Binary Counters

Integer Representations towards Efficient Counting in the Bit Probe Model

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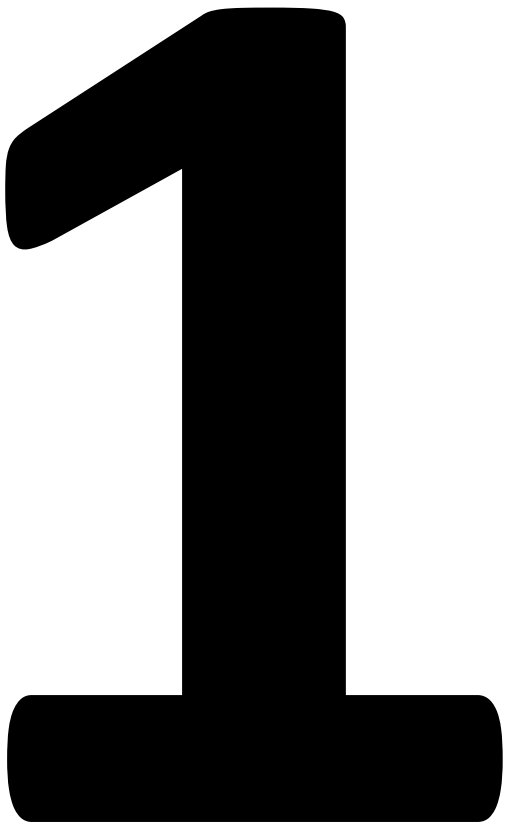


