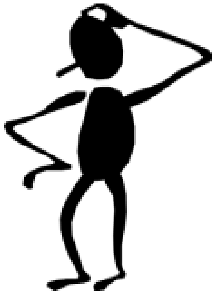
- n **agents**
- A set of **goods** G
- Agent i has **valuation** $u_i(g)$ for good g
- Valuations are additive, i.e.,

$$v_i(S) = \sum_{g \in S} v_i(g)$$

- **Allocation**: a partition $A=(A_1, \ldots, A_n)$ of the goods in G

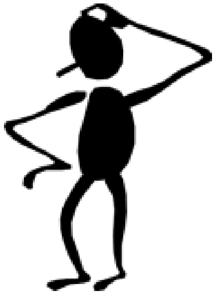
Maxmin fair allocations

What does “**fairly**” mean?



- **Fairness notions**
  - Envy freeness
  - Proportionality
  - **Maxmin share (MmS) allocation**

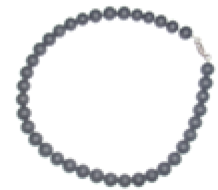
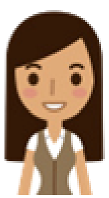
What does “fairly” mean?



- **Fairness notions**
  - **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent
  - **Proportionality**: every agent feels that she gets at least 1/n-th of the goods
  - **Maxmin share (MmS) allocation**: each agent’s value is at least the best guarantee when dividing the goods into n bundles and getting the least valuable bundle

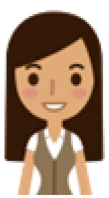
$$\forall i, v_i(A_i) \geq \theta_i = \max_{A'} \min_{j \in N} v_i\left(A'_j\right)$$

MmS: an example

			ROLLS ROYCE		
	$500	$600	$200	$400	$300
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

MmS: an example

Let’s compute the MmS thresholds first

						θ_i
	\$500	\$600	\$200	\$400	\$300	
	\$700	\$700	\$300	\$200	\$100	
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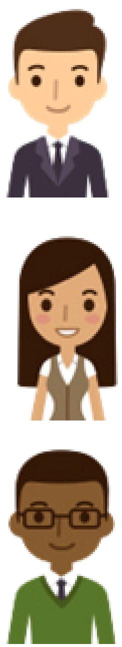
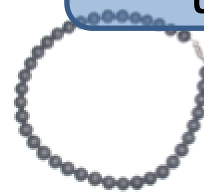
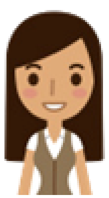


					θ_i
\$500	\$600	\$200	\$400	\$300	\$600
\$700	\$700	\$300	\$200	\$100	\$600
\$900	\$600	\$200	\$200	\$100	\$500

MmS: an example

Now, let's compute the allocation

						θ_i
	\$500	\$600	\$200	\$400	\$300	\$600
	\$700	\$700	\$300	\$200	\$100	\$600
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MmS: an example

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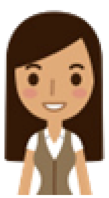
	\$500	**\$600**	\$200	\$400	\$300	\$600
	**\$700**	\$700	\$300	\$200	\$100	\$600
	\$900	\$600	**\$200**	**\$200**	**\$100**	\$500

An implication

- **Theorem**: Proportionality implies MmS

An implication

- **Theorem**: Proportionality implies MmS
- **Proof**: Let A be a proportional allocation. Then,

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$

But the MmS threshold for agent i is

$$\theta_i = \max_{A'} \min_{j \in N} v_i(A'_j) \leq \frac{1}{n} v_i(G)$$

Hence,

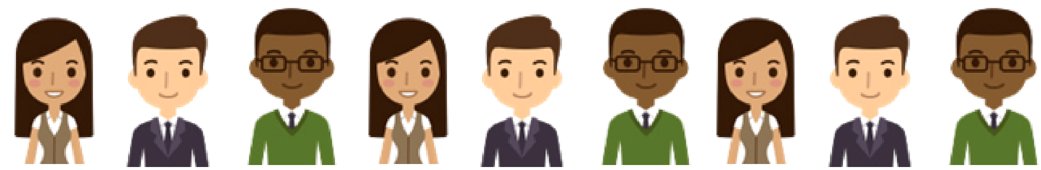
$$\boxed{\forall i, v_i(A_i) \geq \theta_i}$$

