External-Memory Priority Queues and Persistent Search Trees

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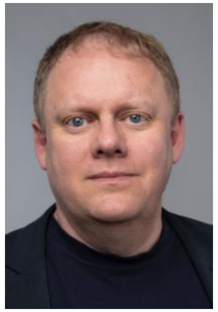
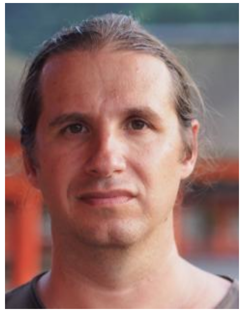
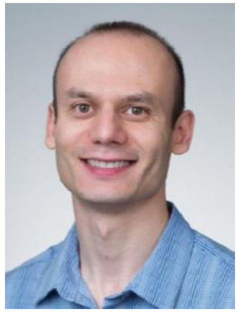




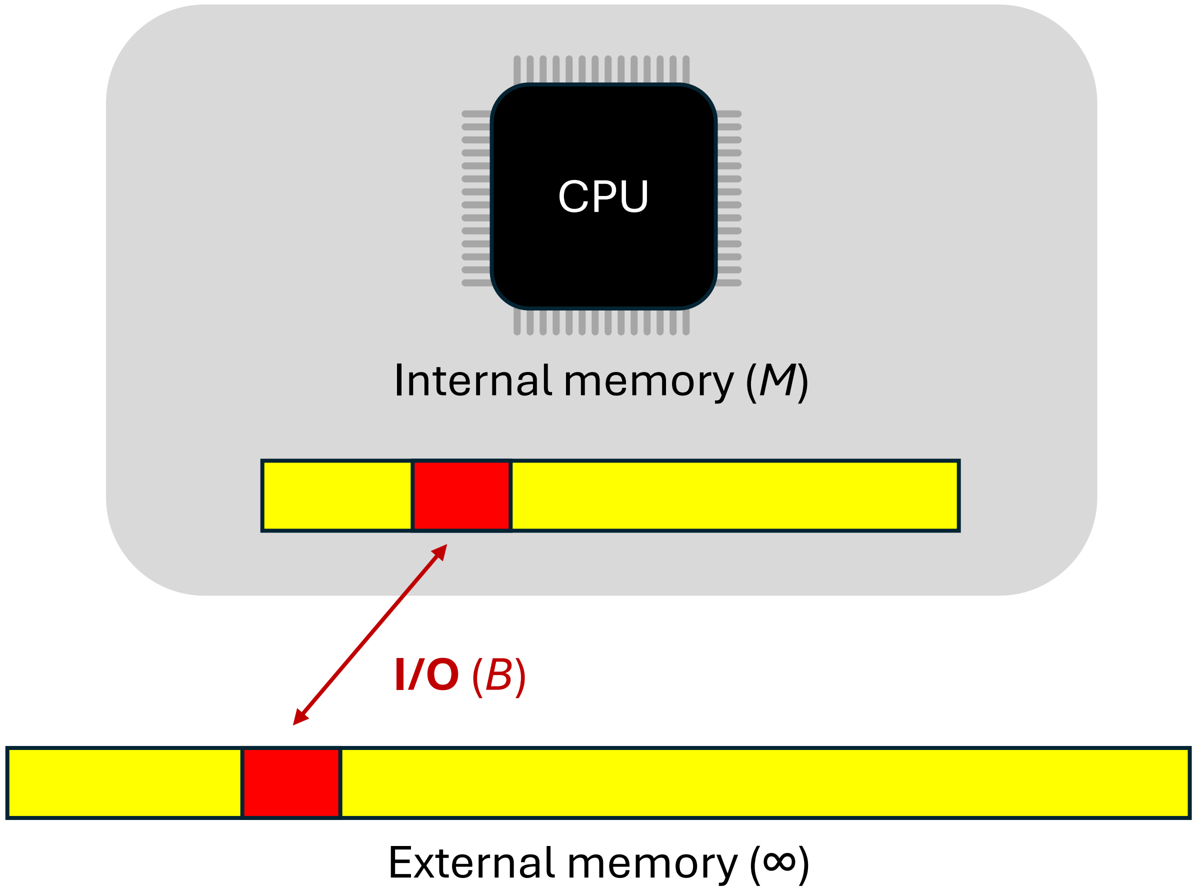


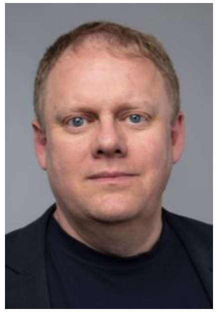

University of Primorska, Koper, Slovenia, December 8, 2025

External-Memory Model

Aggarwal, Vitter [CACM 1988]



Part I
External-Memory Priority Queues with Optimal Insertions



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Michael Goodrich
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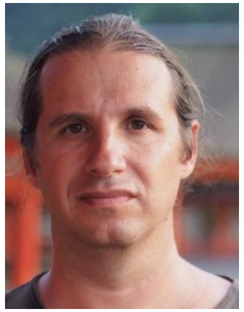