10

11

100

101

110

1 1 1

1000

1001

1010

1011

11100

11101

1110

11111

0000

- we are counting modulo $10000_2 = 16_{10}$

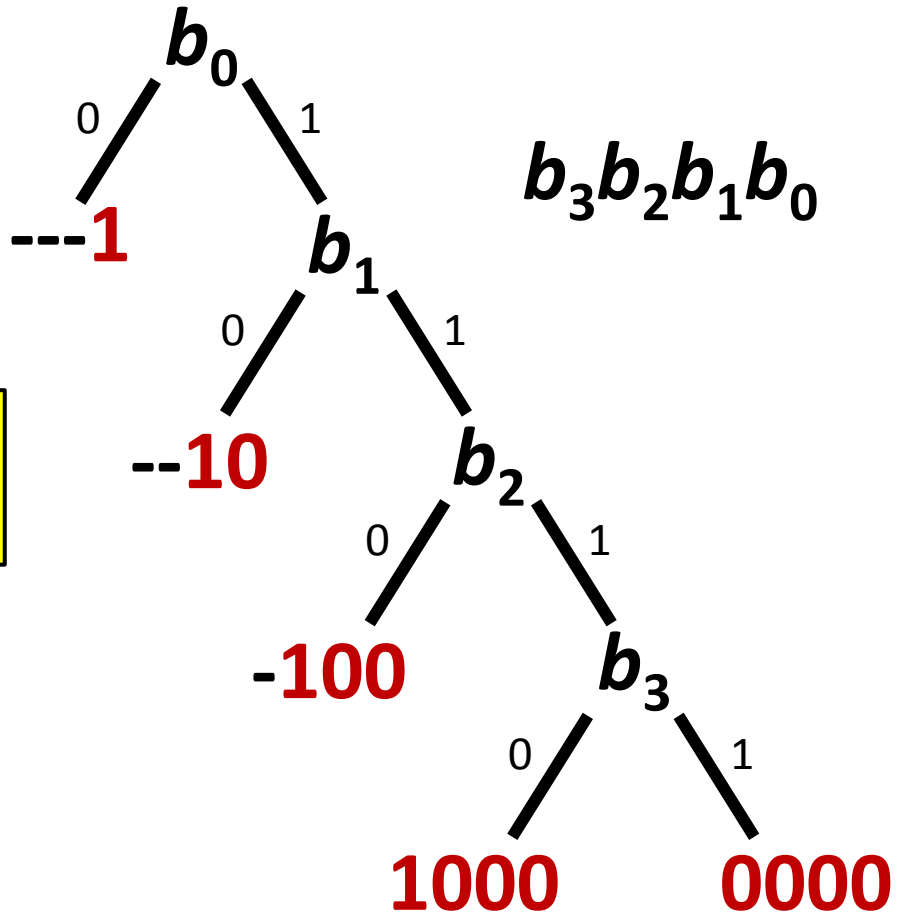
1011

$$1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 8+2+1 = 11_{10}$$

Decimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111
0	0000

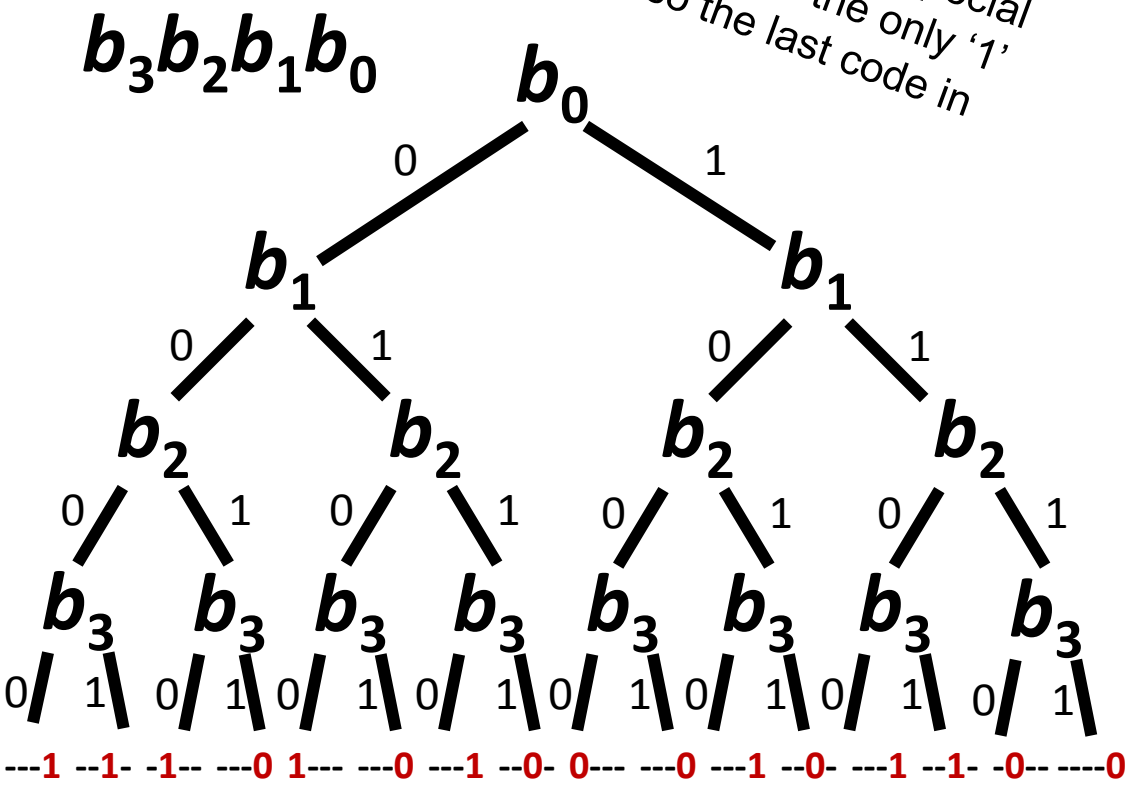
Algorithm

Reads 4 bits
Writes 4 bits



Decimal	Binary	Reflected Gray code
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000
0	0000	0000

"To get the next code, for a code with even parity, flip the rightmost bit. For a code with odd parity, find the rightmost '1' and flip the bit to its left. The only special case is when the last bit b_n is the only '1' in the code. This is also the last code in the sequence."