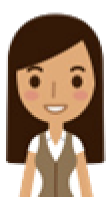
Related work

- MmS concept introduced by Budish (2011)
- **NP-hard** via a reduction from Partition
- MmS allocations **may not exist** but there is always a 2/3-**approximation**
  - Procaccia & Wang (2014)
  - Kurokawa, Procaccia, & Wang (2018)
- Polynomial-time 2/3-approximation algorithms
  - Amanatidis, Markakis, Nikzad, and Saberi (2017)
  - Barman & Murthy (2017)
- Best possible bound: 3/4-approximation algorithm
  - Ghodsi, Hajiaghayi, Seddighin, Seddighin, & Yami (2018)

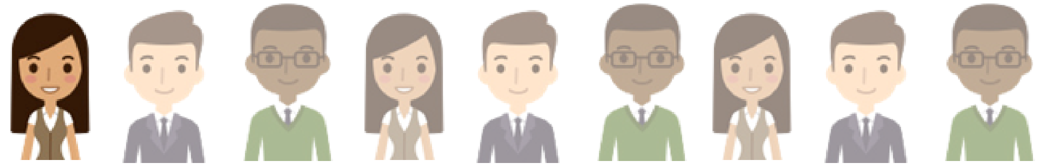
A 1/2-approximate MmS algorithm

The draft mechanism

- Drafting order: 

	\$1200	\$200	\$300	\$200	\$100
	\$800	\$500	\$200	\$300	\$200
	\$800	\$400	\$400	\$200	\$200

The draft mechanism

- Drafting order: 

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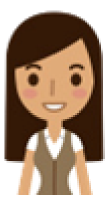	**$800**	$500	$200	$300	$200
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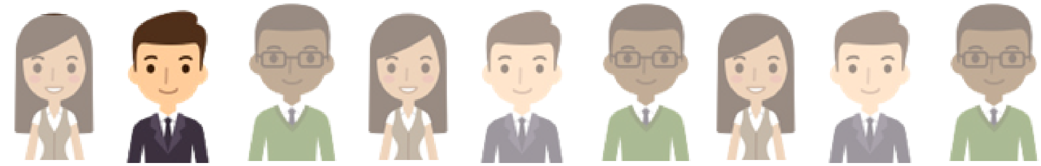
- Drafting order: 

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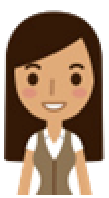	**$800**	$500	$200	$300	$200
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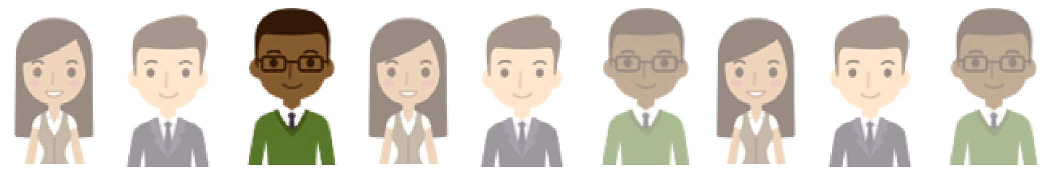
- Drafting order: 

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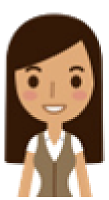	**$800**	$500	$200	$300	$200
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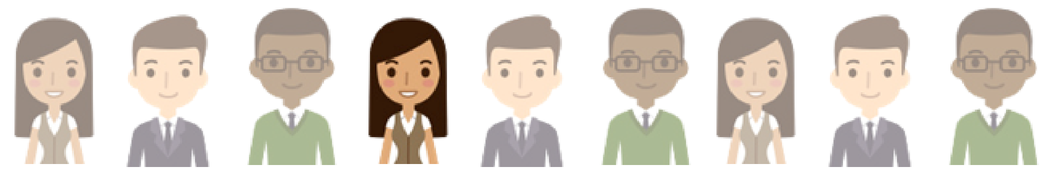
- Drafting order: 