John Iacono
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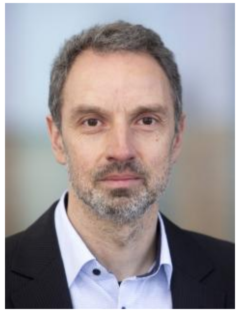
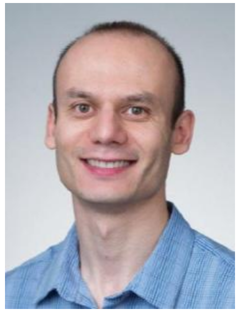


Ulrich Meyer
Goethe University Frankfurt am Main

Victor Pagan
University of Hawai'i at Mānoa



Nodari Sitchinava
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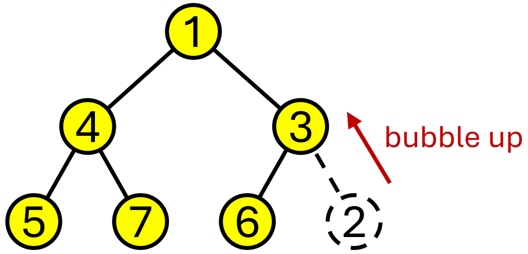
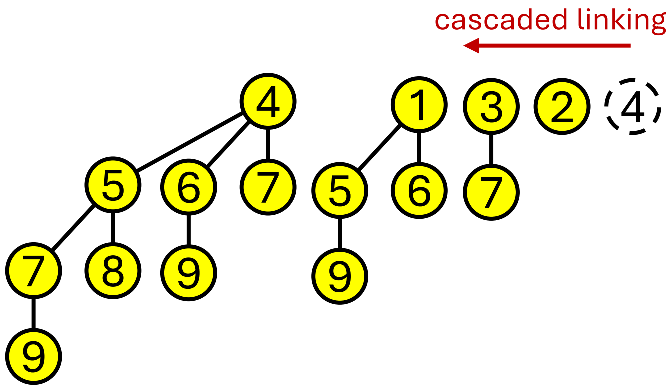
Priority Queues

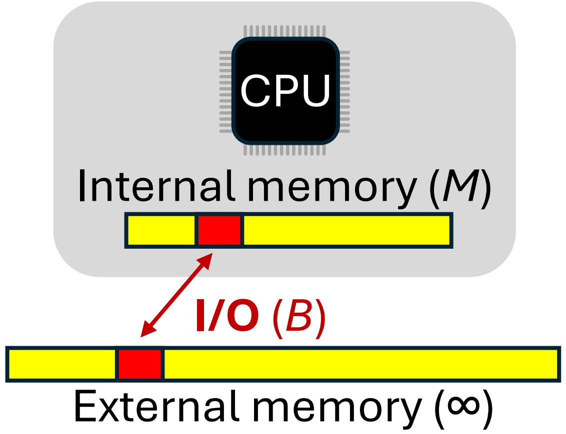
		INSERT(*e*)	DELETEMIN()
Binary heap (Williams 1964)		O(log *N*)	O(log *N*)
Binomial queue (Vuillemin 1978)		$O_A(1)$	O(log *N*)

Can we achieve a similar result in external memory ?

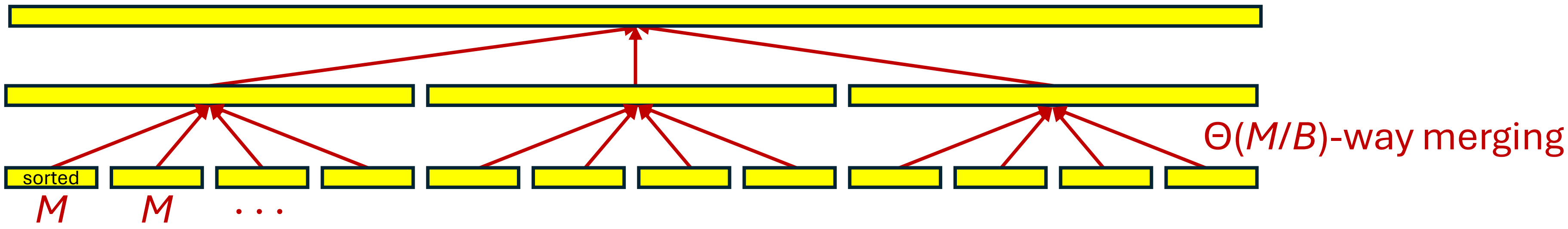
Sorting = *N* × INSERT + *N* × DELETEMIN ⇒ INSERT **or** DELETEMIN Ω(log *N*) comparisons