Always reads 4 bits

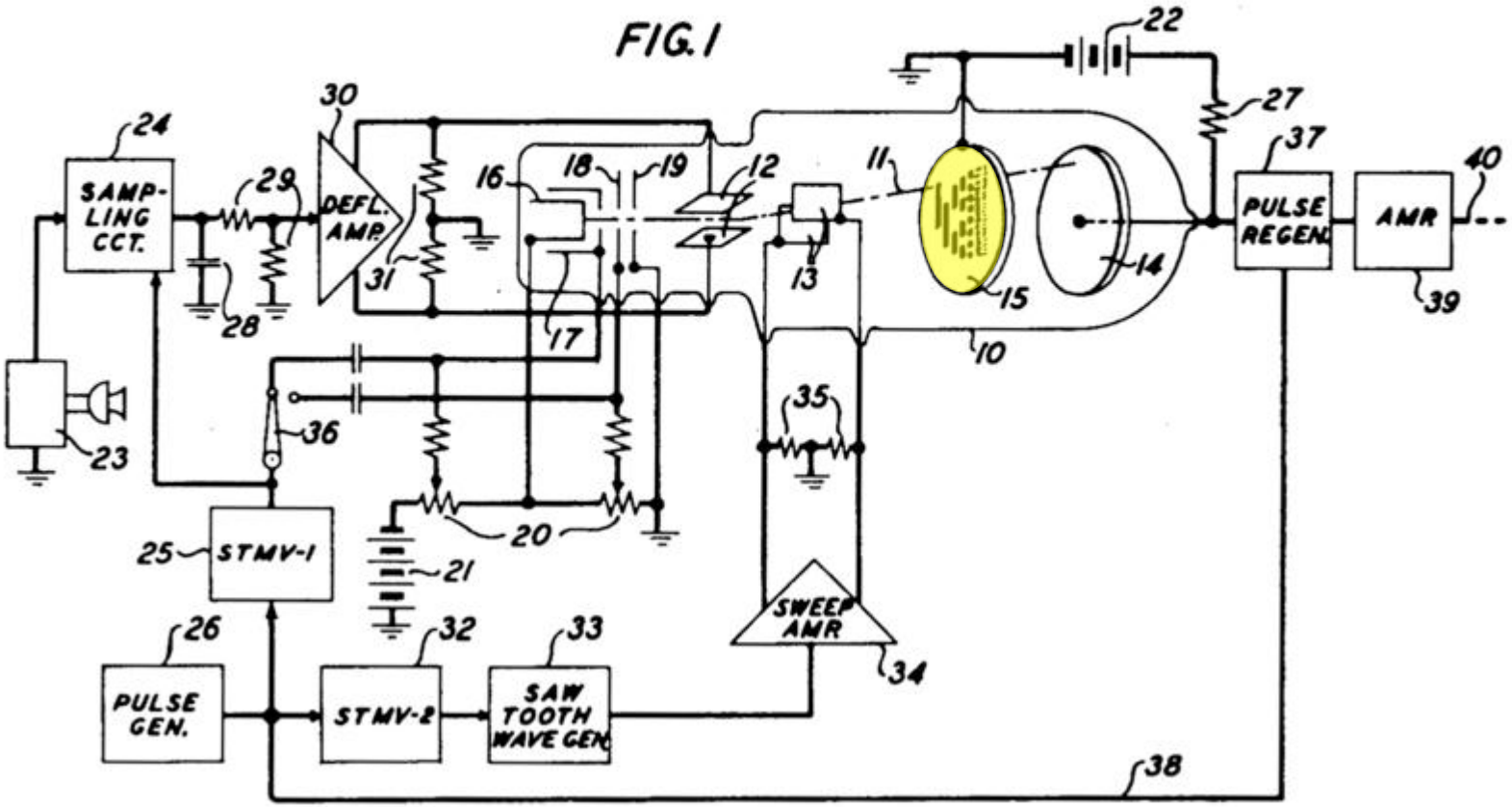
Always writes 1 bit

March 17, 1953 F. GRAY 2,632,058

PULSE CODE COMMUNICATION

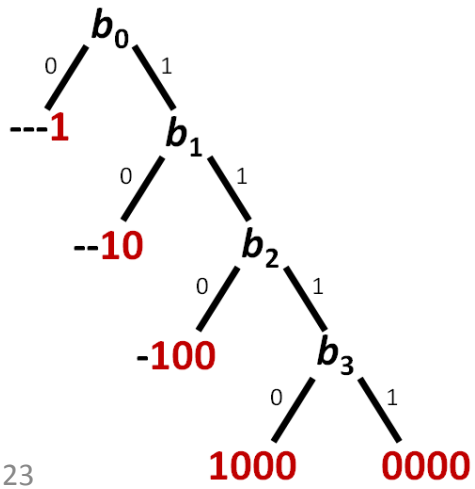
Filed Nov. 13, 1947 4 Sheets-Sheet 1