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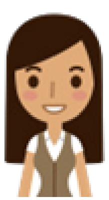	**$800**	$500	$200	**$300**	$200
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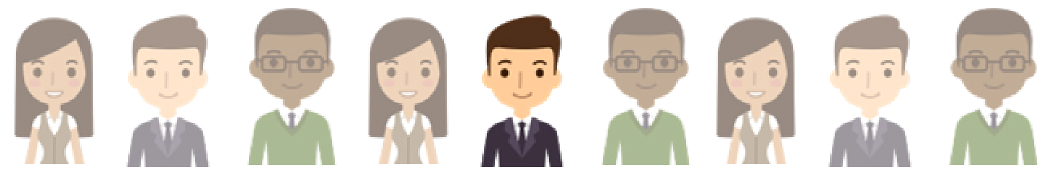
- Drafting order: 

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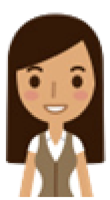	**$800**	$500	$200	**$300**	$200
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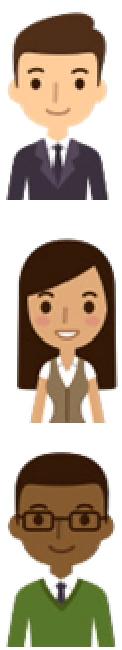
The draft mechanism

- Drafting order: 

Can the draft mechanism compute MmS allocations?

θ_i



$1200	$200	**$300**	$200	**$100**
$800	$500	$200	**$300**	$200
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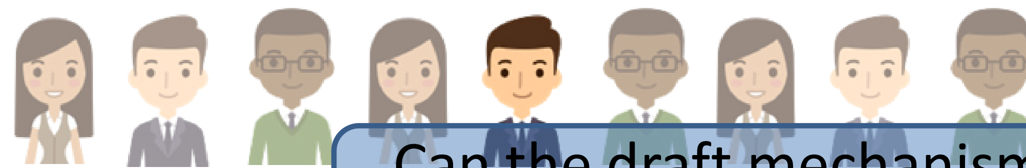
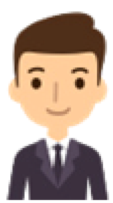
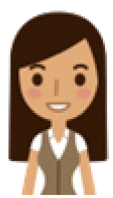
The draft mechanism

- Drafting order: 

Can the draft mechanism compute MmS allocations?

						θ_i
	$1200	$200	**$300**	$200	**$100**	$400
	**$800**	$500	$200	**$300**	$200	$500
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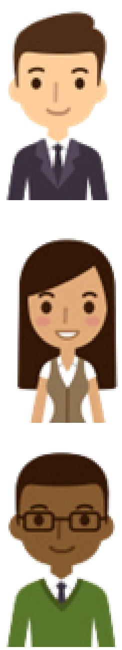

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	$800	**$400**	$400	$200	$200	$600