INSERT ≥ # DELETEMIN

External Memory

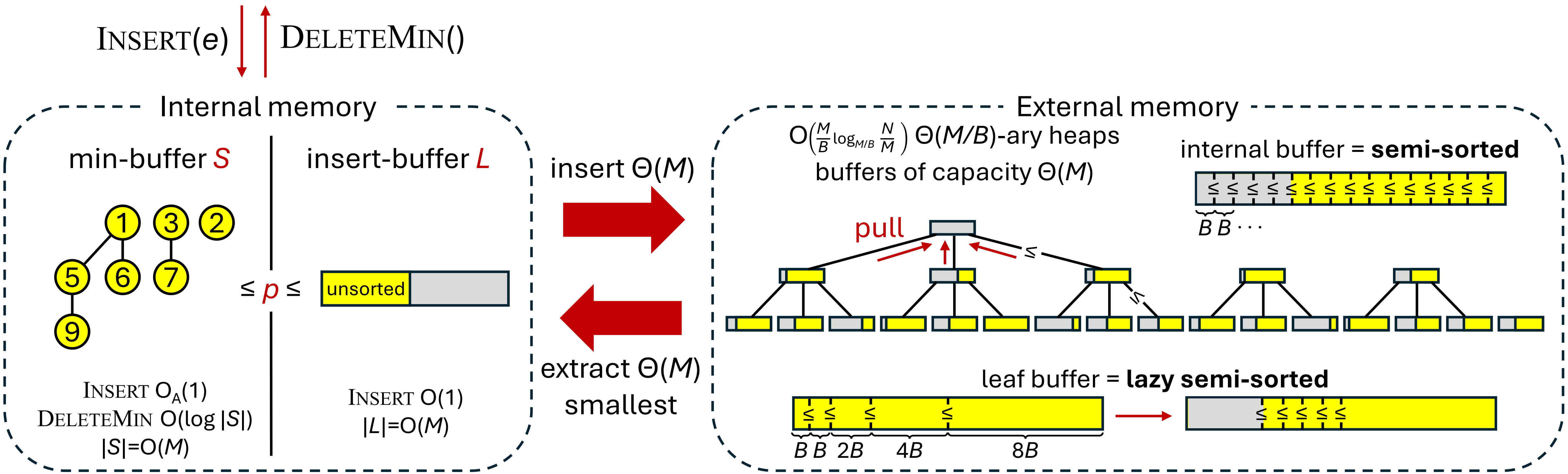


- External-memory model (Aggarwal, Vitter 1988)
- **Sorting** $O\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$ I/Os and $O(N \cdot \log N)$ comparisons



- **Priority queue** $O_A\left(\frac{1}{B}\log_{M/B}\frac{N}{B}\right)$ I/Os and $O_A(\log N)$ comparisons $\begin{cases}\textsc{Insert}\\\textsc{DeleteMin}\end{cases}$
  - **Buffer tree** (Arge 1995)
  - **Θ(*M*/*B*)-ary heap** (Kumar, Schwabe 1996; Fadel, Jakobsen, Katajainen, Teuhola 1997)
  - **Merging sorted lists** (Brengel, Crauser, Ferragina, Meyer 1999; Brodal, Katajainen 1998)
- **This talk** $\textsc{Insert}$ $O_A\left(\frac{1}{B}\right)$ I/Os and $O_A(1)$ comparisons

New External-Memory Priority Queue



Lemma Total $O\left(\frac{N/M}{M/B} + \frac{\#\,\text{deletions}}{M} \cdot \log_{M/B} \frac{N}{B}\right)$ pulls

Lemma Pulling M elements from M/B semi-sorted lists : $O(M/B)$ I/Os and $O\left(M \cdot \log_2 \frac{M}{B}\right)$ comparisons

Theorem Total $O\left(\frac{N}{B} + \frac{\#\,\text{deletions}}{B} \cdot \log_{M/B} \frac{N}{B}\right)$ I/Os and $O(N + \#\,\text{deletions} \cdot \log_2 N)$ comparisons

Summary – Part I

- **New optimal external memory priority queue**

	I/Os	Comparisons
INSERT	$O_A\left(\frac{1}{B}\right)$	$O_A(1)$
DELETEMIN	$O_A\left(\frac{1}{B}\log_{M/B}\frac{N}{B}\right)$	$O_A(\log N)$

- **Open problems**
  - Our data structure is inherently amortized – can it be made **worst-case** ?
  - Structure inherently cache-aware – is there a **cache-oblivious** data structure with matching I/O and comparison bounds ?

