Fairness in allocation problems

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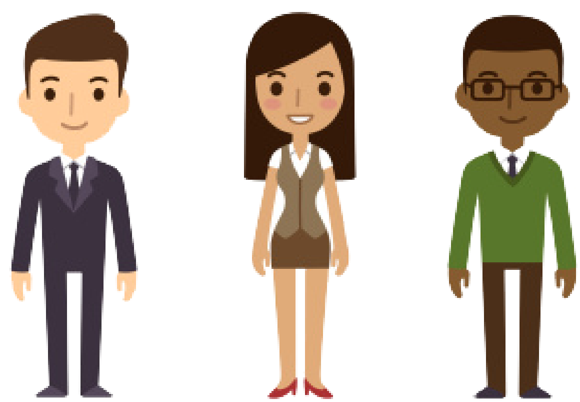
Allocations of goods

- **Indivisible** goods




- **Agents** with additive **valuations** for goods



- Goal: **divide** the goods **fairly**

Formally …

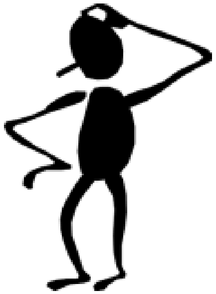
- n **agents**
- A set of **goods** G
- Agent i has **valuation** $v_i(g)$ for good g
- Valuations are additive, i.e.,

$$v_i(S) = \sum_{g \in S} v_i(g)$$

- **Allocation**: a partition $A=(A_1, \ldots, A_n)$ of the goods in G

More fairness notions

What does “fairly” mean?



- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Maxmin share (MmS) allocation
  - Envy-freeness up to one good (EF1)

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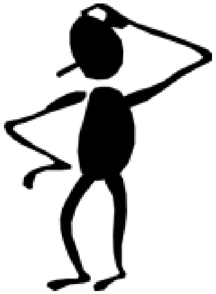

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EF → **Prop** → **MmS**

EF → **EF1**

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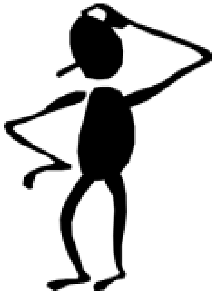

- **Fairness notions**
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  - **Minmax share (mMS) allocation**
  - **Envy-freeness up to any good (EFX)**
  - **Pairwise MmS allocation**

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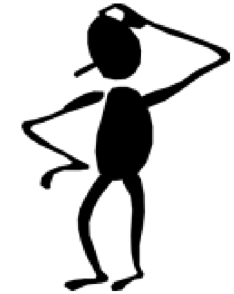

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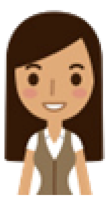
  - **Minmax share (mMS) allocation**: each agent’s value is at least the worst guarantee when dividing the goods into n bundles and getting the most valuable bundle

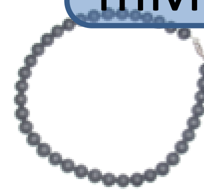
$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i\left(A'_j\right)$$

mMS: an example

			ROLLS ROYCE		
	$500	$600	$200	$400	$300
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

mMS: an example

Let’s compute the mMS thresholds first

						θ_i
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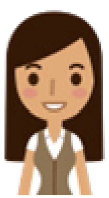	\$700	\$700	\$300	\$200	\$100	
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				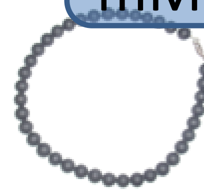		θ_i
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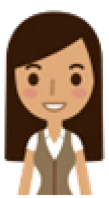	\$700	\$700	\$300	\$200	\$100	\$700
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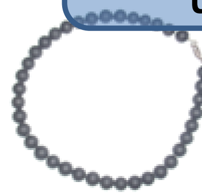
mMS: an example

Now, let’s compute the allocation

			ROLLS ROYCE			θ_i
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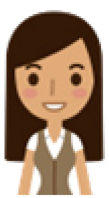	\$700	\$700	\$300	\$200	\$100	\$700
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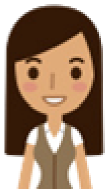	$700	**$700**	**$300**	$200	$100	$700
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An implication

- **Theorem**: EF implies mMS

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- **Proof**: Let A be an EF allocation. Then,

$$\boxed{\forall i, v_i(A_i) \geq \max_{j \in N} v_i(A_j) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)}$$

Another implication

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- **Proof**: Let A be an mMS allocation. Then,