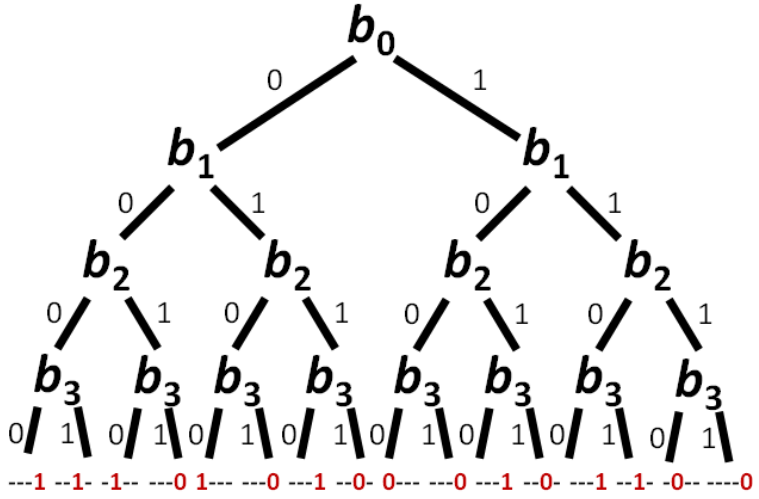
FIG. 1



Question

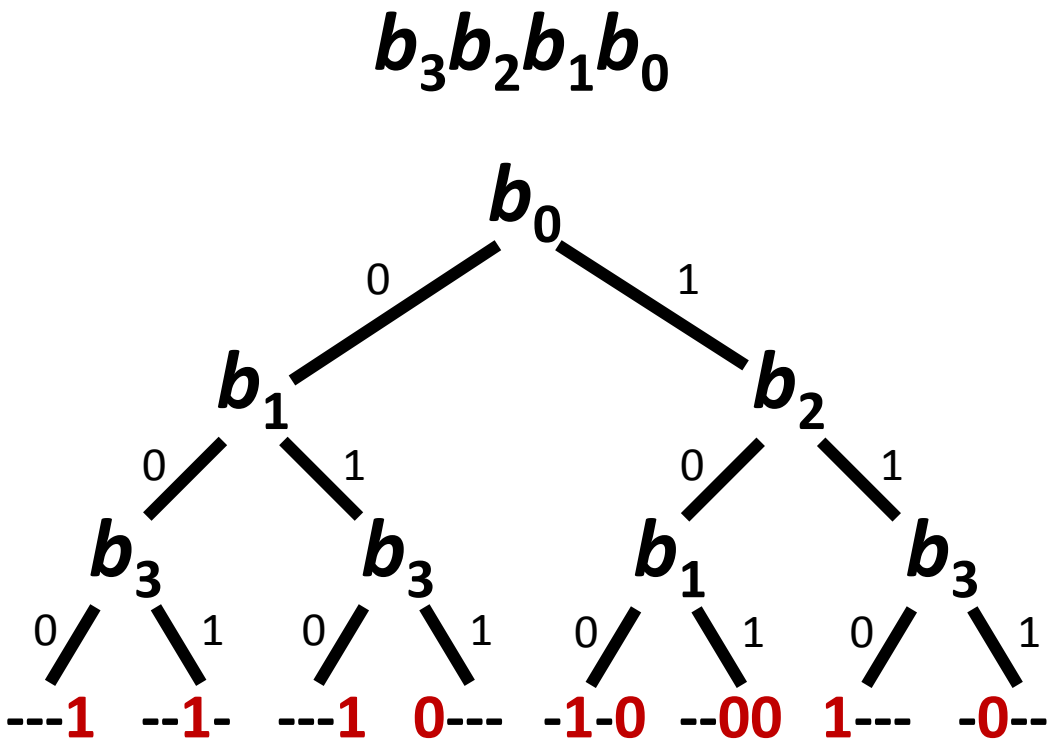
Does there exist a counter where one never needs to read all bits to increment the counter ?