Frigo, Leiserson, Prokop, Ramachandran, *Cache-Oblivious Algorithms*, FOCS 1999

Part II
Buffered Partially-Persistent External-Memory Search Trees

— or, partially-persistent B-trees meet B^{ε}-trees

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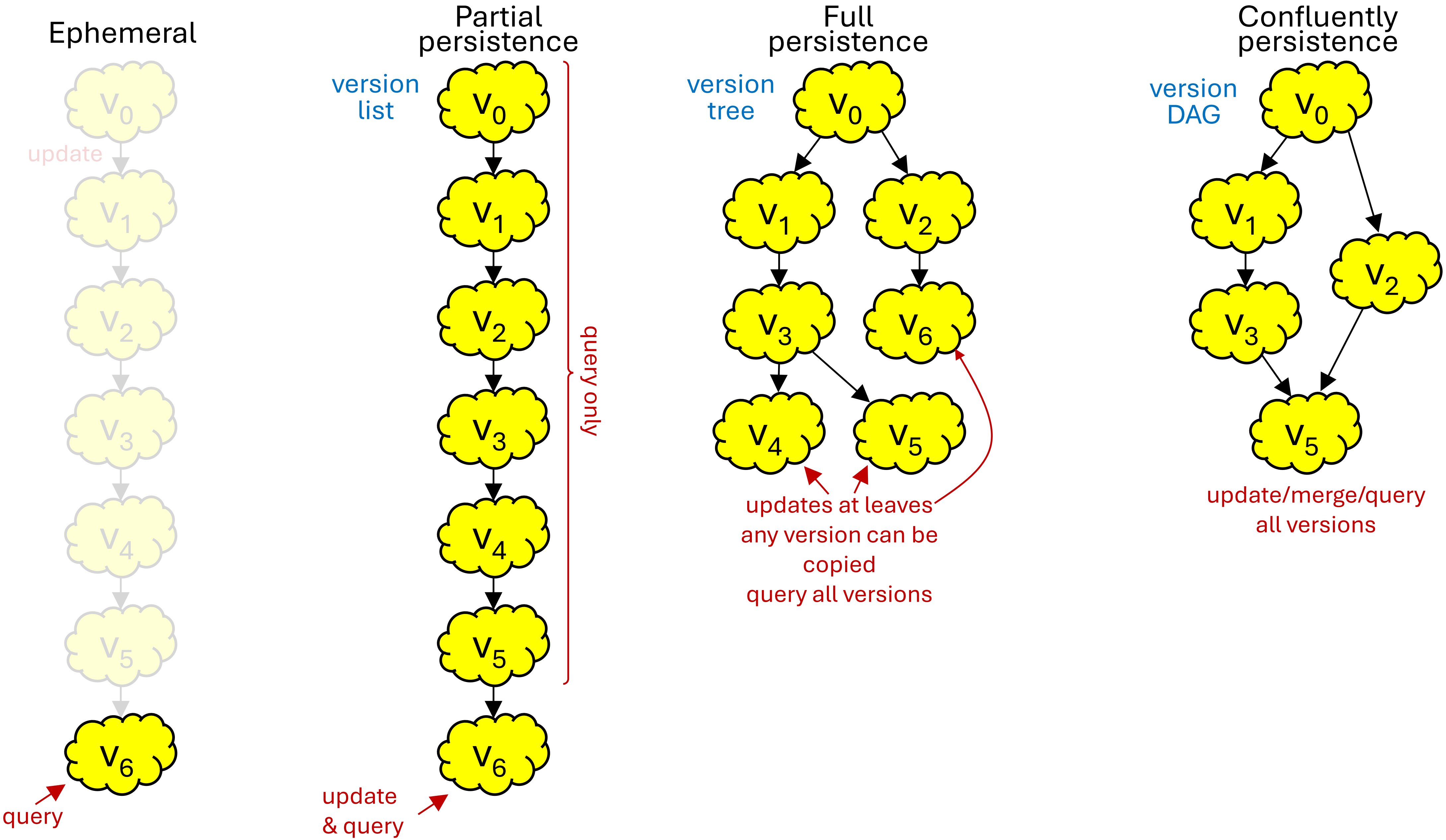
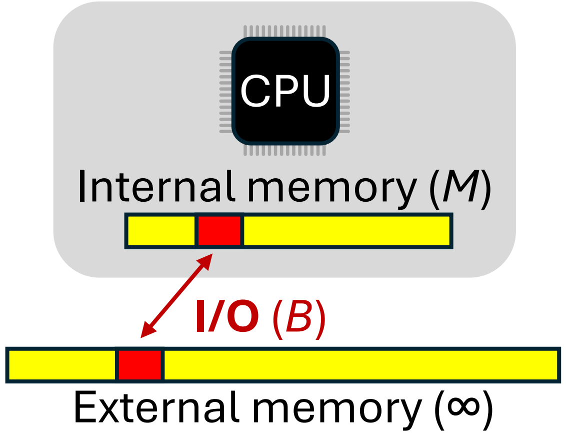
33rd Annual European Symposium on Algorithms (ESA), September 15, 2025, Warsaw, Poland

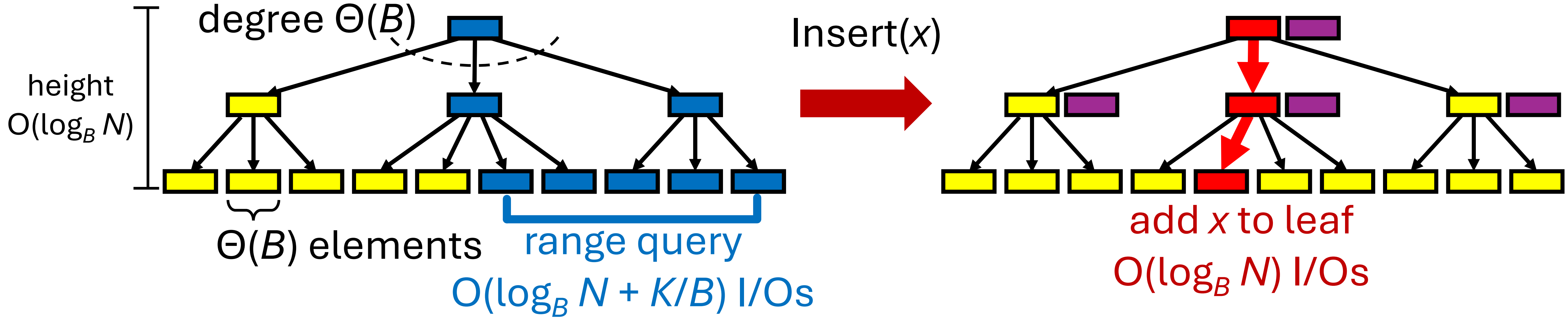
Persistent Data Structures



Buffered partially-persistent external-memory search trees



- Search tree (B-tree, Bayer & McCreight 1972)



