Cache Oblivious Searching and Sorting

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BRICS

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Joint work with Rolf Fagerberg (Århus), Riko Jacob (Munich), Michael A. Bender, Dongdong Ge, Simai He, Haodong Hu (SUNY Stony Brook), John Iacono (Polytechnic, NY), Alejandro López-Ortiz (Waterloo)

IT University of Copenhagen, April 30, 2003

Outline of Talk

- Hardware
- Computational models
  - RAM model (Random Access Machine)
  - IO model
  - Cache oblivious model
- Binary searching and dictionaries
- Sorting
- Priority queues
- Concluding remarks

Hardware

Hardware

- Dell Latitude L400, 700Mhz (January 2002)
- Mobile Intel Pentium III
- Primary 16 Kb instruction cache and 16 Kb write-back data cache
- 256 Kb Level 2 Cache
- 256 Mb SDRAM
- 10 Gb disk

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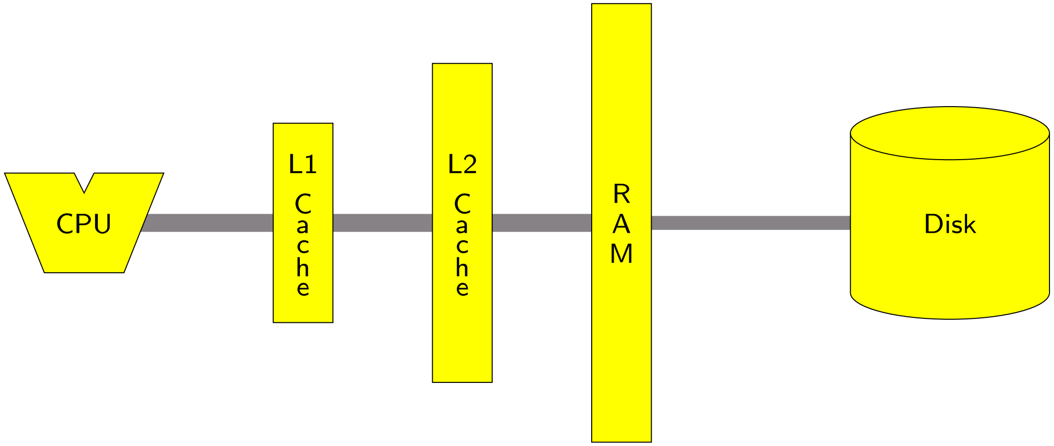
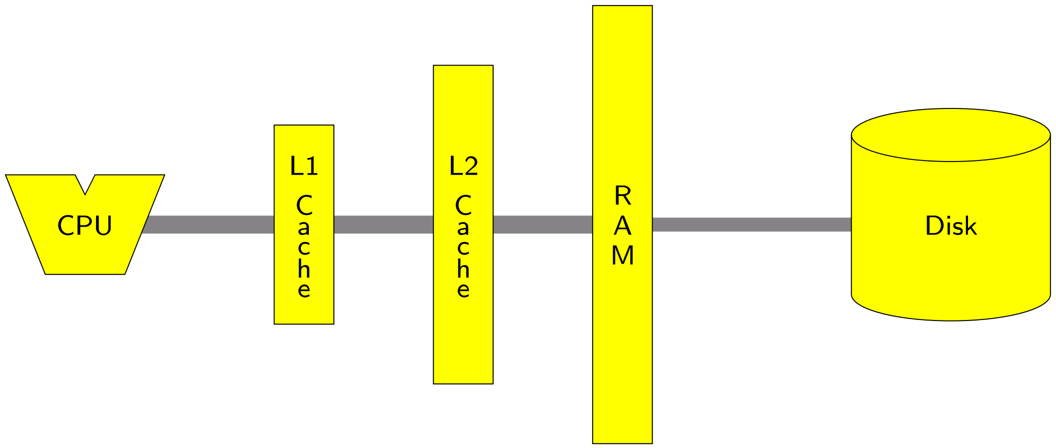


Memory hierarchy

Trends in Implementation Technology



	L1 Cache	L2 Cache	Virtual memory
Block size	4 – 32 bytes	32 – 256 bytes	4 – 16 KB
Hit time (cycles)	1 – 2	6 – 15	10 – 100
Miss penalty (cycles)	8 – 66	30 – 200	700.000 – 6.000.000
Size	1 – 128 KB	256 KB – 16 MB	16 – 8192 MB

Source: *Computer Architecture – A Quantitative Approach*, Hennessy & Patterson, 2nd. Ed. 1996

The Unknown Machine

Algorithm

↓

C program

↓ gcc

Object code

↓ linux

Execution

Can be executed on machines with a specific class of CPUs

Algorithm

↓

Java program

↓ javac

Java bytecode

↓ java

Interpretation

Can be executed on any machine with a Java interpreter

The Unknown Machine

Algorithm	Algorithm
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C program	Java program
↓ gcc	↓ javac
Object code	Java bytecode
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Execution	Interpretation

Can be executed on machines with a specific class of CPUs

Can be executed on any machine with a Java interpreter

Goal Develop algorithms that are optimized w.r.t. memory hierarchies without knowing the parameters

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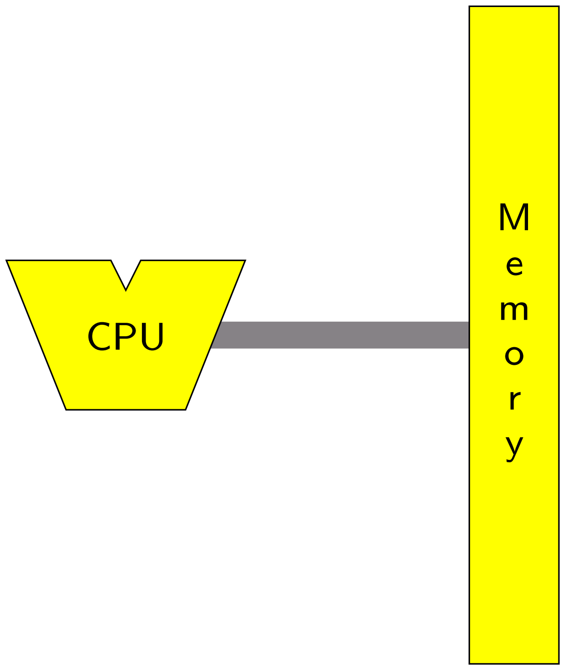
RAM Model
(Random Access Machine)



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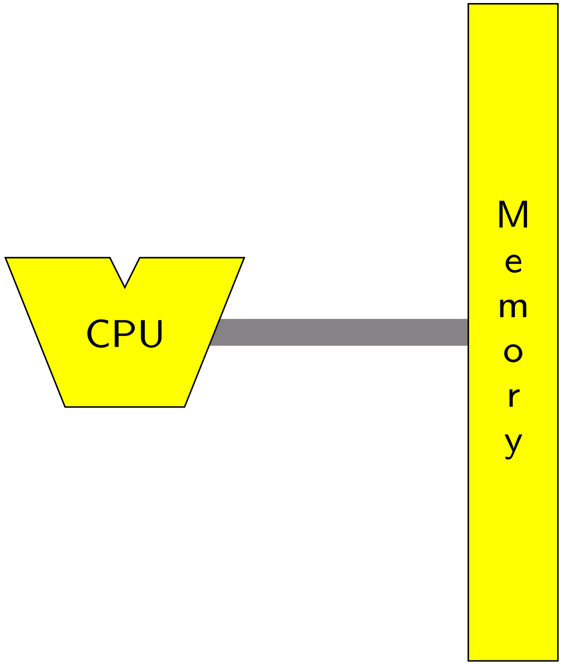

$+ - * / \vee \wedge \neq$ ... $O(1)$ time

Memory access $O(1)$ time

RAM Model
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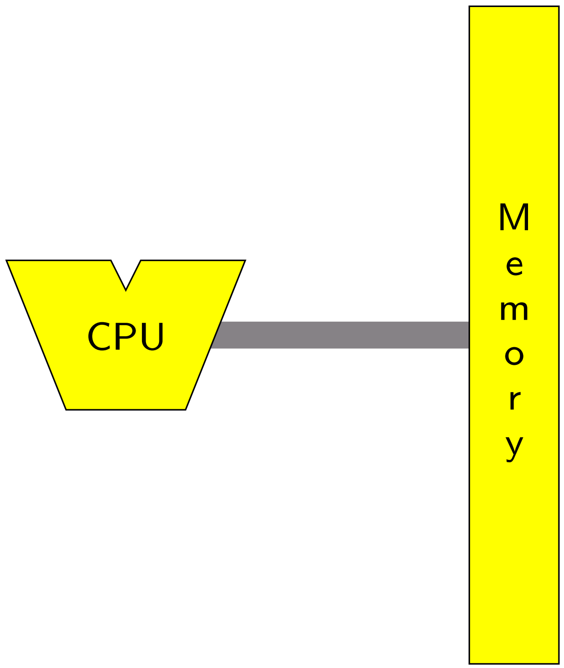


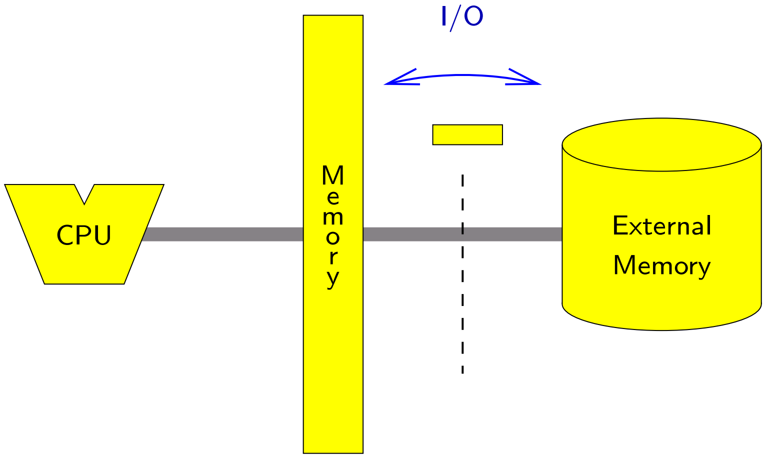
$+ - * \, / \, \vee \wedge \neq$ ... $O(1)$ time

Memory access ~~$O(1)$ time~~

Ignores the presence of memory hierarchies

I/O Model

Aggarwal and Vitter 1988



N = problem size

M = memory size

B = I/O block size

- One I/O moves B consecutive records from/to disk
- Cost: number of I/Os

I/O Model

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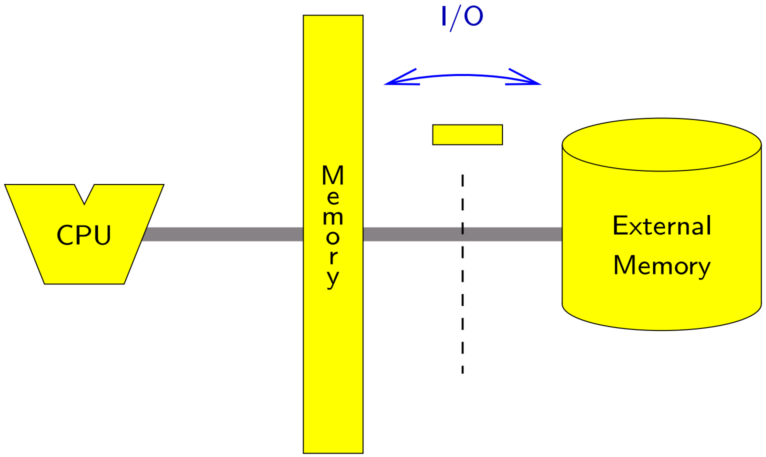

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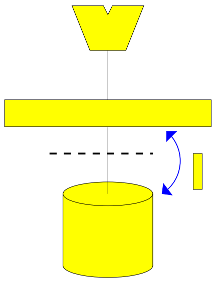
Scan$(N) = O(N/B)$ Sort$(N) = O\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$

Cache Oblivious Model

Frigo, Leiserson, Prokop, Ramachandran 1999

- Program in the RAM model
- Analyze in the I/O model (for arbitrary B and M)
- Optimal off-line cache replacement strategy



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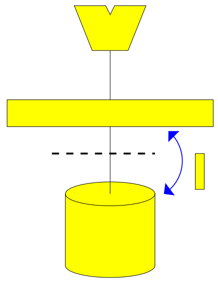
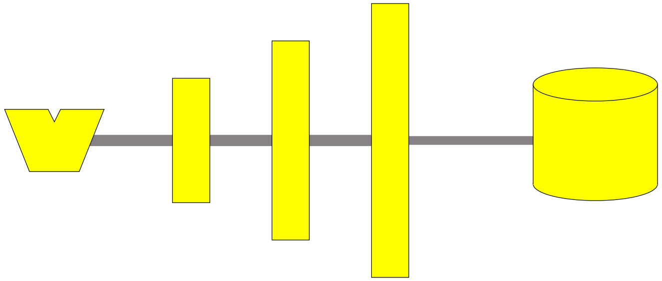


Advantages

- Optimal on arbitrary level $\Rightarrow$ optimal on all levels
- B and M not hard-wired into algorithm



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RAM model : Binary Searching

- Sorted array of n elements
  = static dictionary
- Binary search requires $O(\log_2 N)$ time