$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)$$

But the mMS threshold for agent i is

$$\theta_i = \min_{A'} \max_{j \in N} v_i(A'_j) \geq \frac{1}{n} v_i(G)$$

Hence,

$$\boxed{\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)}$$

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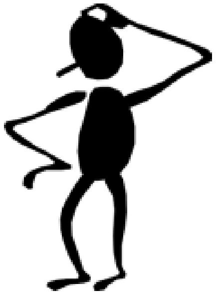

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$$\begin{tikzcd}
\textbf{EF} \arrow[r] \arrow[dr] & \textbf{mMS} \arrow[r] & \textbf{Prop} \arrow[r] & \textbf{MmS} \\
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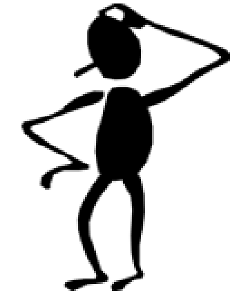

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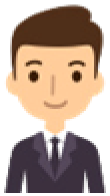
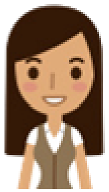
  - **Envy-freeness up to any good (EFX)**: agent i is either not envious of agent j initially or s/he is not envious after removing any good from the bundle of agent j

$$\forall i, j, \forall g \in A_j \text{ with } v_i(g) > 0: v_i(A_i) \geq v_i(A_j - g)$$

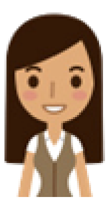
EFX: an example

	$500	$600	$200	**$400**	**$300**
	$700	**$700**	**$300**	$200	$100
	**$900**	$600	$200	$200	$100

EFX: another example

- Drafting order: 

Can the draft mechanism compute EFX allocations?

	\$1200	\$200	**\$300**	\$200	**\$100**
	**\$800**	\$500	\$200	**\$300**	\$200
	\$800	**\$400**	\$400	\$200	\$200

More implications

- **Theorem**: EF implies EFX, which implies EF1

$$\begin{tikzcd}
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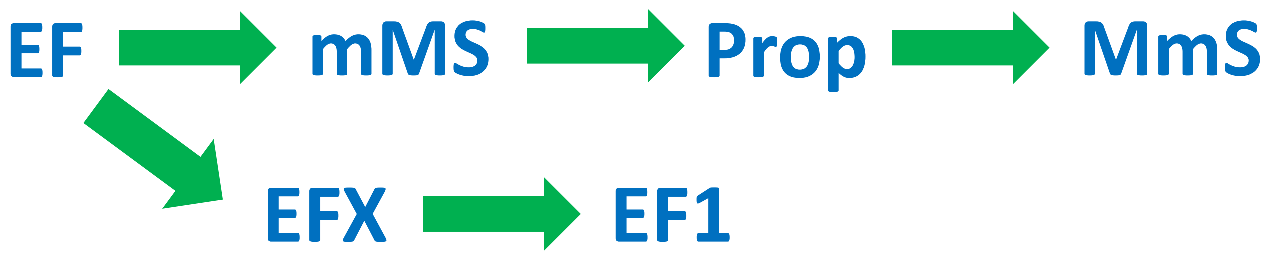

- **Open question**: Does an EFX allocation always exist?
- So, is the implication EFX => EF1 strict?

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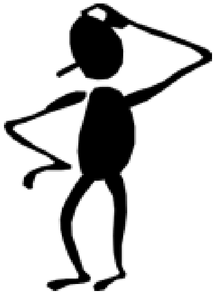

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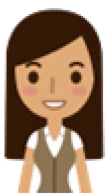
  - **Pairwise MmS allocation**: an allocation A is pairwise MmS if for every pair of agents i and j, the allocation (A_i, A_j) between the two agents is MmS

Pairwise MmS: an example

						θ_i
	\$500	\$600	\$200	**\$400**	**\$300**	\$700
	\$700	**\$700**	**\$300**	\$200	\$100	\$600
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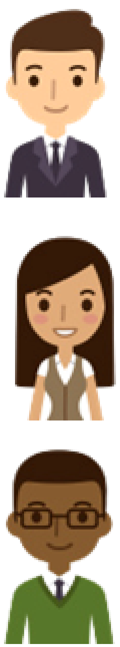

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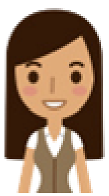	\$700	**\$700**	**\$300**	\$200	\$100	\$700
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Pairwise MmS: another example

- Drafting order: 

Can the draft mechanism compute pMmS allocations?