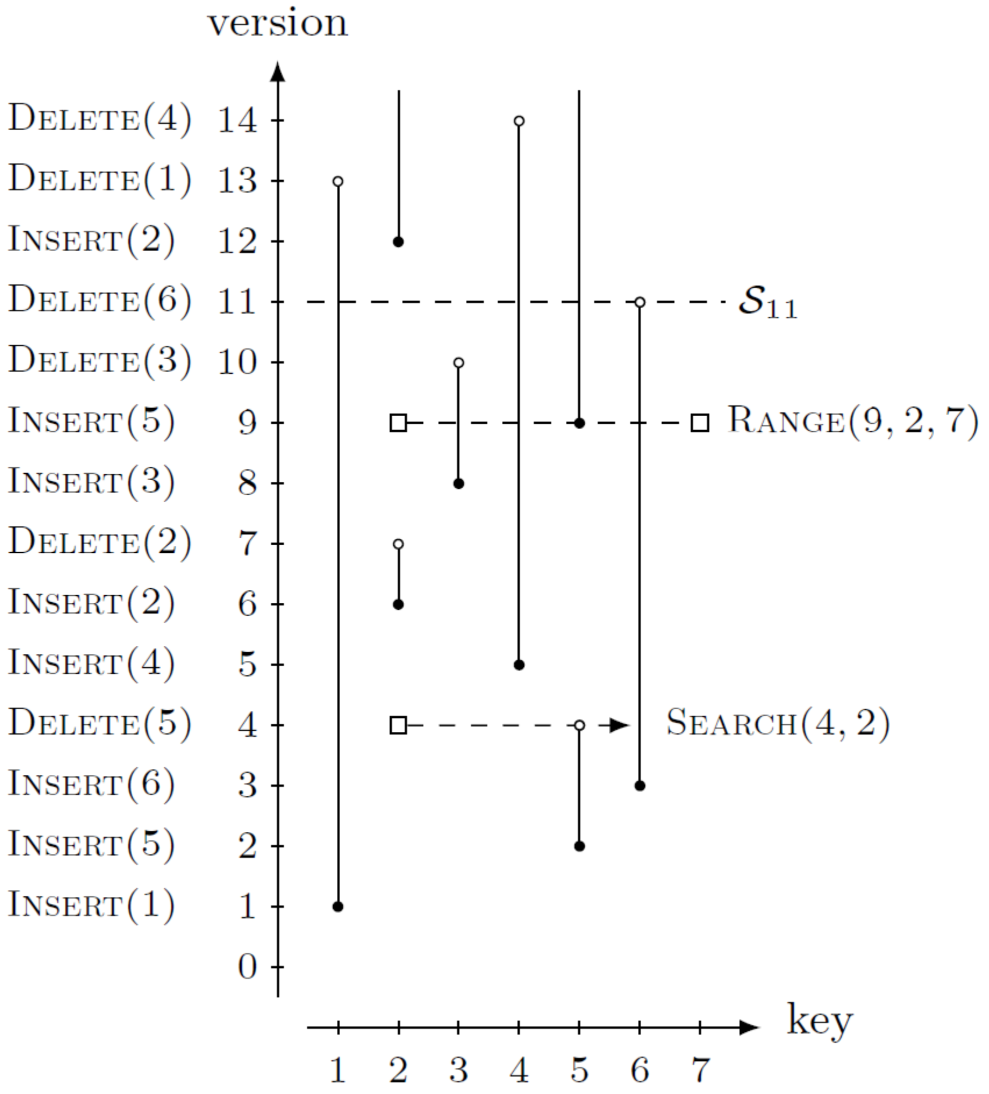
- **Partial persistence** = remember all previous versions
  - **Copy path** from root to updated nodes = space $O(\log_B N)$ blocks per update
  - Associate with each element and pointer a time interval where it is part of the structure (**fat node** and **node copying**)
- ***B*^ε-tree** – reduced degree B^ε, add **buffers** = blocks of delayed updates

I/O results (all linear space)

Ephemeral	**Range query**	**Insert/Delete**	**Assumption**
B-tree Bayer, McCreight 1972	$O\left(\log_B N + \frac{K}{B}\right)$	$O(\log_B N)$	$M \geq 2B$
B^{ε}-tree a) Brodal, Fagerberg 2003 b) Bender, Das, Farach-Colton, Johnson, and Kuszmaul 2020 c) Das, Iacono, and Nekrich 2022	$O\left(\frac{1}{\varepsilon}\log_B N + \frac{K}{B}\right)$	$O\left(\frac{1}{\varepsilon B^{1-\varepsilon}}\log_B N\right)$	a) Amortized, $M \geq 2B$ b) Randomized, $B=\Omega(\log N)$ $M = \Omega(\max\{B^2, \log^{\Theta(1)} N, B^2\})$ c) Worst-case, $M = \Omega(B\log_B N)$
Partial persistence			
Partially-persistent B-tree Becker, Gschwind, Ohler, Seeger, and Widmayer 1996	$O\left(\log_B N + \frac{K}{B}\right)$	$O(\log_B N)$	$M \geq 2B$
This talk	$O\left(\frac{1}{\varepsilon}\log_B N + \frac{K}{B}\right)$	$O\left(\frac{1}{\varepsilon B^{1-\varepsilon}}\log_B N\right)$	a) Amortized, $M \geq 2B$ b) Worst-case, $M = \Omega(B^{1-\varepsilon}\log_2 N)$