Decimal	
0	0000
1	0001
2	0100
3	0101
4	1101
5	1001
6	1100
7	1110
8	0110
9	0111
10	1111
11	1011
12	1000
13	1010
14	0010
15	0011
0	0000

[B., Greve , Pandey, Rao 2011]

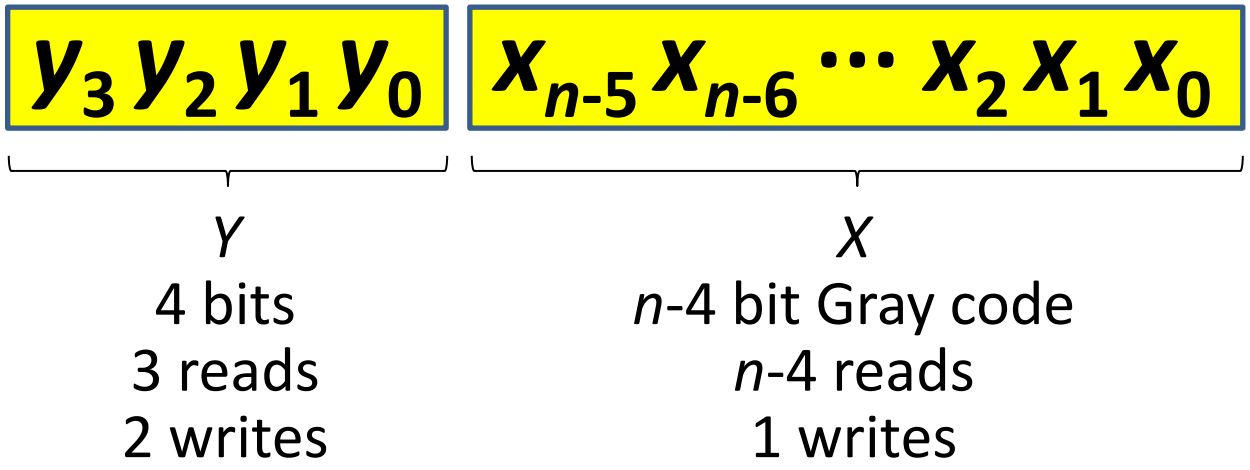


Always reads 3 bits

Always writes ≤ 2 bits

[B., Greve , Pandey, Rao 2011]

Generalization to *n* bit counters



metode Increment(*YX*)

inc(*X*)

if (*X* == 0) inc(*Y*)