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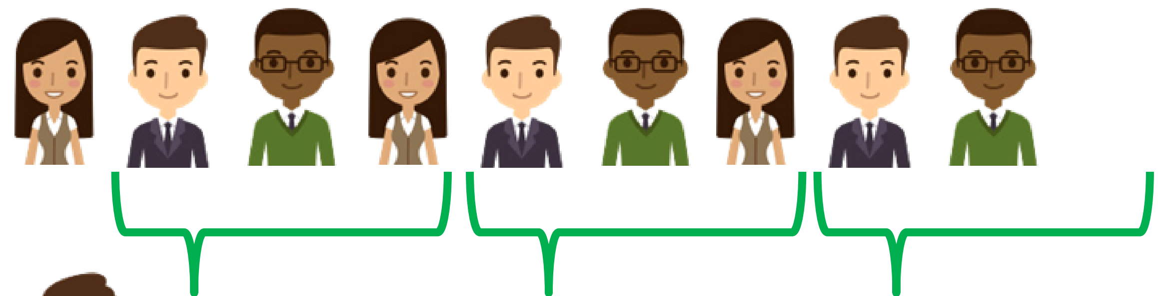
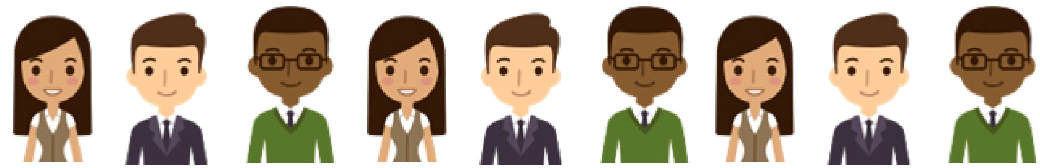
- Drafting order: 
- **Phases** for agent 
- In each phase,  prefers the good he gets to the good every other agent gets
- So,

$$v_i(A_i) \geq \frac{1}{n} v_i(G) - \max_{g \in G} v_i(g)$$

The draft mechanism

- Drafting order: 

$$v_i(A_i) \geq \frac{1}{n} v_i(G) - \max_{g \in G} v_i(g)$$

- So, if $\max v_i(g) \leq \theta_i/2$, then $v_i(A_i) \geq \theta_i/2$ and the draft mechanism yields a 1/2-approximate MmS allocation

A 1/2-approximate MmS algorithm

- If there is an agent i and an good g with $v_i(g) \geq v_i(G)/2n \geq \theta_i/2$, **allocate good g to agent i**
- **Remove agent i and good g** from the instance and **repeat** until $v_i(g) \leq v_i(G')/2n'$ for every good g in the remaining instance with n' agents and set of goods G'
- **Run the draft mechanism** in the remaining instance

A 1/2-approximate MmS algorithm

- Key idea in the analysis:
  - **The MmS threshold increases** as we remove agents and goods

EF1: a relaxed version of EF

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Share Rent

Split Fare

Assign Credit

spliddit

- Fairness hierarchy
  1. Envy-freeness
  2. Proportionality
  3. Maxmin share guarantee
- Previous spliddit protocol
  - Find best fairness criterion
  - Maximize **social welfare** (subject to that criterion)

spliddit

Spliddit Feedback - [redacted]

admin@spliddit.org Jan 7

to admin

Hi! Great app :) We're 4 brothers that need to divide an inheritance of 30+ furniture items. This will save us a fist fight ;) I played around with the demo app and it seems there are non-optimal results for at least two cases where everyone distributes the same amount of value onto the same goods. Try it with either 3 people distributing 1000 points to good A and 0 to the 5 remaining goods, OR try 3 people, 5 goods, with everyone placing 200 on every good. The first case gives 0 to one person, 1 to another and 5 to the third. The second case gives 3 to one person and 1 to each of the others. Why is that? All the best, [redacted]

spliddit

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to admin

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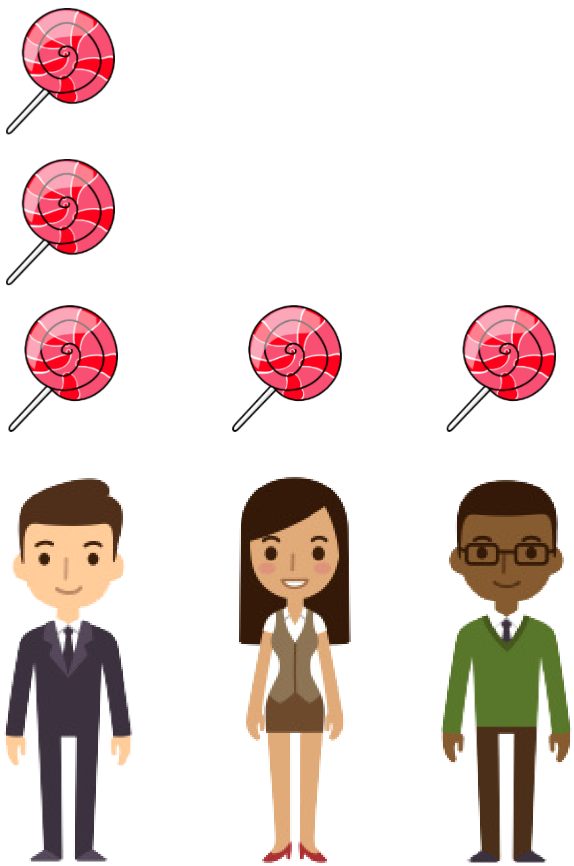
… *try 3 people, 5 goods, with everyone placing 200 on every good*.