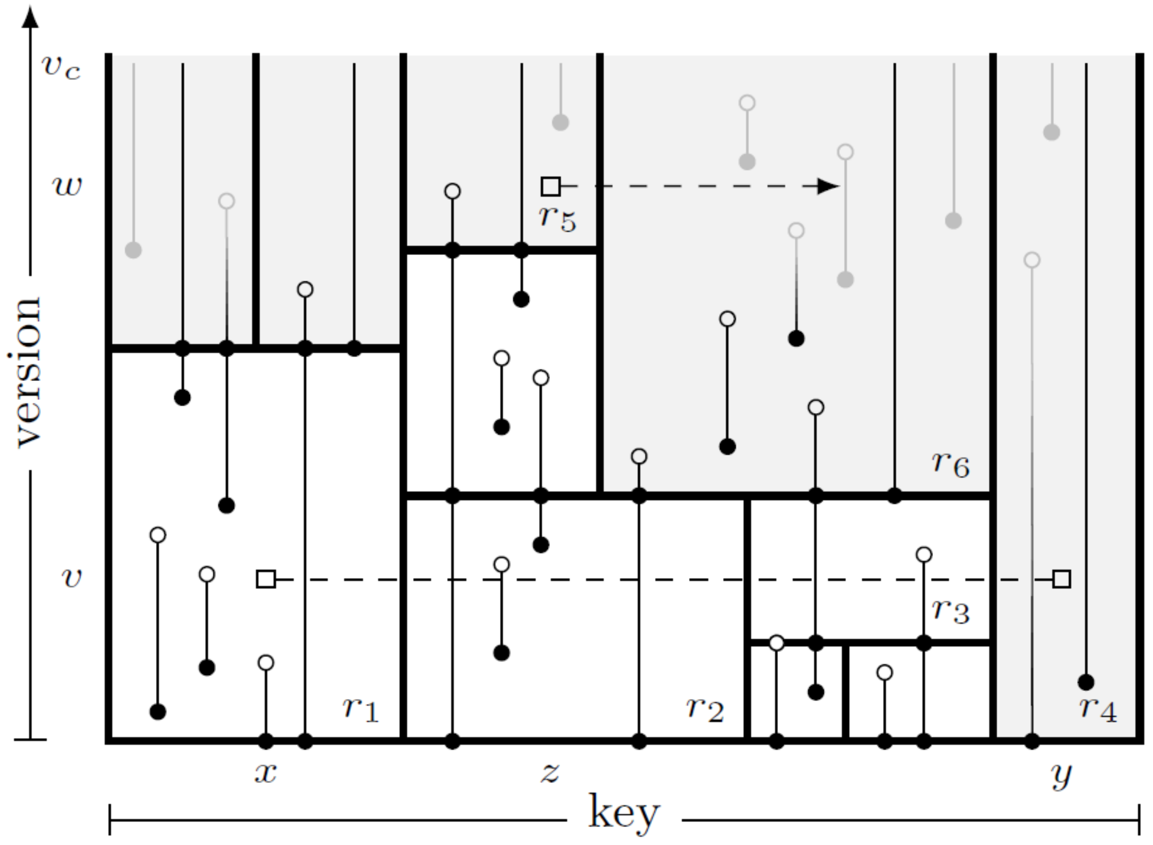
Geometric interpretation

- **Insertions** and **deletions** are endpoints of vertical segments
- **Search** and **range queries** are horizontal ray shooting and segment intersection queries