2	3	5	6	10	12	15	16	18	19	23	24	28	29	31

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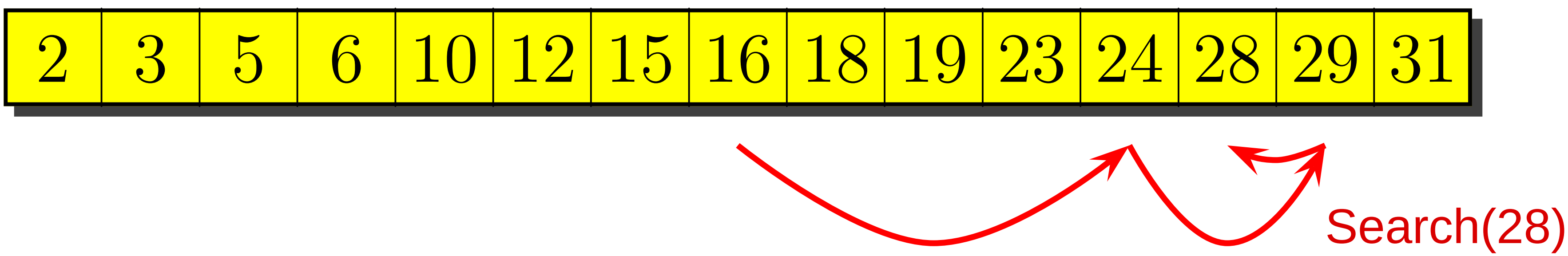

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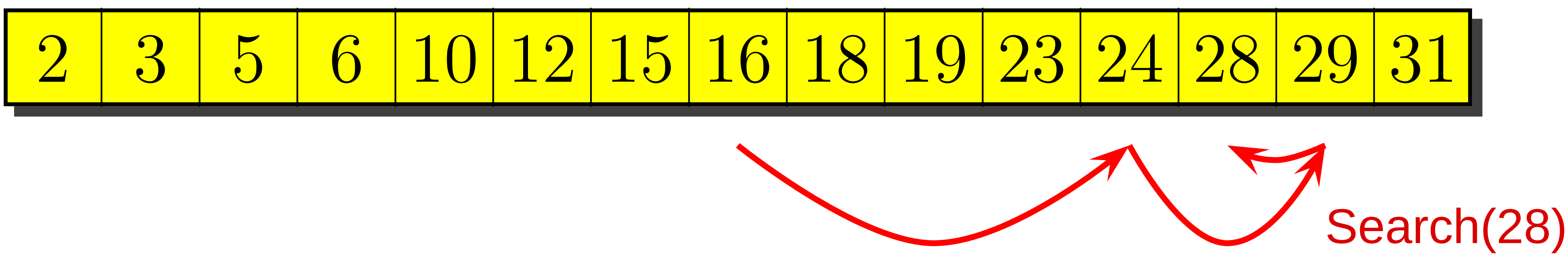
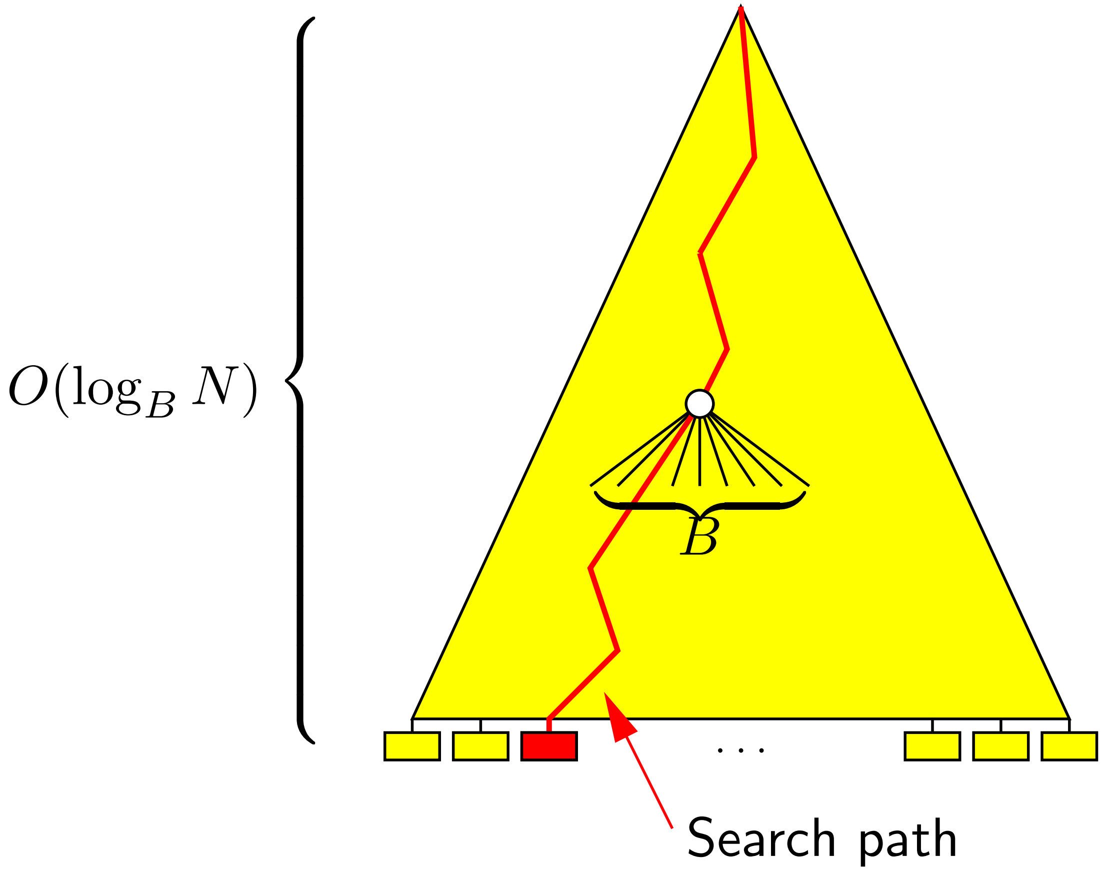


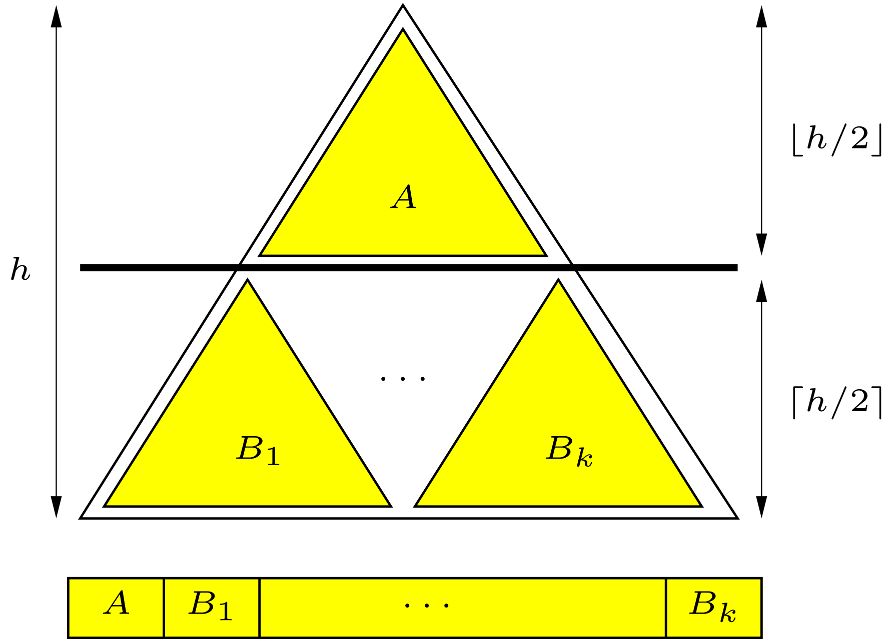
A binary search is cache oblivious and uses $O\left(\log_2 \frac{N}{B}\right)$ I/Os

IO model : B-trees



- Each node stores B keys and has degree $B + 1$
- Searches use $O(\log_B N)$ I/Os

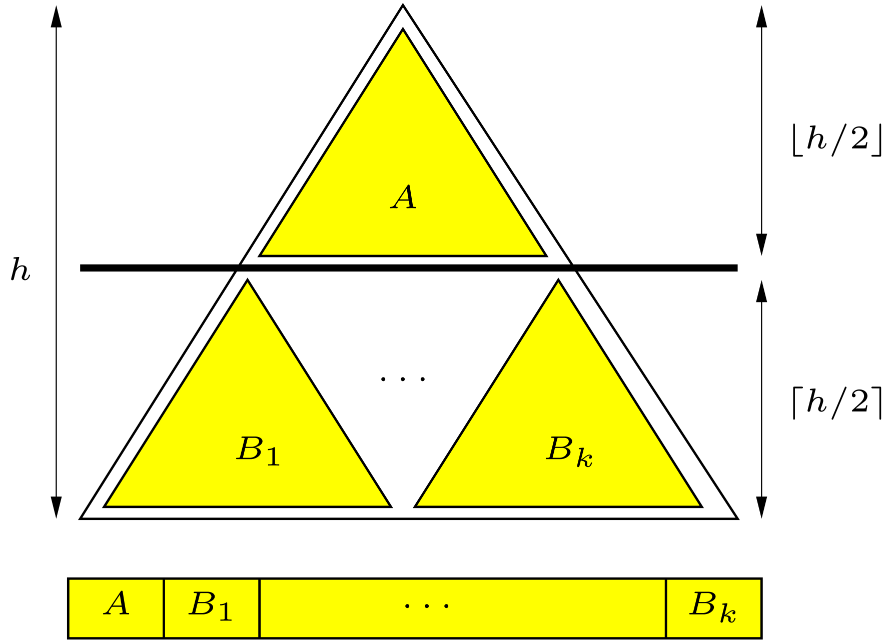
Static Cache Oblivious Dictionary

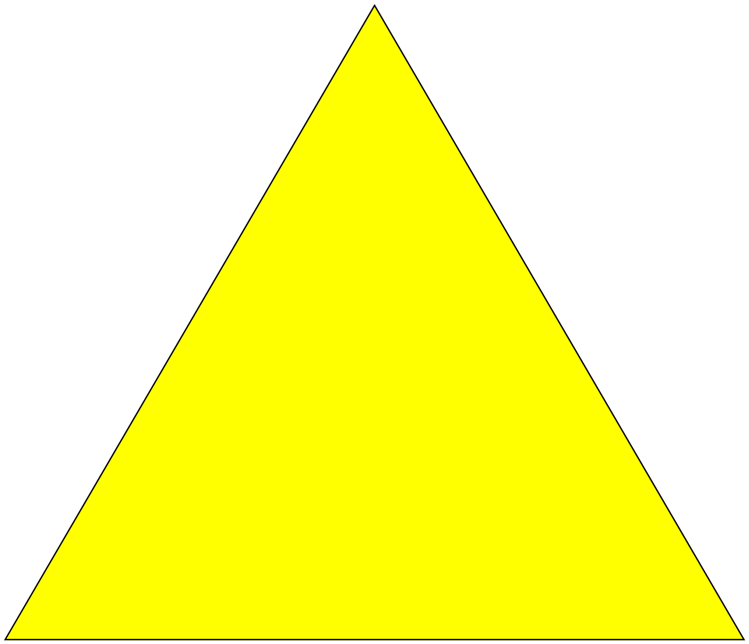


Recursive layout of binary tree

$\equiv$ van Emde Boas layout

Static Cache Oblivious Dictionary





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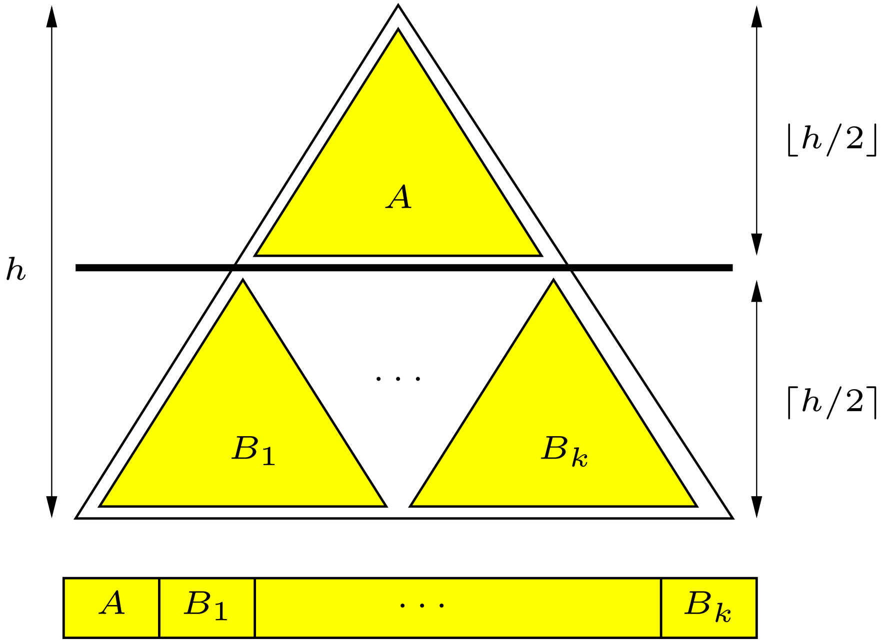