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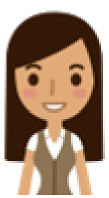
	\$1200	\$200	**\$300**	\$200	**\$100**	\$700
	**\$800**	\$500	\$200	**\$300**	\$200	\$800
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  Let A be a pMmS allocation that is not EFX.

  I.e., there are agents i, j so that for a good $g \in A_j$ with $v_i(g)>0$, it holds that $v_i(A_i) < v_i(A_j\text{-}g)$.

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  Then, the pairwise MmS threshold for agent i should be higher than either $v_i(A_i+g)$ or $v_i(A_j\text{-}g)$.

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  Then, the pairwise MmS threshold for agent i should be higher than either $v_i(A_i+g)$ or $v_i(A_j\text{-}g)$.

  This contradicts the assumptions that A is pMmS.

Yet another implication

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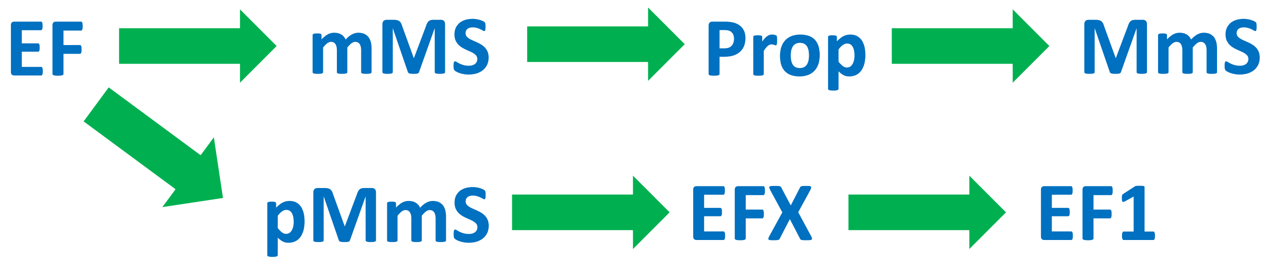

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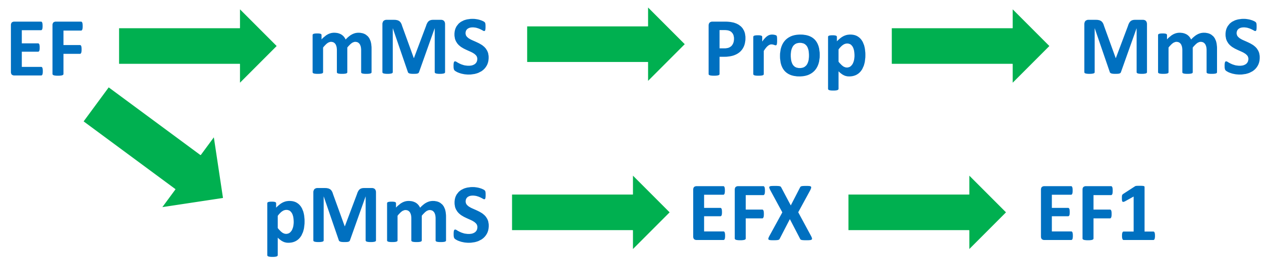


- **Open question**: Does a pairwise MmS allocation always exist?
- So, is the implication pMmS => EFX strict?

Further reading

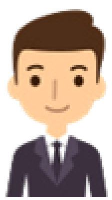
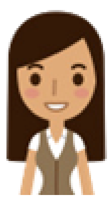
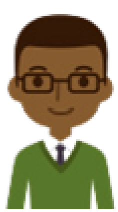
- **Fairness notions**
  - EF, Proportionality: folklore
  - MmS, EF1: Budish (2011)
  - mMS: Bouveret & Lemaitre (2016)
  - EFX, pairwise MmS: C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)
  - EFX: Plaut & Roughgarden (2018)
  - Approximate notions of EF: Amanatidis, Markakis, & Birmpas (2018)

Fairness, knowledge, and social constraints

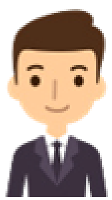
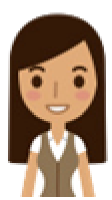
Fairness and knowledge

- What kind of **knowledge** do the agents need to have?
- Knowledge about the **goods** and the **number of agents** only:
  - Proportionality, MmS, mMS
- Knowledge about the whole **allocation**:
  - EF, EFX, EF1, pairwise MmS

Envy-freeness?