… *gives 3 to one person and 1 to each of the others*. Why is that?

…



Relaxing EF

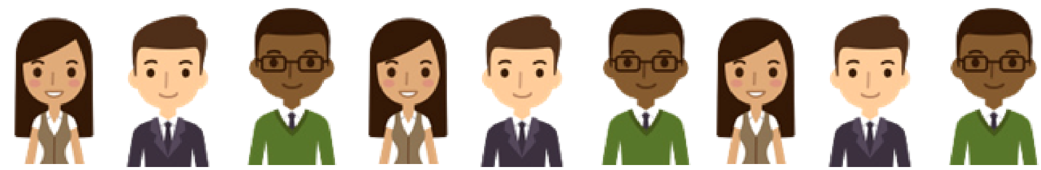
- **Envy-freeness up to one good (EF1)**:
  - There is a good that can be removed from the bundle of agent j so that any envy of agent i for agent j is eliminated

$$\forall i, j, \exists g \in A_j : v_i(A_i) \geq v_i(A_j - g)$$

Relaxing EF

- **Envy-freeness up to one good (EF1)**:
  - There is a good that can be removed from the bundle of agent j so that agent i is not envious for agent j
  - Budish (2011)
  - Easy to achieve: **draft mechanism**
  - Also: Lipton, Markakis, Mossel, and Saberi (2004)

The draft mechanism

- Drafting order: 

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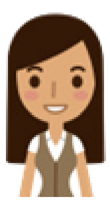	\$800	\$500	\$200	\$300	\$200
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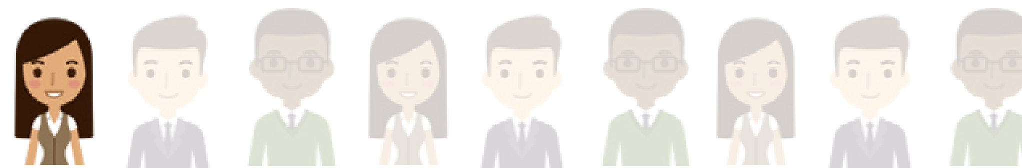
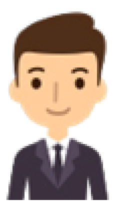
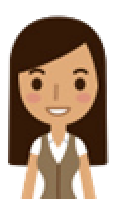
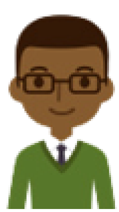
- Drafting order: 

	$1200	$200	$300	$200	$100
	**$800**	$500	$200	$300	$200
	$800	$400	$400	$300	$100

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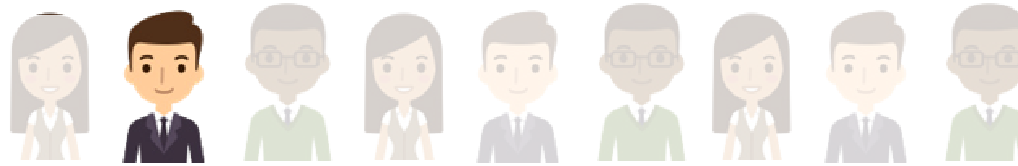
- Drafting order: 