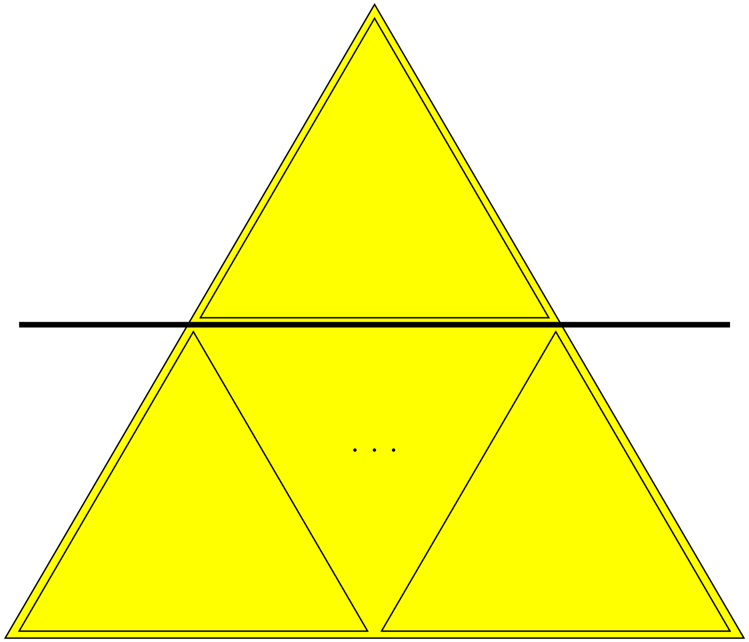

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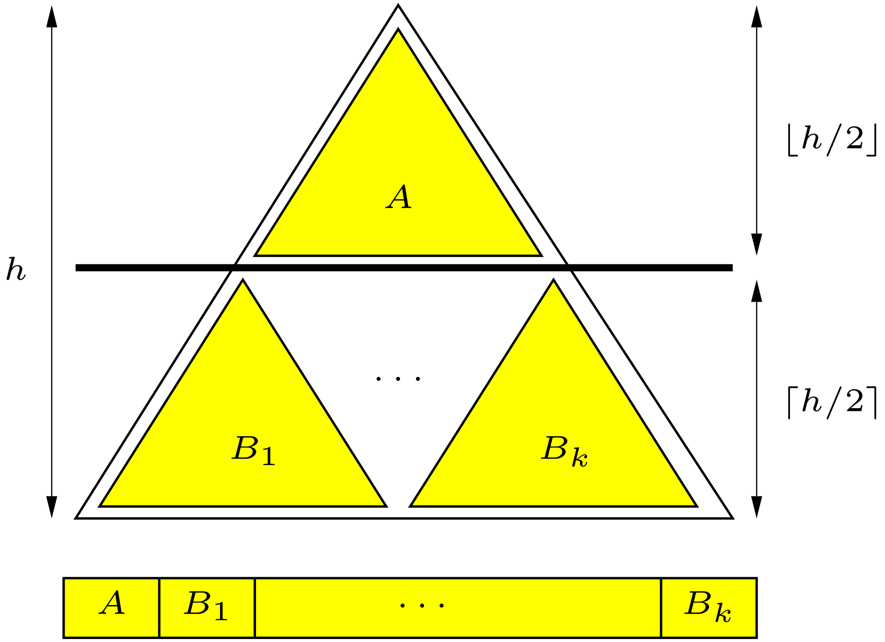

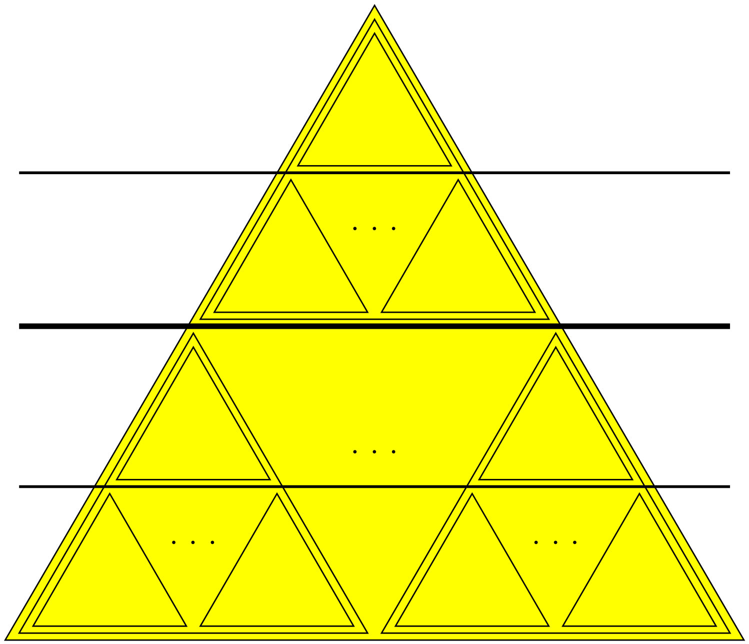

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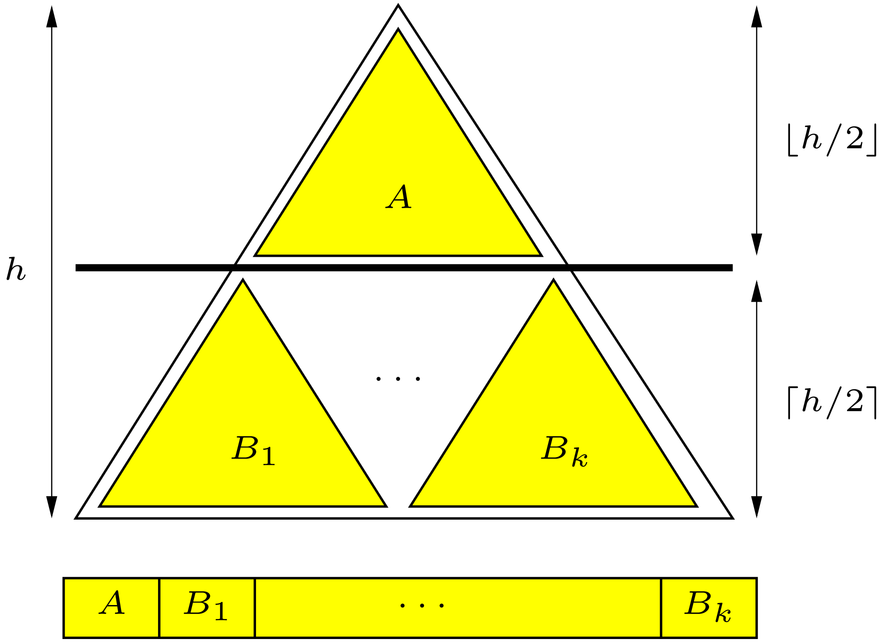

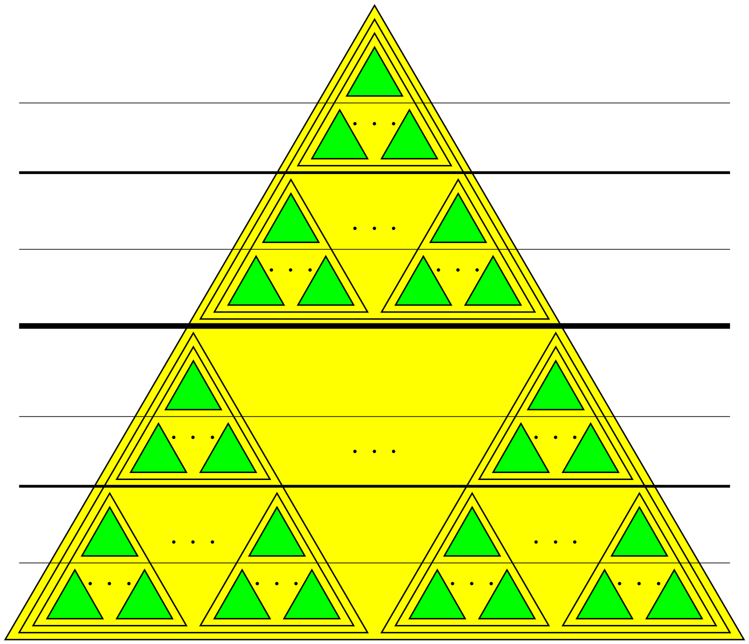

Recursive layout of binary tree

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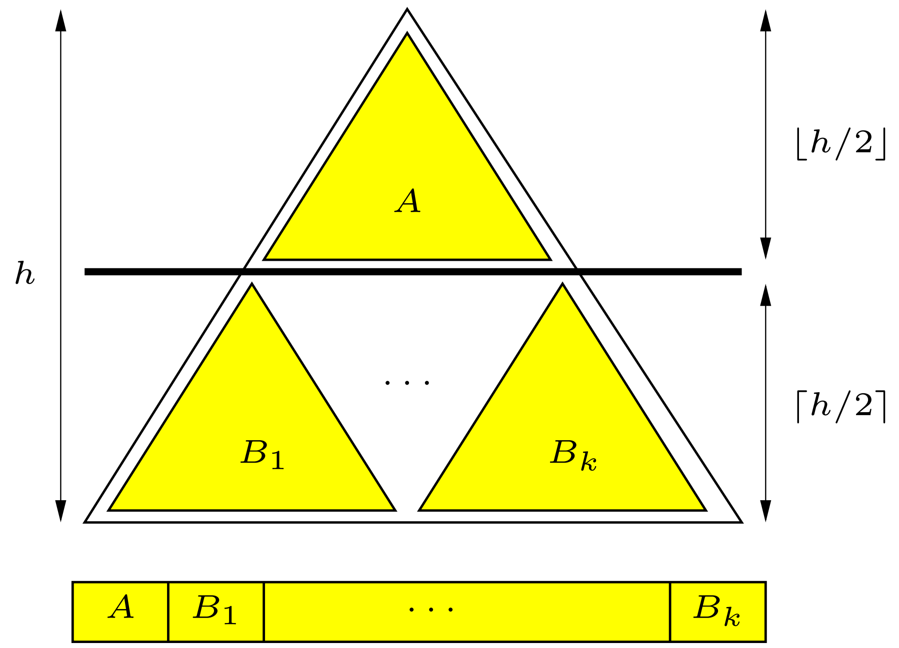

Recursive layout of binary tree
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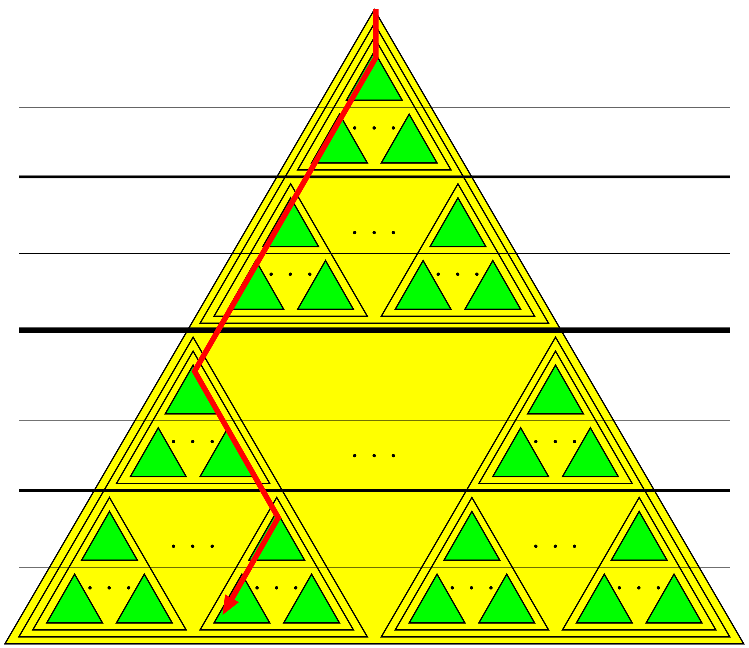
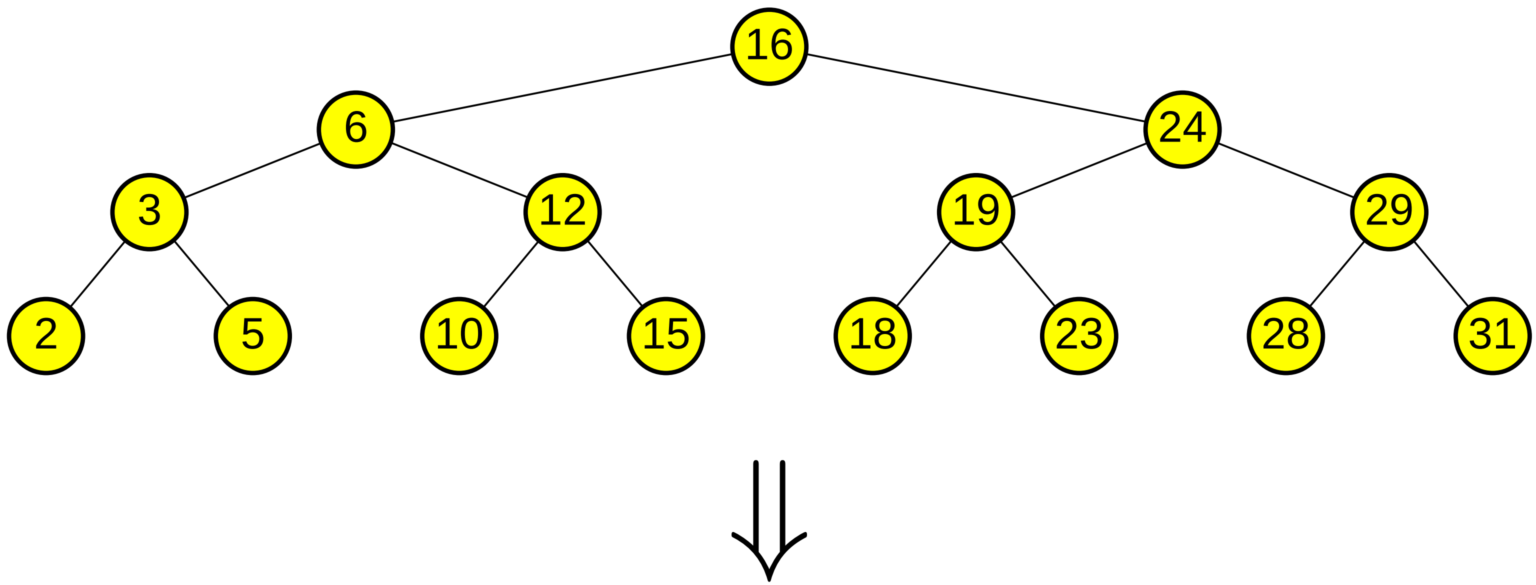