				
	$1000	$600	$600	$100
	$1000	$600	$600	$100
	$100	$600	$600	$1000

Epistemic envy-freeness (EEF)

			ROLLS ROYCE	
	**$1000**	$600	$600	$100
	$1000	**$600**	**$600**	$100
	$100	$600	$600	**$1000**

Epistemic envy-freeness (EEF)

- Informally: a **relaxation of EF** with a definition that uses only knowledge about goods and number of agents
- Formal definition:
  - the allocation (A_1, A_2, …, A_n) is **EEF** if, for every agent i, there is a **reallocation** (**B_1, …, B_{i-1}, A_i, B_{i+1}, …, B_n**) in which agent i is not envious, i.e., **$v_i(A_i) \geq v_i(B_j)$** for every other agent j
- Aziz, C., Bouveret, Giagkousi, & Lang (2018)

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- **Theorem**: EF implies EEF, which implies mMS

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- **Theorem**: EF implies EEF, which implies mMS
- **Proof**: EF trivially implies EEF (with B = A).

Also, $v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)$

Epistemic envy-freeness (EEF)

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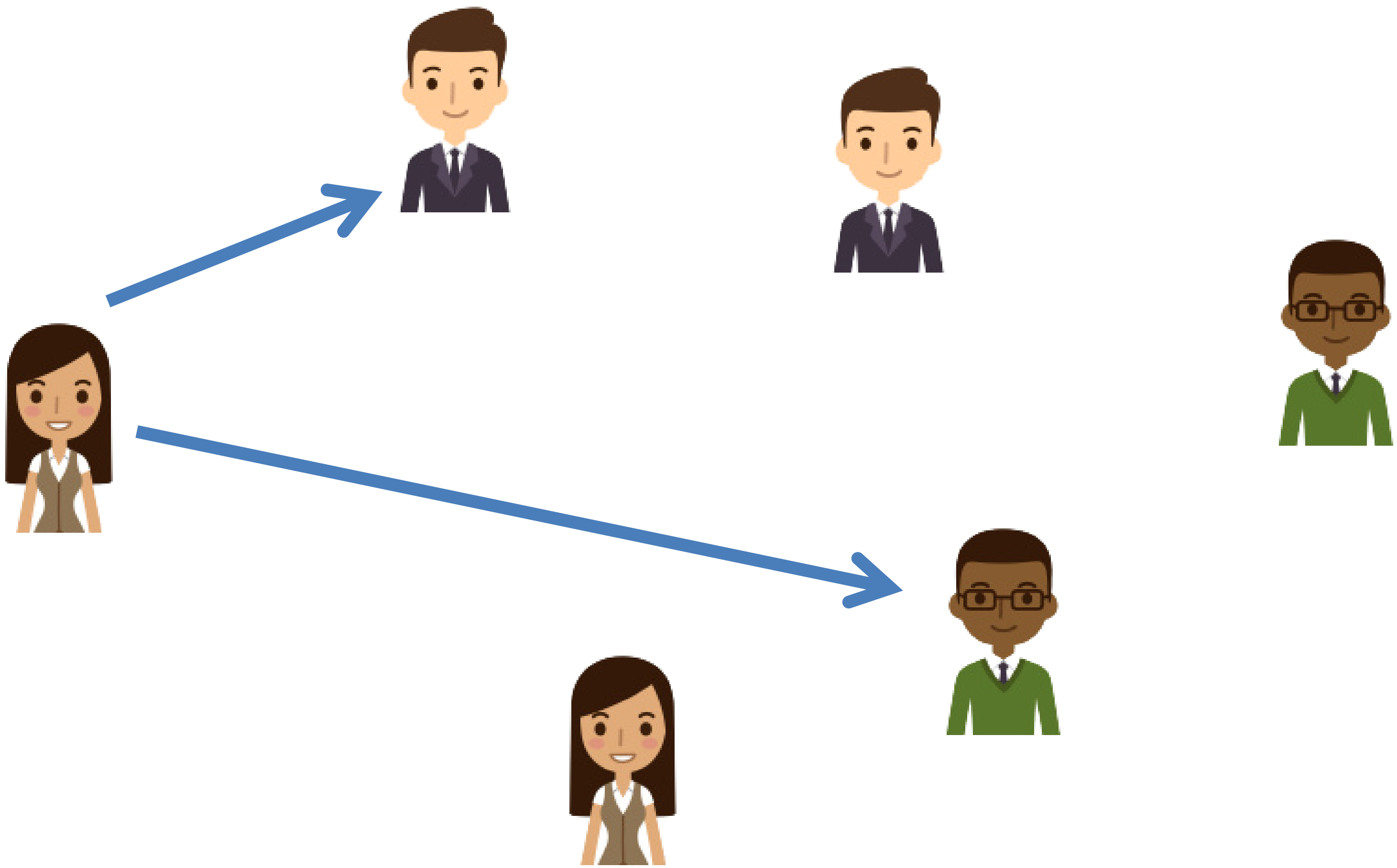
\end{tikzcd}
$$