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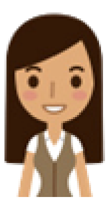	**$800**	$500	$200	$300	$200
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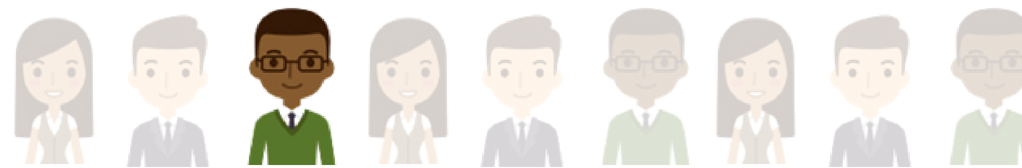
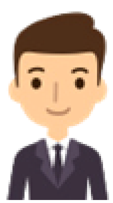
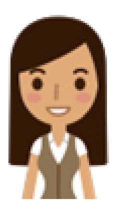
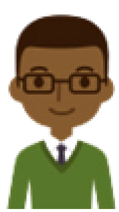
- Drafting order: 

	$1200	$200	**$300**	$200	$100
	**$800**	$500	$200	$300	$200
	$800	**$400**	$400	$300	$100

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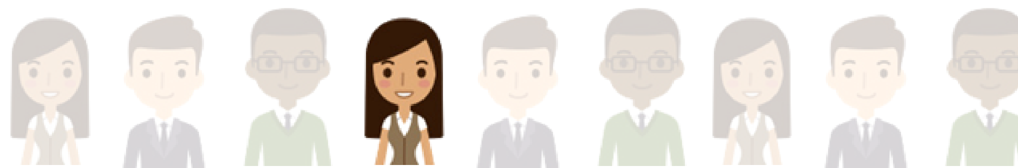
- Drafting order: 

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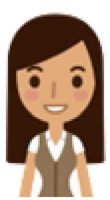	**$800**	$500	$200	**$300**	$200
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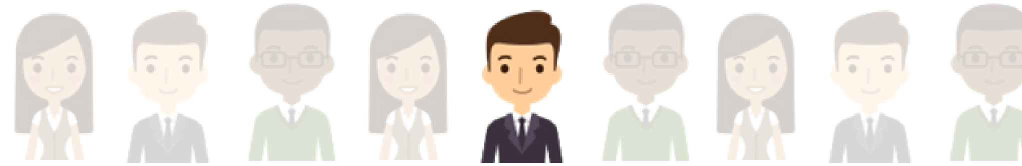
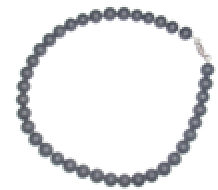
- Drafting order: 

			ROLLS ROYCE		
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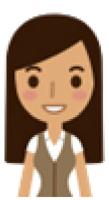	**$800**	$500	$200	**$300**	$200
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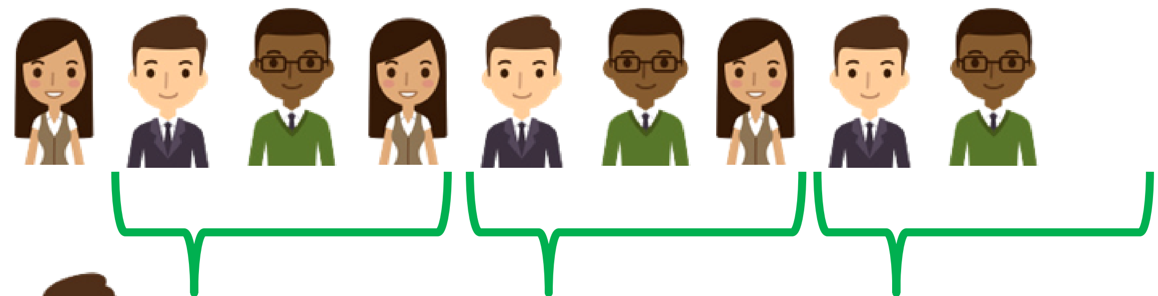
- Drafting order: 
- **Phases** for agent 
- In each phase,  prefers the good he gets to the good every other agent gets
- So, ignoring the good picked by an agent at the very beginning of the sequence,  is EF