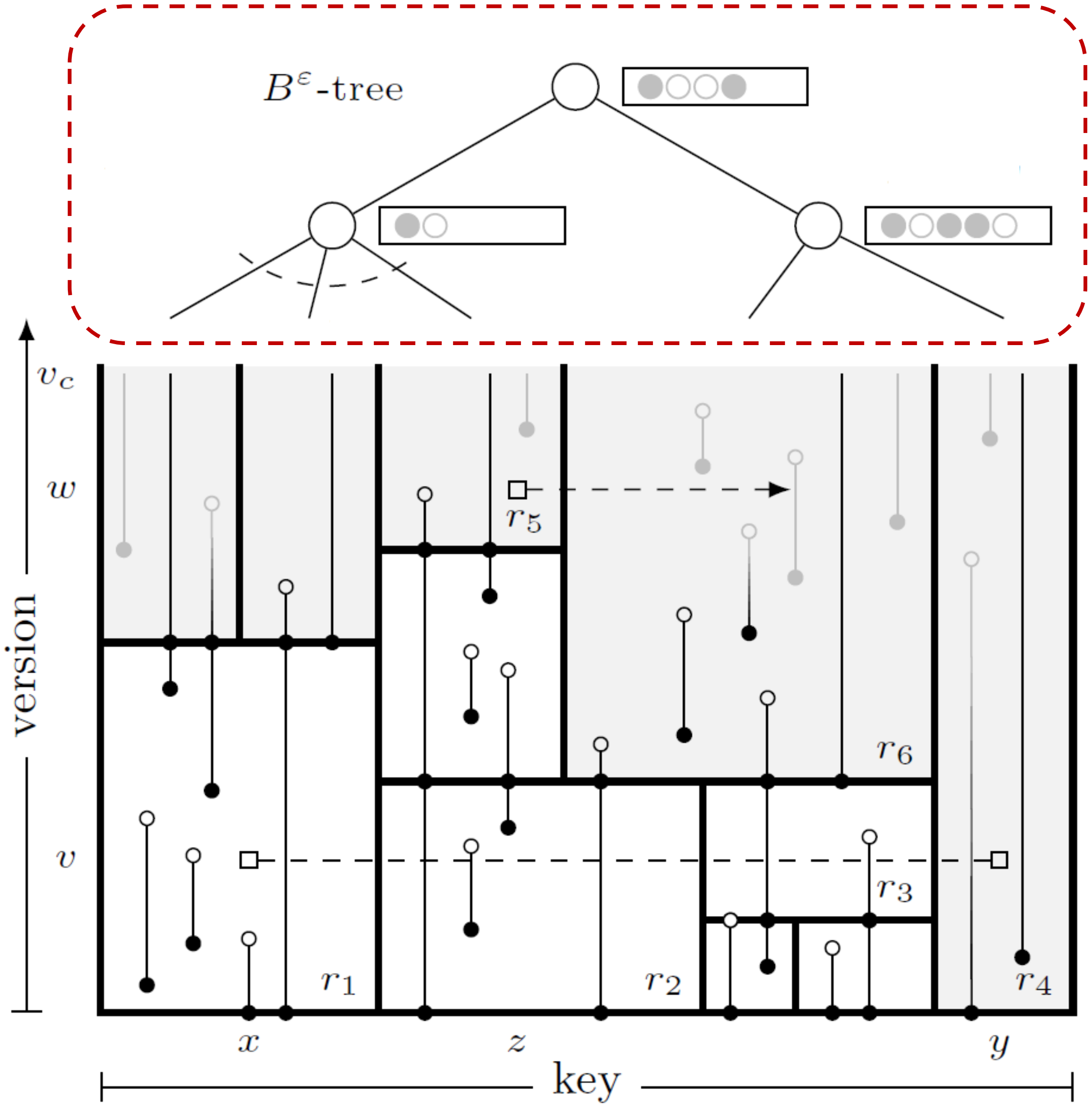
Rectangle partition

- Partition plan into **rectangles**
- Topmost rectangles are **open** (can received more updates)
- **Segments** crossing multiple rectangles are **split**
- Rectangles store **$\Theta(B \cdot \log_B N)$ endpoints** where $\Omega(B \cdot \log_B N)$ segments **span** bottom-to-top
- **Global rebuild** when N doubles/halves
- Segments in a rectangle are **sorted by key**