Searches use $O(\log_B N)$ I/Os

- Each green tree has height between $(\log_2 B)/2$ and $\log_2 B$
- Searches visit between $\log_B N$ and $2\log_B N$ green trees, i.e. perform at most $4\log_B N$ I/Os (misalignment)

Example : Recursive Layout



16	6	24	3	2	5	12	10	15	19	17	23	29	28	31

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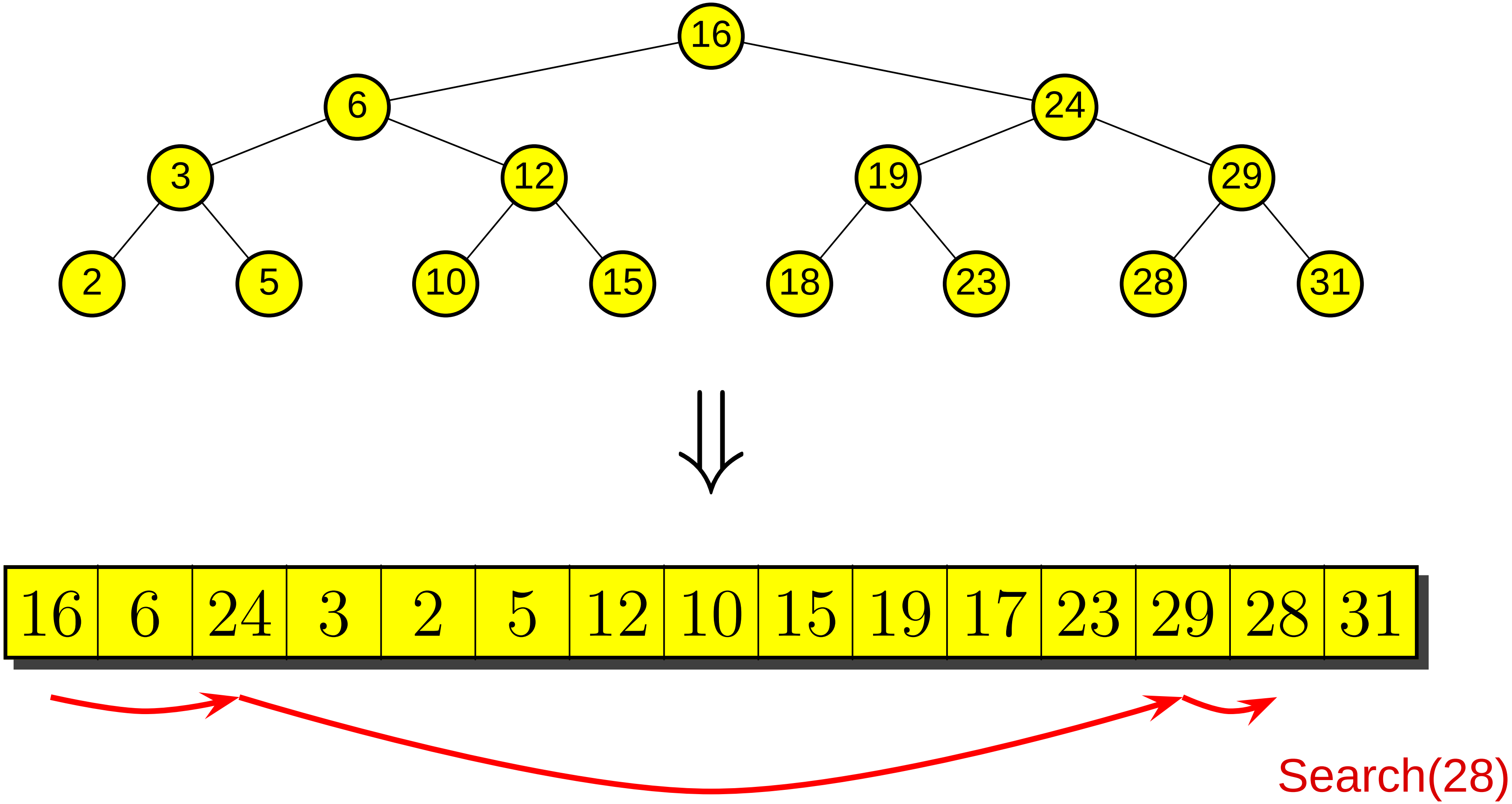


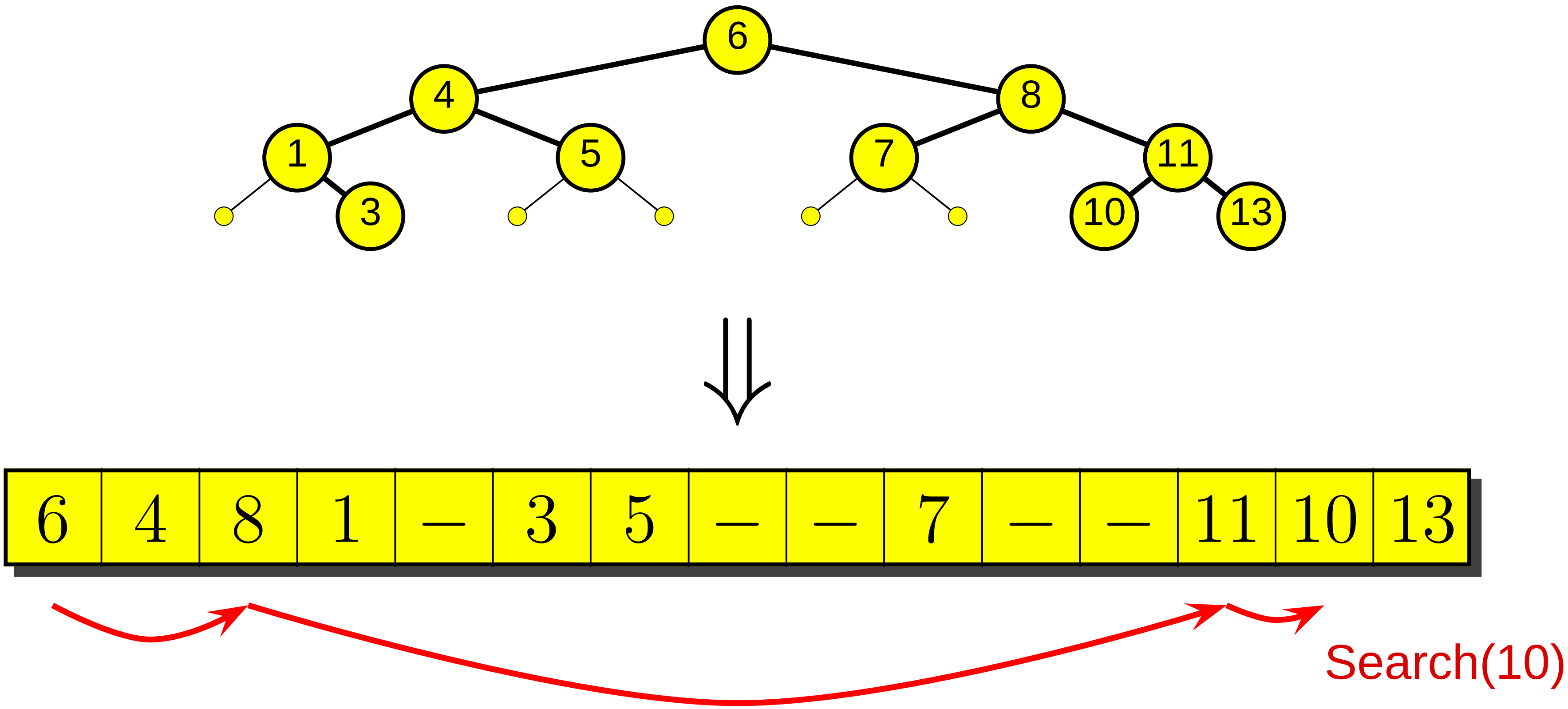
Dynamic Dictionaries

RAM model :	Balanced binary search trees, e.g. AVL-trees and red-black trees
IO model :	B-trees
Cache oblivious model :	?

Dynamic Cache Oblivious Dictionaries

Brodal and Fagerberg 2002

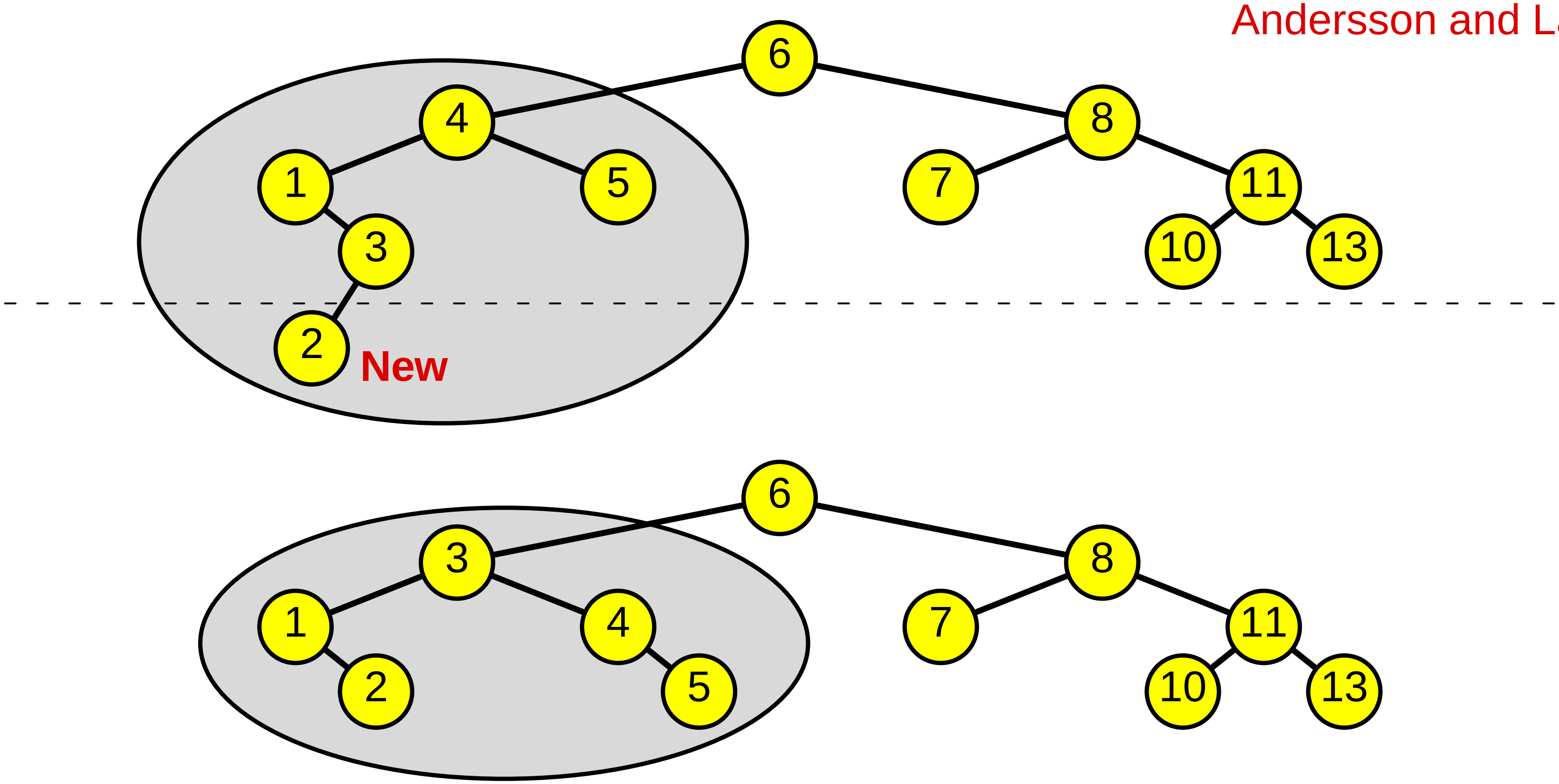
- Embed a dynamic height $\log_2 N + O(1)$ tree in a complete tree
- Static van Emde Boas layout



Dynamic Binary Trees of Small Height

Itai, Konheim and Rodeh 1981

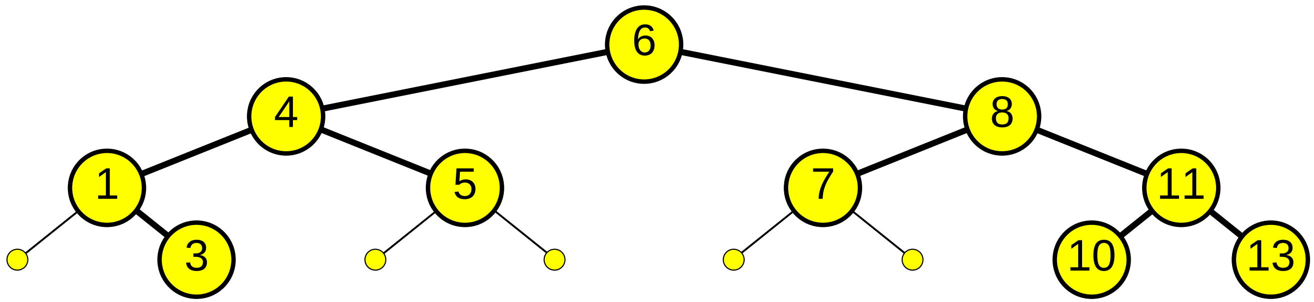
Andersson and Lai 1990



- If an insertion causes non-small height then rebuild subtree at nearest ancestor with suffi cient few descendents
- Insertions require amortized $O(\log^2 N)$ time