Fairness with social constraints

- Existence of an **underlying social graph**



Fairness with social constraints

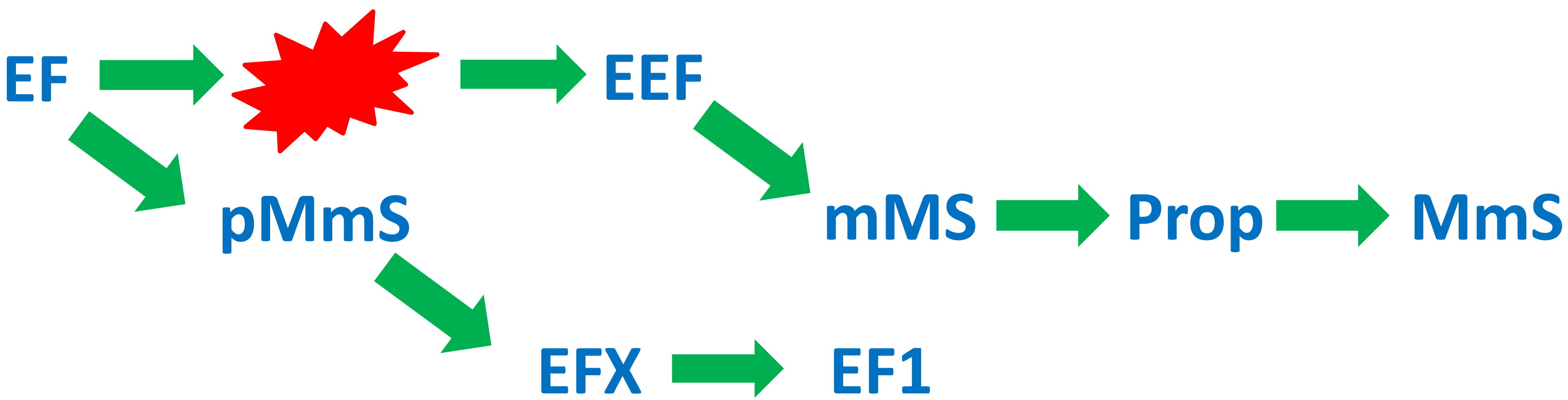
- Existence of an **underlying social graph**, which represents the knowledge each agent has for the bundles allocated to other agents
- Recent related papers (graph-EF/Proportionality):
  - Abebe, Kleinberg, & Parkes (2017)
  - Bei, Qiao, & Zhang (2017)
  - Chevaleyre, Endriss, & Maudet (2017)
  - Aziz, C., Bouveret, Giagkousi, & Lang (2018)

Graph-EEF

- **Social graph G**: directed graph having the agents as nodes
- **G-EEF**:
  - agent i is **EF wrt her neighbors** and
  - **EEF wrt to her non-neighbors**
- G-EEF is
  - **EF** if **G is the complete graph** (or every node has degree ≥ n-2)
  - **EEF** if **G is the empty graph**

More implications

- Social graphs G and H over the same set of nodes
  - Rich hierarchy of fairness notions between EF and EEF
  - If **G is a subgraph of H**, then **H-EEF implies G-EEF**
  - Otherwise, there is an n-agent allocation instance that has an **H-EEF but no G-EEF** allocation



More fairness notions

- **G-PEF**
  - Again, using a social graph G
  - P stands for **proportionality**
  - Combined with **EF**
- See also:
  - Aziz, C., Bouveret, Giagkousi, & Lang (2018)

Summary

- What have we covered today?
  - Minmax (mMS) share allocations
  - EFX
  - Pairwise MmS
  - Epistemic envy-freeness
  - Fairness and social constraints

Last slide

- Please, send me any questions, remarks, or proofs at **caragian@ceid.upatras.gr**

Last slide

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