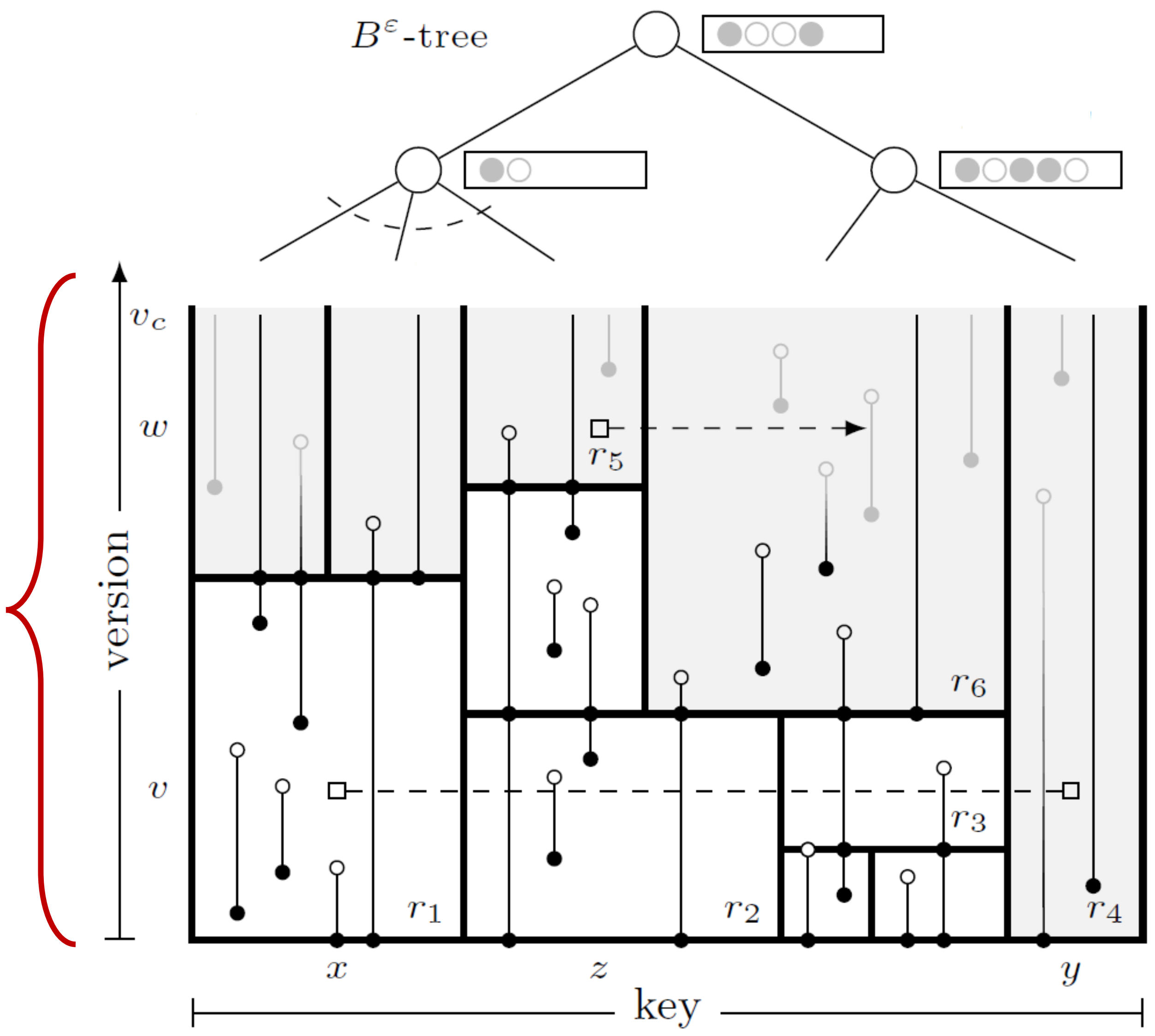
Top structure

- B^{ε}-tree with updates heading for the **open rectangles**
- Degree B^{ε}, buffer capacity B
- **Close** a rectangle when it has received $\Theta(B\log_B N)$ updates (move all buffered updates to it)
- **Join** and **split** open rectangles like in a B-tree (after closing them)

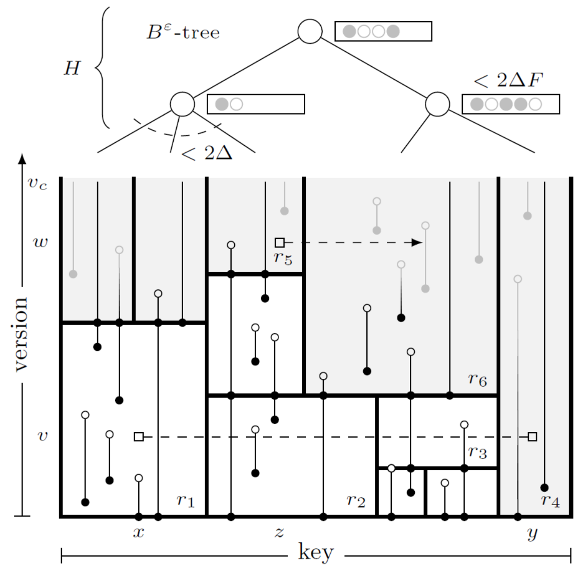


Point location

- **Queries** need to locate relevant rectangles
- For each version have a **B-tree** with the rectangles left-to-right intersection the version
- Make it partially-persistent using simple **path copying** for updates
- A **query** to an **open rectangle** moves all buffered updates to the rectangle



Achieving worst-case I/O bounds



- Buffer overflow = incremental **flush a single path**, pushing $B^{1\text{-}\varepsilon}$ updates to the child with most updates (ensures never more than $B^{1\text{-}\varepsilon} \cdot \log_2 B$ updates in a buffer heading for a single child)
- Incrementally **closing** an open rectangle, collect all buffered updates in internal memory, requires $\mathbf{M = \Omega(B^{1\text{-}\varepsilon} \cdot \log_2 N)}$
- **Incremental global rebuild** for changing N
- **Never merge internal nodes** in top structure (avoids flushing buffers, incremental global rebuilding bounds height)

Summary – Part II

- Linear space buffered partially-persistent B-trees, with I/O bounds

Range queries	**Insert/Delete**	
$O\left(\frac{1}{\varepsilon}\log_B N + \frac{K}{B}\right)$	$O\left(\frac{1}{\varepsilon B^{1-\varepsilon}}\log_B N\right)$	a) Amortized, $M \geq 2B$ b) Worst-case, $M = \Omega(B^{1-\varepsilon}\log_2 N)$

- **Open problems**
  - Worst-case for $M \geq 2B$
  - Let fully-persistent B-trees meet B^{ε}-trees
    (Brodal, Rysgaard, Svenning, STOC 2023, $O(\log_B N)$ I/Os queries & updates)

Thanks

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- Brodal, Goodrich, Iacono, Lo, Meyer, Pagan, Sitchinava, Svenning, *External-Memory Priority Queues with Optimal Insertions*, ESA 2025
- Brodal, Rysgaard, Svenning, *Buffered Partially-Persistent External-Memory Search Trees,* ESA 2025