Dynamic Cache Oblivious Dictionaries

Brodal and Fagerberg 2002



Search	$O(\log_B N)$
Updates	$O\left(\log_B N + \frac{\log^2 N}{B}\right)$

- Updates can be improved to $O(\log_B N)$ I/Os by buckets of size $\Theta(\log_2 N)$ and one level of indirection

Lower bounds

(Comparison) RAM model : $\log n$ comparisons
(decision tree argument)

IO model : $\log_{B+1} N$ I/Os
(reduction to RAM model)

Cache oblivious model : $\log_{B+1} N$ I/Os
(follows from IO model)

$\log_2 e \cdot \log_B N \approx 1.443 \log_B N$ I/Os

Bender et al. 2003

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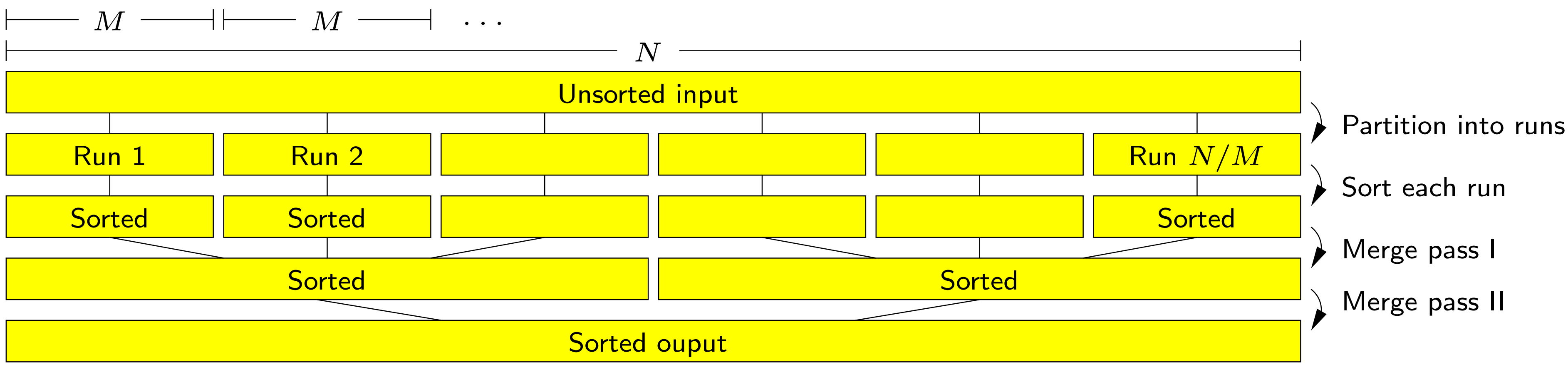
Sorting

RAM model : Binary MergeSort takes $O(N \log_2 N)$ time

IO model : $\Theta\left(\frac{M}{B}\right)$-way MergeSort achieves optimal $O(\text{Sort}(N)) = O\left(\frac{N}{B} \log_{M/B} \frac{N}{B}\right)$ I/Os

Aggarwal and Vitter 1988

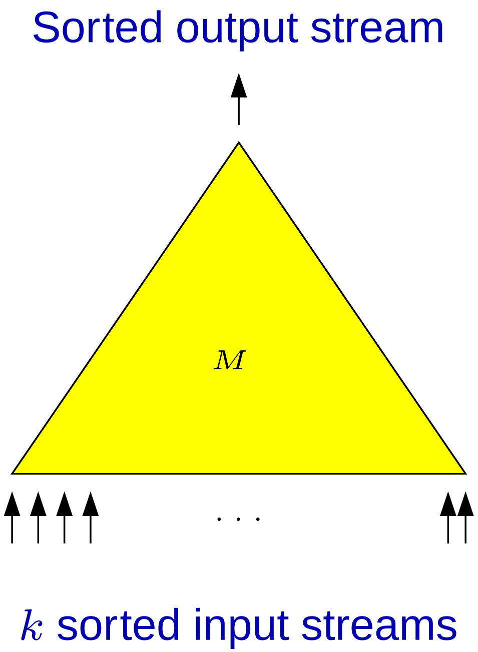


Cache oblivious : FunnelSort achieves $O(\text{Sort}(N))$ I/Os

Frigo, Leiserson, Prokop and Ramachandran 1999
Brodal and Fagerberg 2002

k-merger

Frigo et al., FOCS'99



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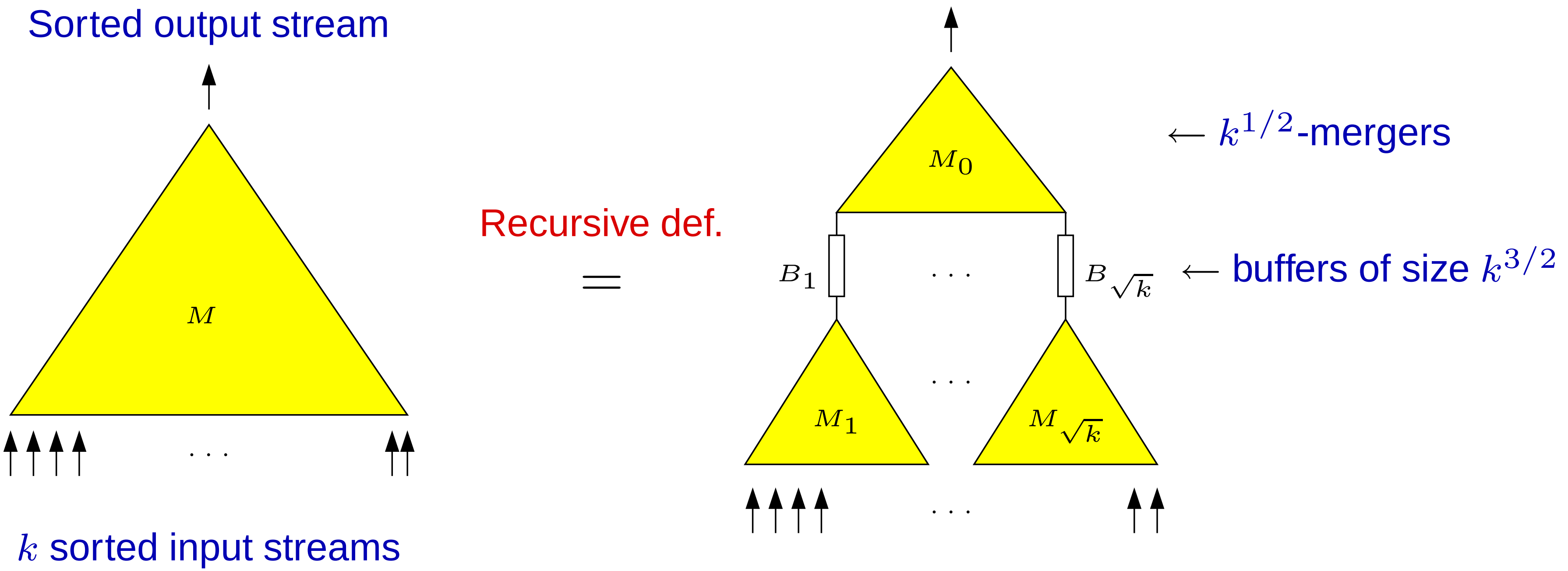

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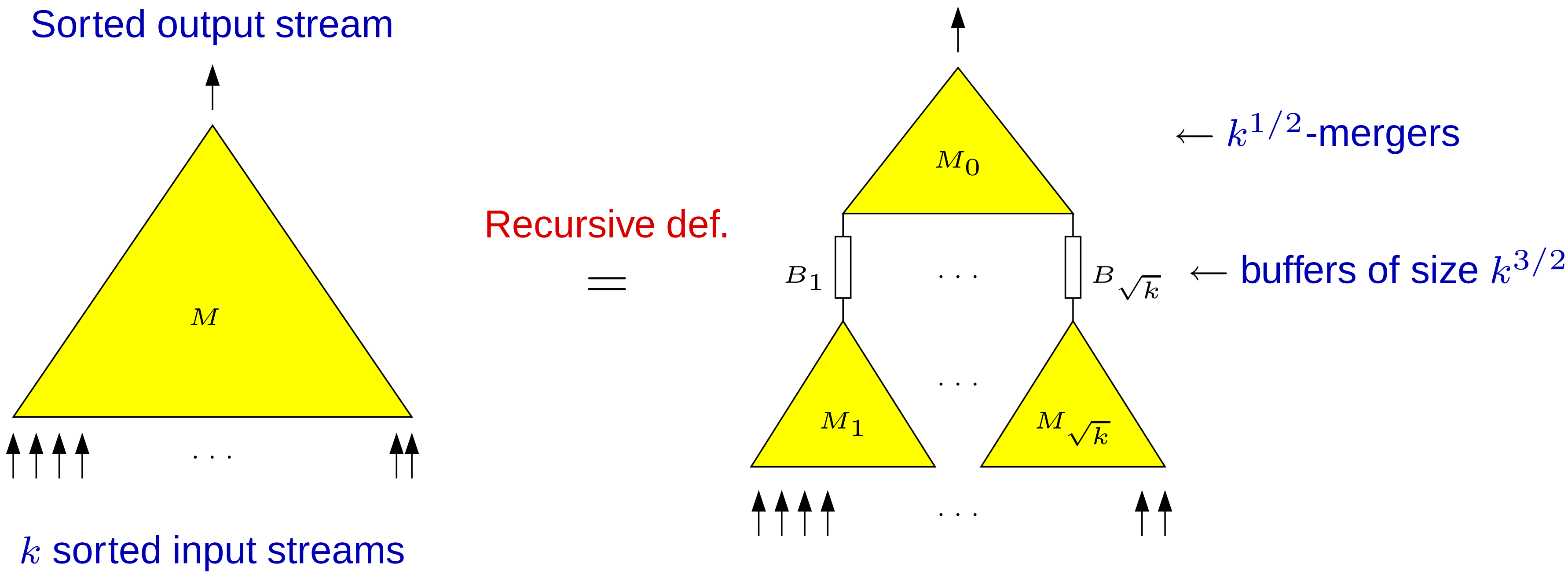
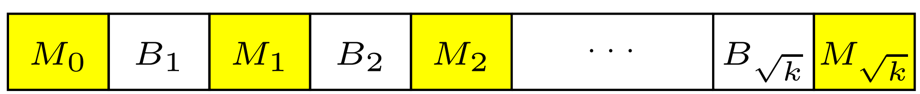
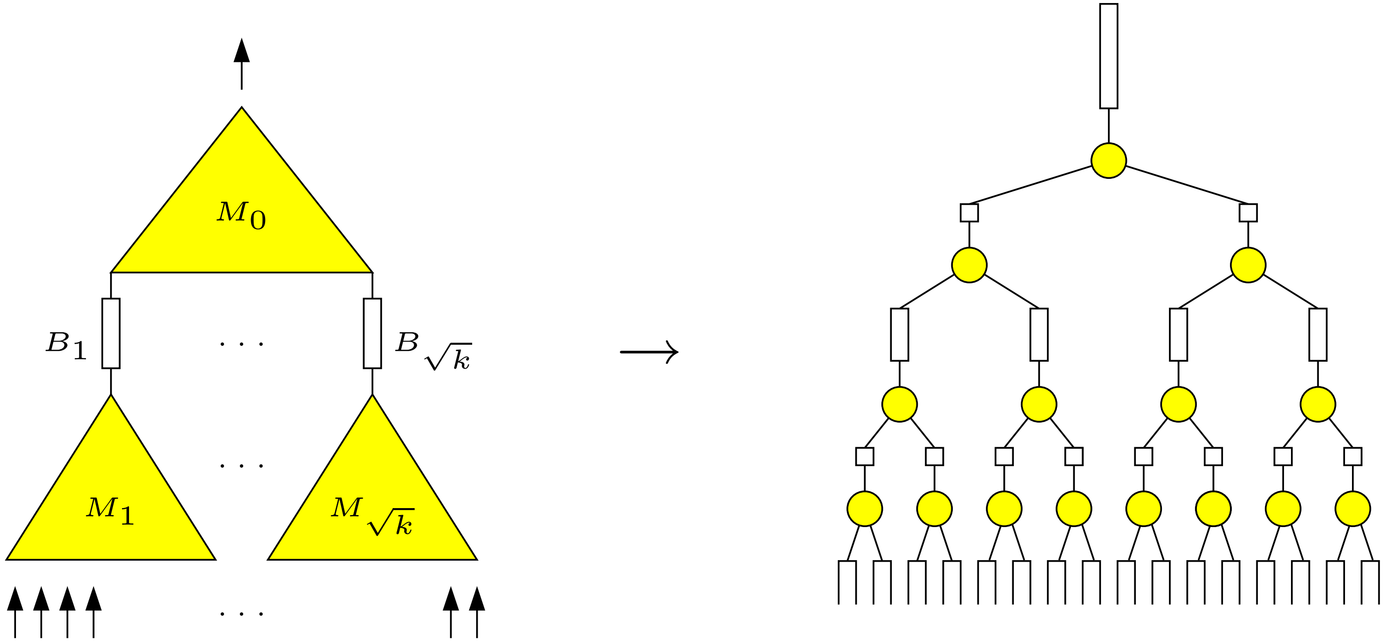