test needs to read all bits of *X*

Always reads *n*-1 bits

Always writes ≤ 3 bits

Theorem

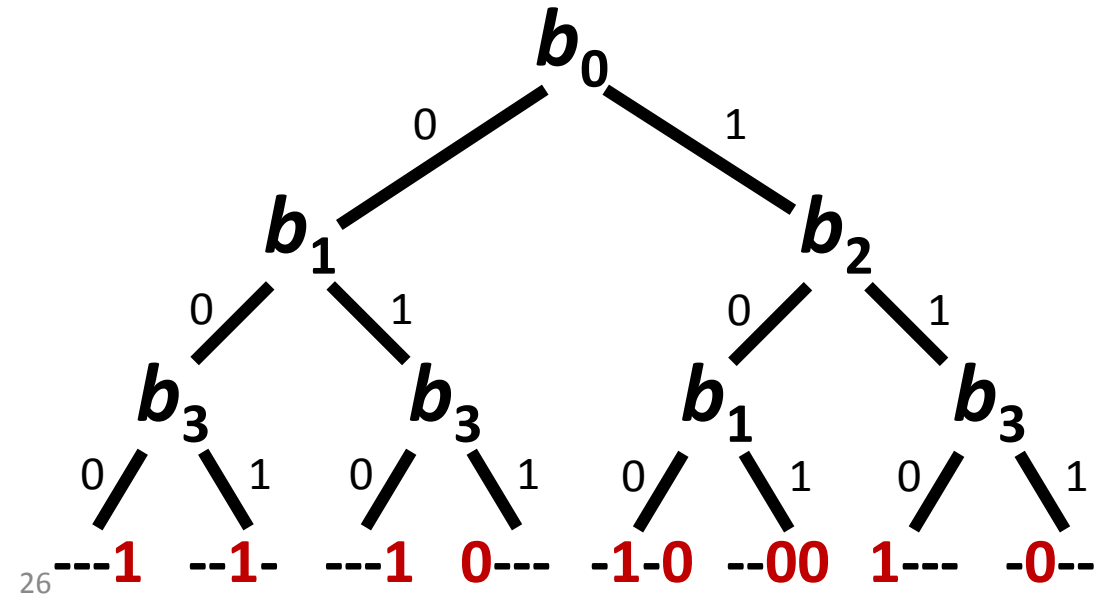
4-bit counter 3 reads and 2 writes

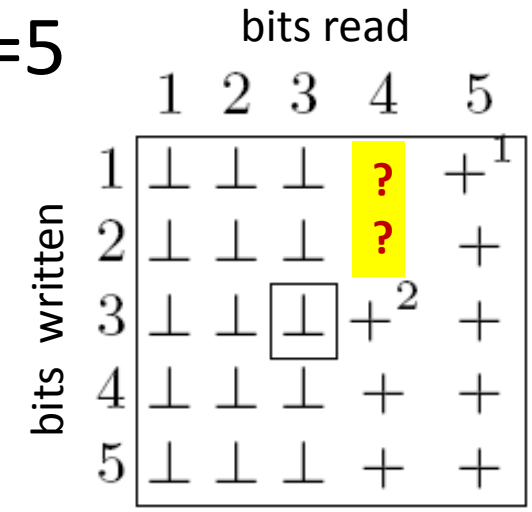
n-bit counter *n*-1 reads and 3 writes

Open problems

n-1 reads and 2 writes, *n*>4?

« *n* reads and writes ? [number of reads at least $\log_2 n$]





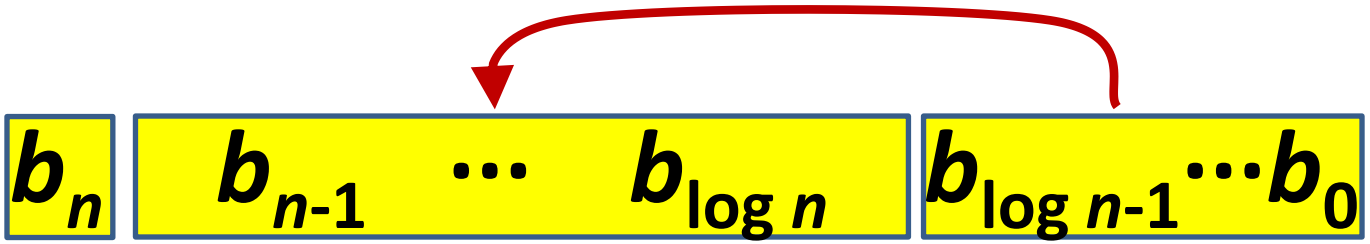
Redundant Counters

Represent ***L*** different values
using ***d*** > log *L* bits

Efficiency ***E*** = $L / 2^d$

[B., Greve , Pandey, Rao 2011]

Redundant counter with $E = ½$ ($L = 2^n$)



carry
1 bit
1 read
1 write

X_H
n-log n bits
1 read
1 write

standard binary counter with delayed increment

X_L
log n bits Gray code
log n reads
1 write

Idea: Each increment of X_L performs one step of the delayed increment of X_H

n+1 bits	2^n values	log n + 2 reads	3 writes

Redundant counter with $E = 1/2$

increment

Value	carry	X_H					X_L		
0	1	0	1	0	1	1	0	0	0
1	1	0	1	0	1	0	0	0	1
2	1	0	1	0	0	0	0	1	1
3	0	0	1	1	0	0	0	1	0
4	0	0	1	1	0	0	1	1	0
5	0	0	1	1	0	0	1	1	1
6	0	0	1	1	0	0	1	0	1
7	0	0	1	1	0	0	1	0	0
8	1	0	1	1	0	0	0	0	0
10	0	0	1	1	0	1	0	0	1