Local search

- **Allocate goods one by one**
- In each step j:
  - Allocate good j **to an agent that nobody envies**
  - If this creates a "cycle of envy", **redistribute the bundles along the cycle**
- Crucial property:
  - Envy can be eliminated by removing just a **single good**
  - Implies **EF1**
- Lipton, Markakis, Mossel, & Saberi (2004)

Adding an efficiency objective

- **Pareto optimality (PO)**:
  - No alternative allocation exists that makes some agent better off without making any agents worse off
  - An allocation $A = (A_1, A_2, \ldots, A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, \ldots, B_n)$ such that $v_i(B_i) \geq v_i(A_i)$ for every agent i and $v_{i'}(B_{i'}) > v_{i'}(A_{i'})$ for some agent i'
- Easy to achieve: give each good to the agent that values it the most

EF1+PO?

EF1+PO?

- **Maximum Nash welfare (MNW)** allocation:
  – the allocation that maximizes the Nash welfare (**product of agent valuations**)
- **Theorem**: the MNW solution is EF1 and PO
  – C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)

Theorem: MNW solution is EF1+PO

Theorem: MNW solution is EF1+**PO**

- **PO** is trivial since MNW maximizes $\prod_{i \in N} v_i(A_i)$

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1
- Agent i envies agent j even after any single good is removed from j's bundle

Theorem: MNW solution is **EF1**+PO

- Assume MNW is not EF1
- Agent i envies agent j even after any single good is removed from j's bundle
- For good $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

  we have $\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \operatorname*{argmin}_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

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- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g)$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \operatorname*{argmin}_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

Theorem: MNW solution is **EF1**+PO

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$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g)$$

Theorem: MNW solution is EF1+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \dfrac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)$$

Theorem: MNW solution is **EF1**+PO

- Recall that $g^* = \underset{g \in A_j : v_i(g) > 0}{\operatorname{argmin}} \frac{v_j(g)}{v_i(g)}$

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$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)$$

- Hence, $\boxed{v_i(g^*) v_j(A_j) - v_j(g^*) v_i(A_j) \geq 0}$

Theorem: MNW solution is **EF1**+PO

$$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$$

$$\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$$

Theorem: MNW solution is **EF1**+PO

$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$ $\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$

$v_i(A_i)\ v_j(A_j)$

Theorem: MNW solution is **EF1**+PO

$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$ $\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$

$$v_i(A_i)\, v_j(A_j)$$

$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

Theorem: MNW solution is **EF1**+PO

$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$ $\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$

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$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

$$< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)$$

Theorem: MNW solution is EF1+PO

$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$ $\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$

$$
\begin{aligned}
& v_i(A_i)\, v_j(A_j) \\
& \leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \\
& < v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*) \\
& = (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))
\end{aligned}
$$

Theorem: MNW solution is **EF1**+PO

$\boxed{v_i(A_i) < v_i(A_j) - v_i(g^*)}$ $\boxed{v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0}$

$$v_i(A_i)\, v_j(A_j)$$

$$\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)$$

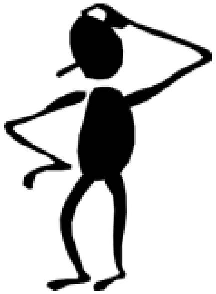
$$< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)$$

$$= (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))$$

- So A is not a MNW solution, a contradiction.

- **QED**

Computational issues



- **EF1**+**PO** in **polynomial time**?
  - Yes for two agents (using a restricted MNW solution)
  - Open for more agents (e.g., three agents)
  - Several attempts (e.g., rounding a fractional MNW solution) miserably failed
  - Some progress in very recent work by Barman, Murthy, & Vaish (2018)

Summary

- What have we covered today?
  - Maxmin fair share
  - EF1
  - Draft mechanism
  - Local search
  - Max Nash Welfare mechanism