Recursive Layout

Lazy k-merger

Brodal and Fagerberg 2002



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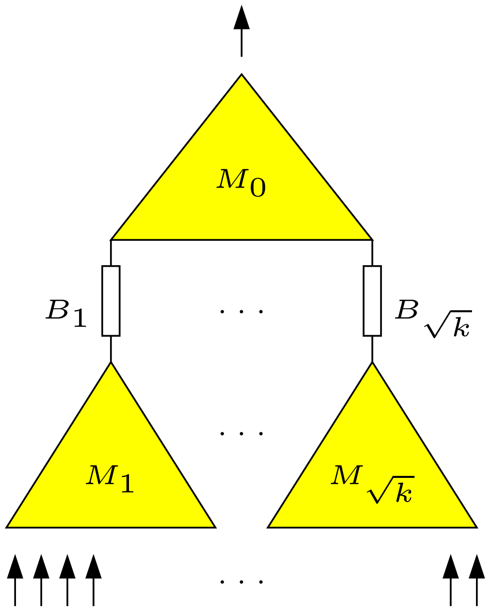

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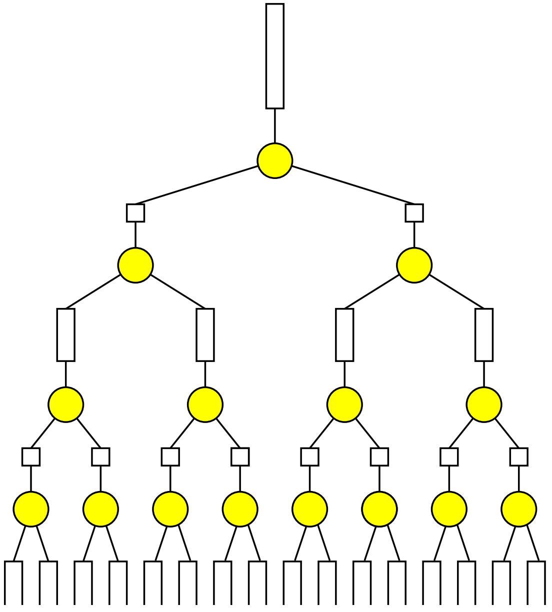
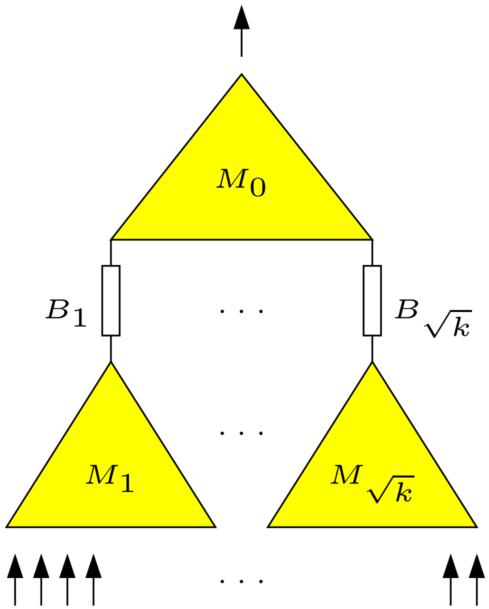


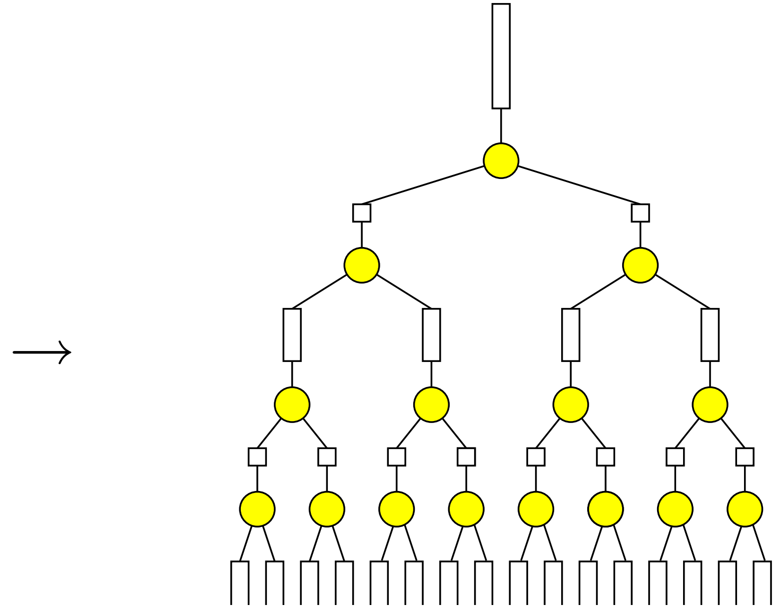
Procedure **Fill**(v)

```
while out-buffer not full
    if left in-buffer empty
        Fill(left child)
    if right in-buffer empty
        Fill(right child)
    perform one merge step
```

Lazy k-merger

Brodal and Fagerberg 2002





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Lemma

If $M \geq B^2$ and output buffer has size k^3 then $O(\frac{k^3}{B}\log_M(k^3)+k)$ I/Os are done during an invocation of **Fill**(root).

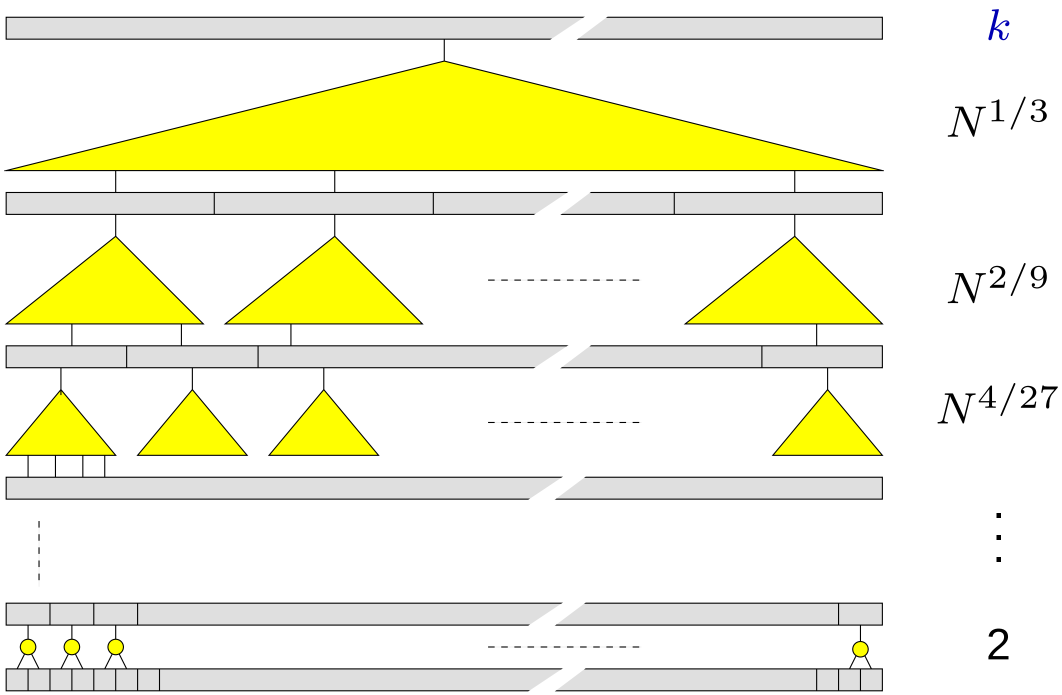
FunnelSort

Brodal and Fagerberg 2002
Frigo, Leiserson, Prokop and Ramachandran 1999

Divide input in $N^{1/3}$ segments of size $N^{2/3}$

Recursively **MergeSort** each segment

Merge sorted segments by an $N^{1/3}$-merger



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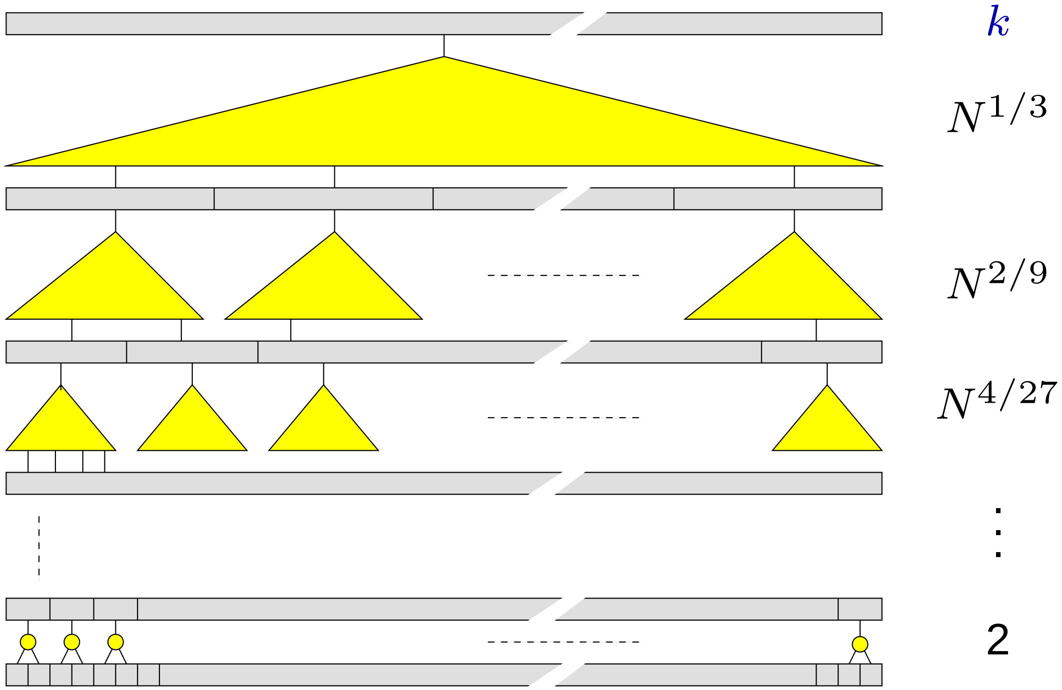

Theorem Provided $M \geq B^2$ (tall cache assumption), FunnelSort performs optimal $O(\mathrm{Sort}(N))$ I/Os

Computational Geometry

Brodal and Fagerberg 2002

Cache oblivious $O(\mathrm{Sort}(N))$ distribution sweeping algorithms for

- Maxima for point set (3D)
- Measure of a set of axis-parallel rectangles (2D)
- Visibility of non-intersecting line segments from a point (2D)
- All nearest neighbors for point set (2D)

Cache oblivious $O(\mathrm{Sort}(N) + \frac{\mathrm{output}}{B})$ algorithms for

- Orthogonal line segment intersection reporting (2D)
- Batched orthogonal range queries on point set (2D)
- Pairwise intersections of axis-parallel rectangles (2D)

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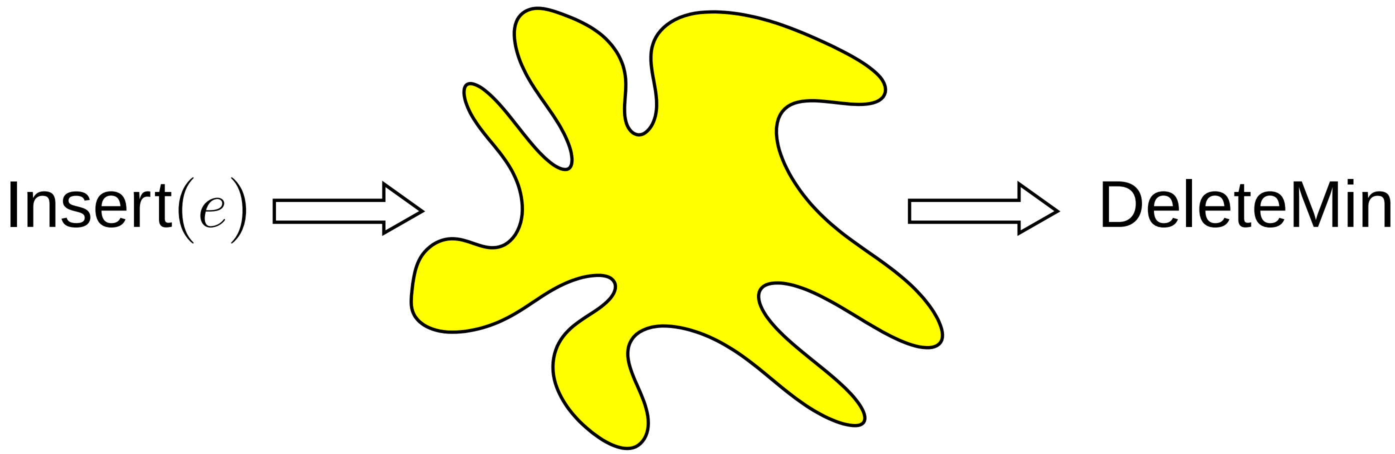
Priority Queues