$$\text{Value} = \text{Val}(X_L) + 2^{|X_L|} \cdot (\text{Val}(X_H) + \text{carry} \cdot 2^{\text{Val}(X_L)})$$

n+1 bits	2^n values	log n + 2 reads	3 writes

Redundant counter with $E = 1/2$

increment

delayed reset

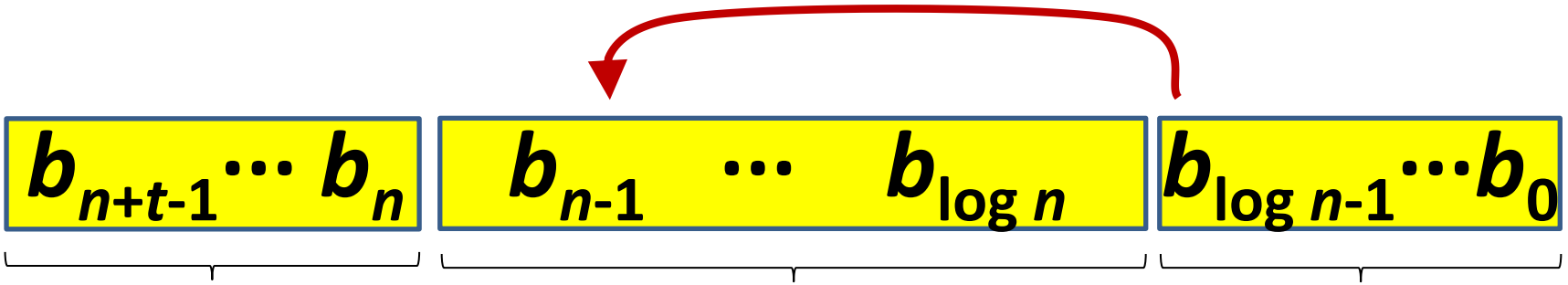
delayed reset

Value	carry	X_H					X_L		
0	1	0	1	0	1	1	0	0	0
1	1	0	1	0	1	0	0	0	1
2	1	0	1	0	0	0	0	1	1
3	1	0	1	1	0	0	0	1	0
4	0	0	1	1	0	0	1	1	0
5	0	0	1	1	0	0	1	1	1
6	0	0	1	1	0	0	1	0	1
7	0	0	1	1	0	0	1	0	0
8	1	0	1	1	0	0	0	0	0
10	1	0	1	1	0	1	0	0	1

n+1 bits **2^n values** **log n + 3 reads** **2 writes**

[B., Greve , Pandey, Rao 2011]