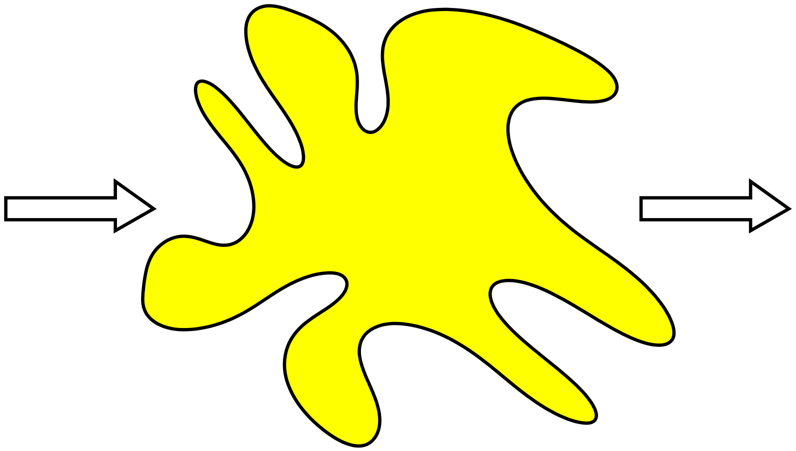


Classic RAM:

- Heap: $O(\log_2 n)$ time — Williams 1964

Priority Queues

Insert(e)

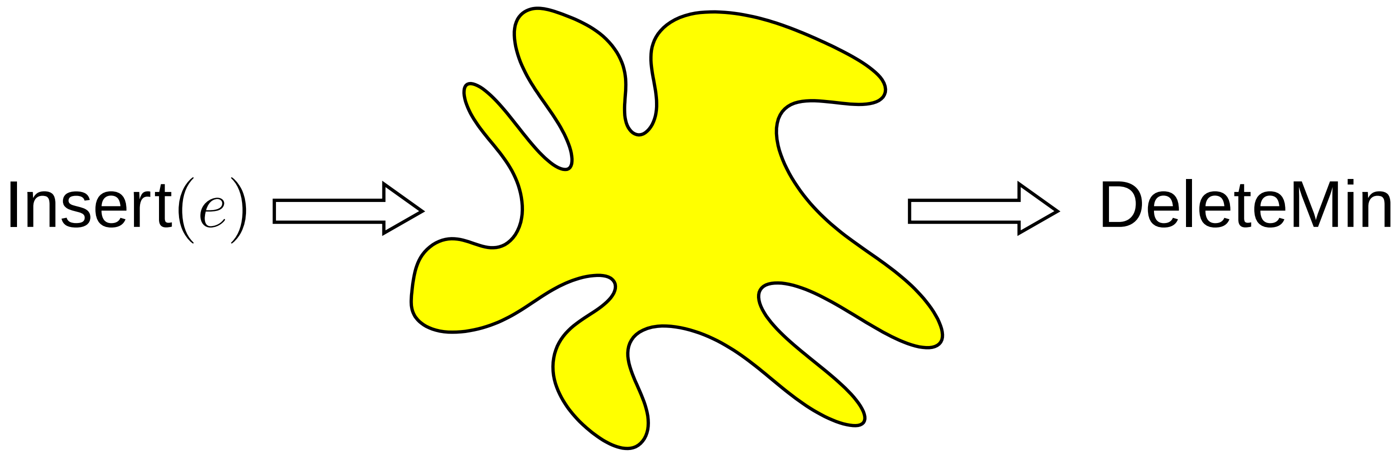


DeleteMin

Classic RAM:

- Heap: $O(\log_2 n)$ time, $O\left(\log_2 \frac{N}{M}\right)$ I/Os   Williams 1964

Priority Queues



Classic RAM:

- Heap: $O(\log_2 n)$ time, $O\left(\log_2 \frac{N}{M}\right)$ I/Os — Williams 1964

I/O model:

- Buffer tree: $O\left(\frac{1}{B} \log_{M/B} \frac{N}{B}\right) = O\left(\frac{\text{Sort}(N)}{N}\right)$ I/Os — Arge 1995

Cache-Oblivious Priority Queues

- $O\left(\frac{1}{B}\log_{M/B}\frac{N}{B}\right)$ I/Os

  Arge, Bender, Demaine, Holland-Minkley and Munro 2002

  – Uses sorting and selection as subroutines

  – Requires tall cache assumption, $M \geq B^2$

- Funnel heap

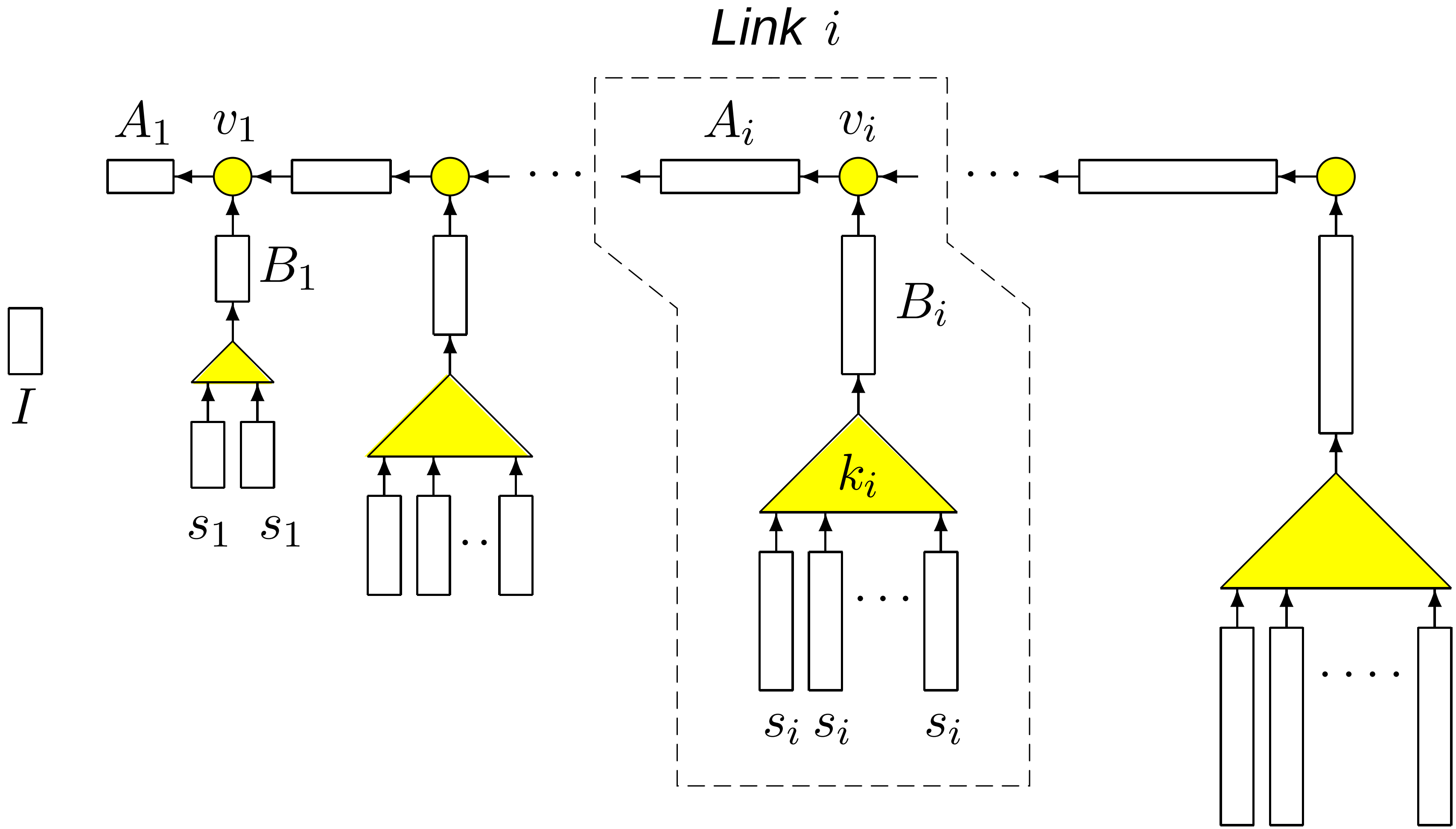
  Brodal and Fagerberg 2002

  – Uses only binary merging

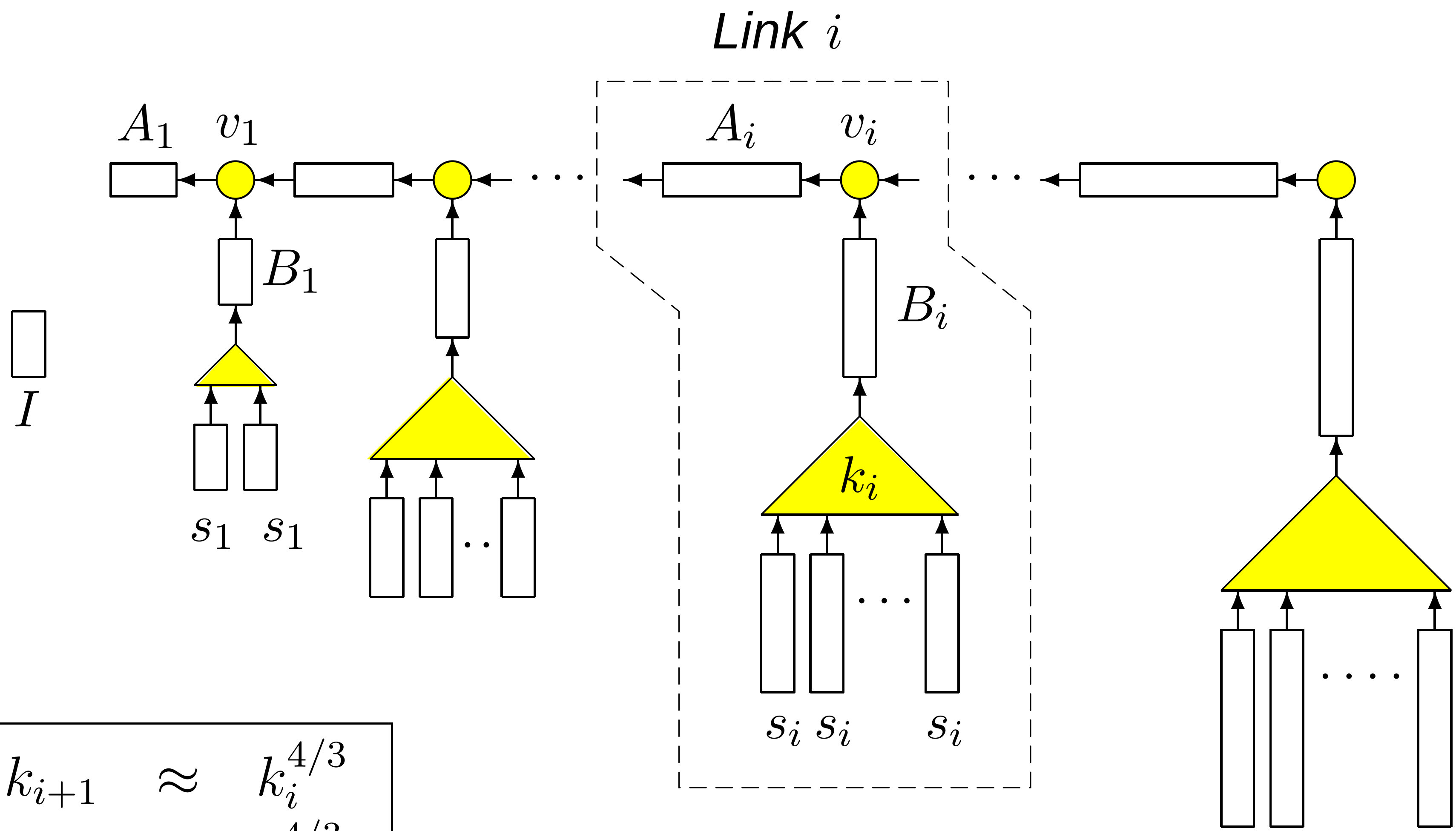
  – Profi le adaptive, i.e. $O\left(\frac{1}{B}\log_{M/B}\frac{N_i}{B}\right)$ I/Os

  N_i is either the size profi le, max depth profi le, or #insertions during the lifetime of the ith inserted element

The Priority Queue

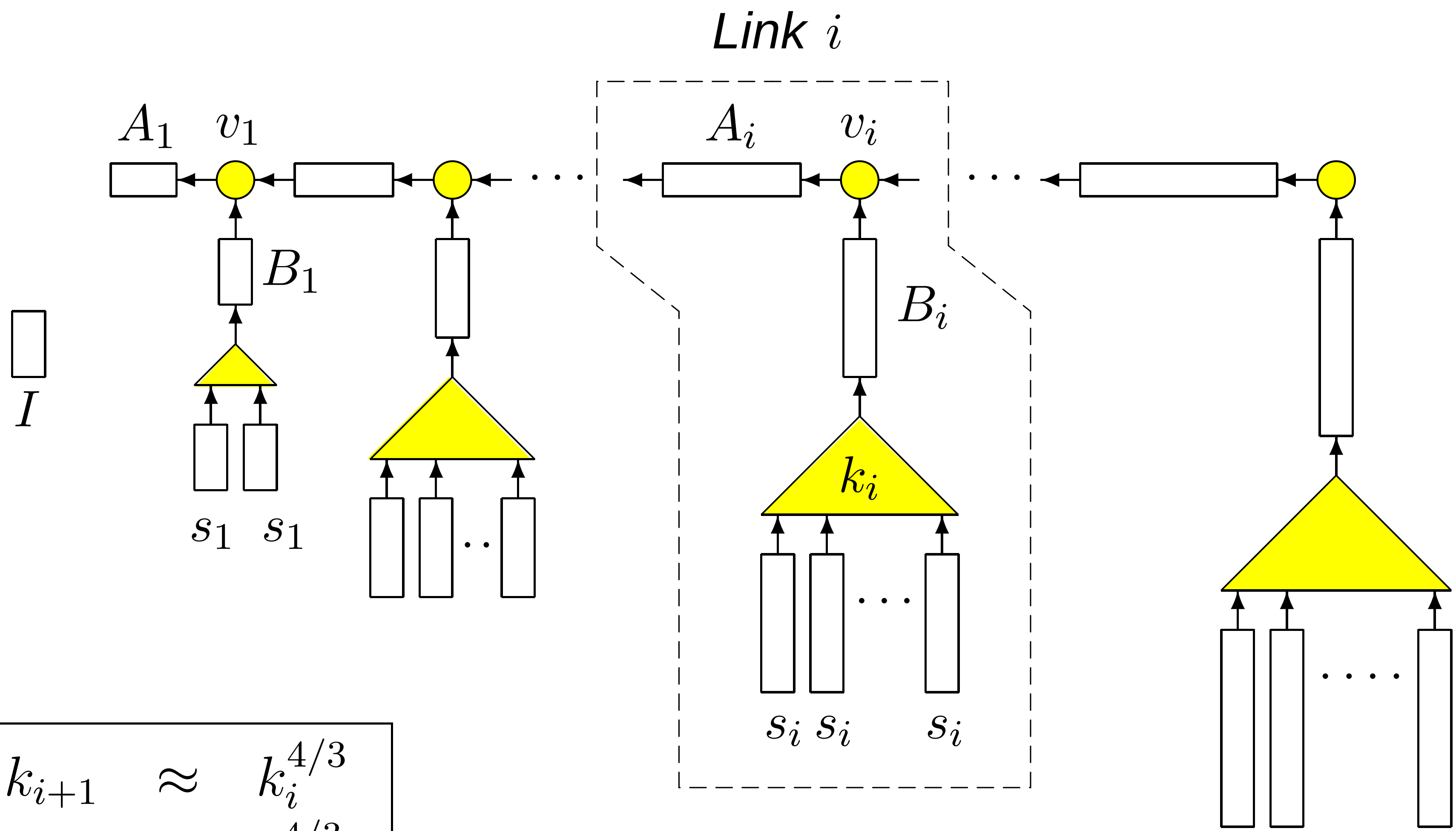


The Priority Queue



$$k_{i+1} \approx k_i^{4/3}$$
$$s_{i+1} \approx s_i^{4/3}$$
$$k_i \approx s_i^{1/3}$$

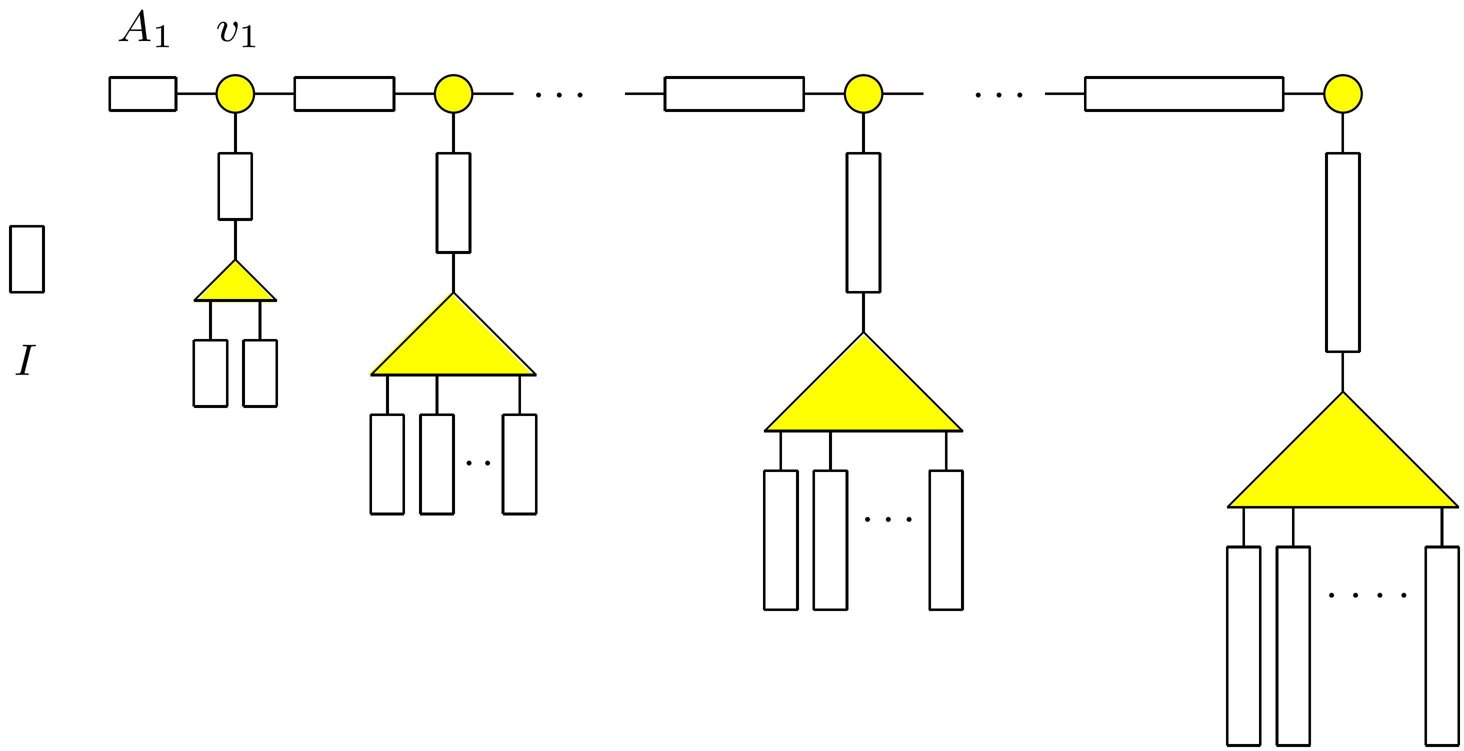
The Priority Queue



$$k_{i+1} \approx k_i^{4/3}$$
$$s_{i+1} \approx s_i^{4/3}$$
$$k_i \approx s_i^{1/3}$$

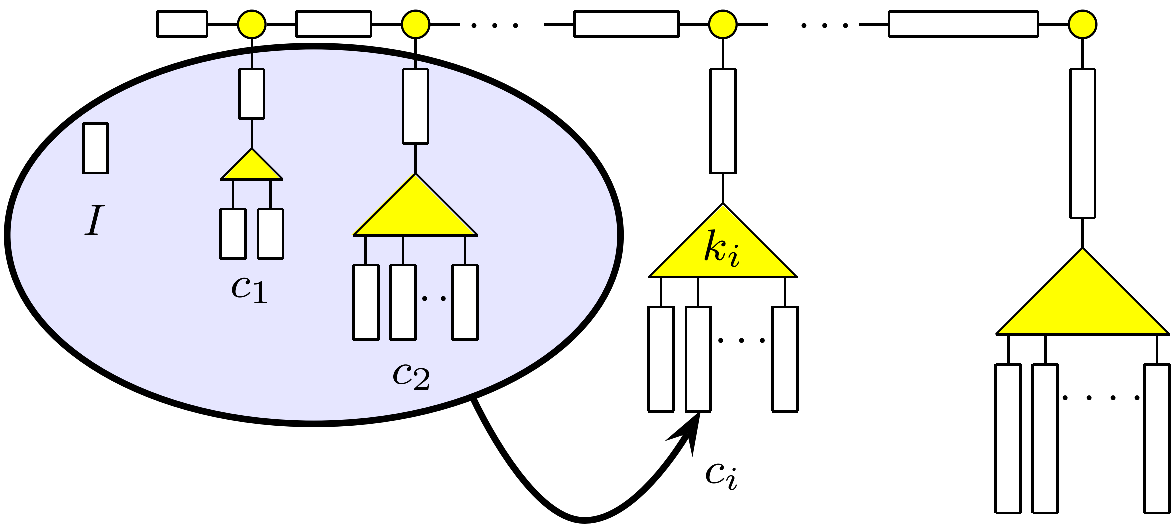
In total: A single binary merge tree

Operations — DeleteMin



- If A_1 is empty, call Fill(v_1)
- Search I and A_1 for minimum element

Operations — Insert

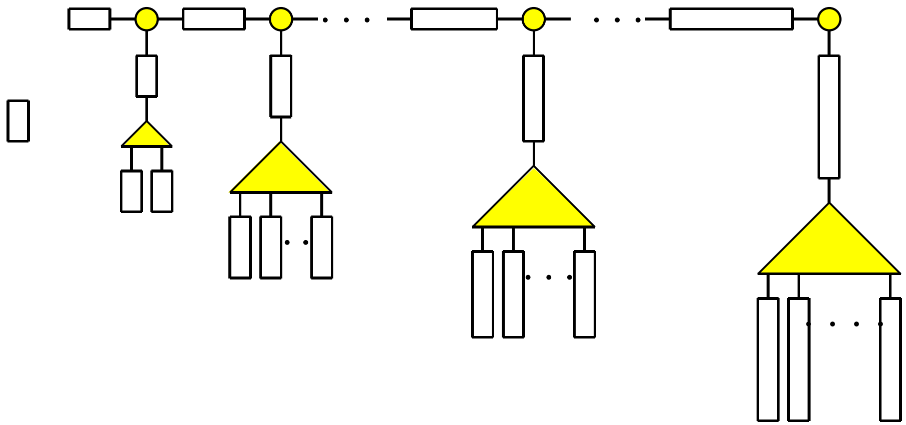


- Insert in I
- If I overflows, call Sweep(i) for first i where $c_i \leq k_i$

Sweep $\approx$ addition of one to number $c_1 c_2 .. c_i .. c_{\max}$

$$s_i = s_1 + \sum_{j=1}^{i-1} k_j s_j$$

Analysis



We can prove:

- Number N of insertions performed: $s_{i_{\max}} \leq N$
- Number of I/Os per Insert for link i: $O\left(\frac{1}{B}\log_{M/B} s_i\right)$
- By the doubly-exponentially growth of s_i,
  the total number of I/Os per Insert is

$$O\left(\sum_{k=0}^{\infty} \frac{1}{B} \log_{M/B} N^{(3/4)^k}\right) = O\left(\frac{\mathrm{Sort}(N)}{N}\right)$$

- DeleteMin is amortized for free

Outline of Talk

- Hardware
- Computational models
  - RAM model (Random Access Machine)
  - IO model
  - Cache oblivious model
- Binary searching and dictionaries
- Sorting
- Priority queues
- ▶ Concluding remarks

Some Cache-Oblivious Results

- Scanning $\Rightarrow$ stack, queue, median finding,. . . .
- Sorting, matrix multiplication, FFT

  Frigo, Leiserson, Prokop, Ramachandran, FOCS'99
- Cache oblivious search trees

  Prokop 99

  Bender, Demaine, Farach-Colton, FOCS'00

  Rahman, Cole, Raman, WAE'01

  Bender, Duan, Iacono, Wu and Brodal, Fagerberg, Jacob, SODA'02
- Priority queue and graph algorithms

  Arge, Bender, Demaine, Holland-Minkley, Munro, STOC'02

  Brodal, Fagerberg, ISAAC'02
- Computational geometry

  Brodal, Fagerberg, ICALP'02

  Bender, Cole, Raman, ICALP'02
- Scanning dynamic sets

  Bender, Cole, Demaine, Farach-Colton, ESA'02

Cache Oblivious Technics

- Scanning
- Sorting
- Recursion
- Recursive layout (van Emde Boas layout)
- Merging (FunnelSort, distribution sweeping, FunnelHeap)

Conclusions

- Cache oblivious model : Simpel and general
- Algorithms exist for many problems
  - stacs, queues, dictionaries, priority queues, sorting, selection, permuting, matrix multiplicataion, FFT, graph algorithms, computational geometry...
- Limitations
  - searching costs a factor $\log_2 e$
  - sorting and priority queues requires a tall cache

Brodal and Fagerberg 2003

Open problems

- Other algorithms ...
- Cache obliviousness *vs* parallel disks ?
- Implementations and experiments ?
- Libraries ?
- ...

References

- **The Cost of Cache-Oblivious Searching**, Michael A. Bender, Gerth Stølting Brodal, Rolf Fagerberg, Dongdong Ge, Simai He, Haodong Hu, John Iacono, and Alejandro López-Ortiz. Submitted.
- **On the Limits of Cache-Obliviousness**, Gerth Stølting Brodal and Rolf Fagerberg. To appear in *Proc. 35th Annual ACM Symposium on Theory of Computing*, 2003.
- **Funnel Heap - A Cache Oblivious Priority Queue**, Gerth Stølting Brodal and Rolf Fagerberg. In *Proc. 13th Annual International Symposium on Algorithms and Computation*, volume 2518 of *Lecture Notes in Computer Science*, pages 219-228. Springer Verlag, Berlin, 2002.
- **Cache Oblivious Distribution Sweeping**, Gerth Stølting Brodal and Rolf Fagerberg. In *Proc. 29th International Colloquium on Automata, Languages, and Programming*, volume 2380 of *Lecture Notes in Computer Science*, pages 426-438. Springer Verlag, Berlin, 2002.
- **Cache-Oblivious Search Trees via Trees of Small Height**, Gerth Stølting Brodal, Rolf Fagerberg, and Riko Jacob. In *Proc. 13th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 39-48, 2002.