Redundant counter with $E = 1\text{-}1/2^t$



"carry"
t bits
t read
1 write

X_H
n-log n bits
1 read
1 write

X_L
log *n* bit Gray code
log *n* reads
1 write

delayed standard binary counter

"Carry" : part of counter = 0.. 2^t-3, set = 2^t-2, clear 2^t-1

***n+t* bits**	**$(2^t\text{-}1)\cdot 2^n$ values**	**log *n*+*t*+1 reads**	**3 writes**

Redundant Counters

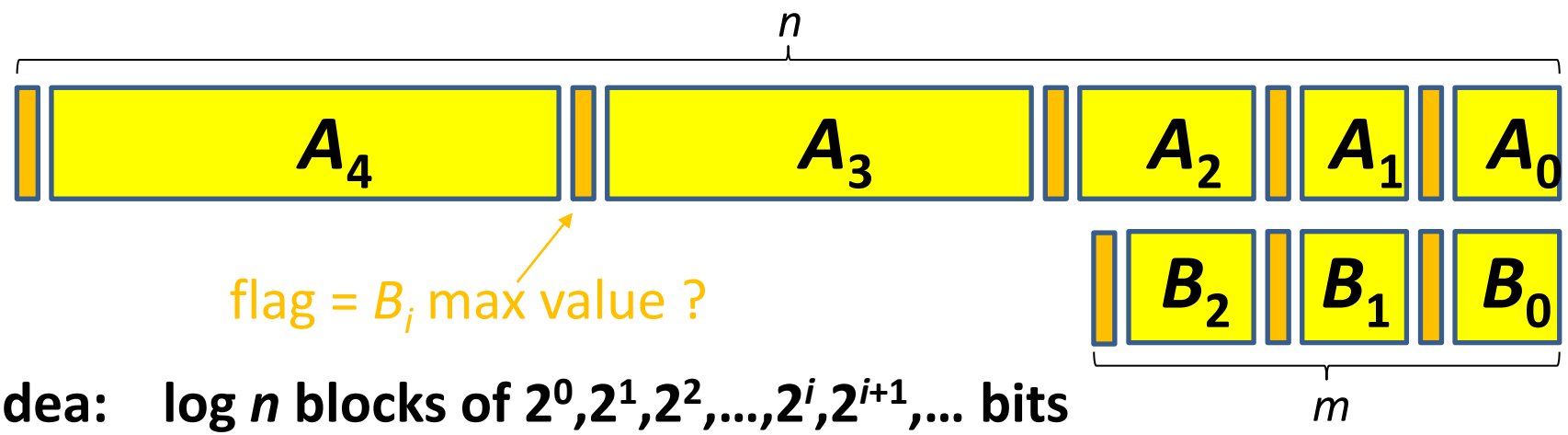
Efficiency	Space	Reads	Writes
1/2	$n + 1$	$\log n + 2$	3
		$\log n + 3$	2
$1 - 1/2^t$	$n + t$	$\log n + t + 1$	3
		$\log n + t + 2$	2

Open problem 1 write and « n reads ?

Addition of Counters

Numbers in the range $0..2^n-1$ and $0..2^m-1$ ($m \le n$)

Space	Reads	Writes
$n + O(\log n)$	$\Theta(m + \log n)$	$\Theta(m)$
$n + O(\log\log n)$	$\Theta(m + \log n \cdot \log\log n)$	
$n + O(1)$	$\Theta(m + \log^2 n)$	



Idea: **log *n* blocks of $2^0,2^1,2^2,\ldots,2^i,2^{i+1},\ldots$ bits**

Space (d)	Space efficiency	Bits read (B_R)		Bits written (B_W)	Inc. & Dec.	Ref.
		Average-case	Worst-case	Worst-case		
n	1	$2 - 2^{1-n}$	n	n	Y	Binary
		n		1	Y	[3]
		$6 \log n$		1	Y	[1]
		$O(\log^{(2c-1)} n)$		c	N	[1]
$n+1$	1/2	$O(1)$	$\log n + 4$	4	Y	[4]
$n + O(t \log n)$	$1 - O(n^{-t})$	$O(\log^{(2c)} n)$	$O(t \log n)$	$2c+1$	N	[1]

Space (d)	Space efficiency	Average-case		Worst-case		Inc. & Dec.	Ref.
		B_R	B_W	B_R	B_W		
4	1	3	1.25	3	2	Y	Th. 1
n		$6 \log(n-4) + O(2^{-n})$	$1 + O(2^{-n})$	$n-1$	3		Th. 2
$n+1$	1/2	$O(\log \log n)$	$1 + O(n^{-1})$	$\log n + 2$	3	N	Th. 3
				$\log n + 3$	2		Th. 4
		$O(\log n)$		$\log n + 3$	3	Y	Th. 6
				$\log n + 4$	2		
n	$1 - \frac{1}{2^{t-1}}$	$O(\log \log n)$	$1 + O(n^{-1})$	$\log n + t + 1$	3	N	Th. 5
				$\log n + t + 2$	2		
		$O(\log n)$		$\log n + t + 2$	3	Y	Cor. 1
				$\log n + t + 3$	2		

[1] Bose, Carmi, Jansens, Maheshwai, Morin, Smid, SWAT 2010
[3] Gray, Patent 1953 [4] Rahman, Munro, Algorithmica 2010


Thank You
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