Time-Space Trade-Offs for 2D Range Minimum Queries

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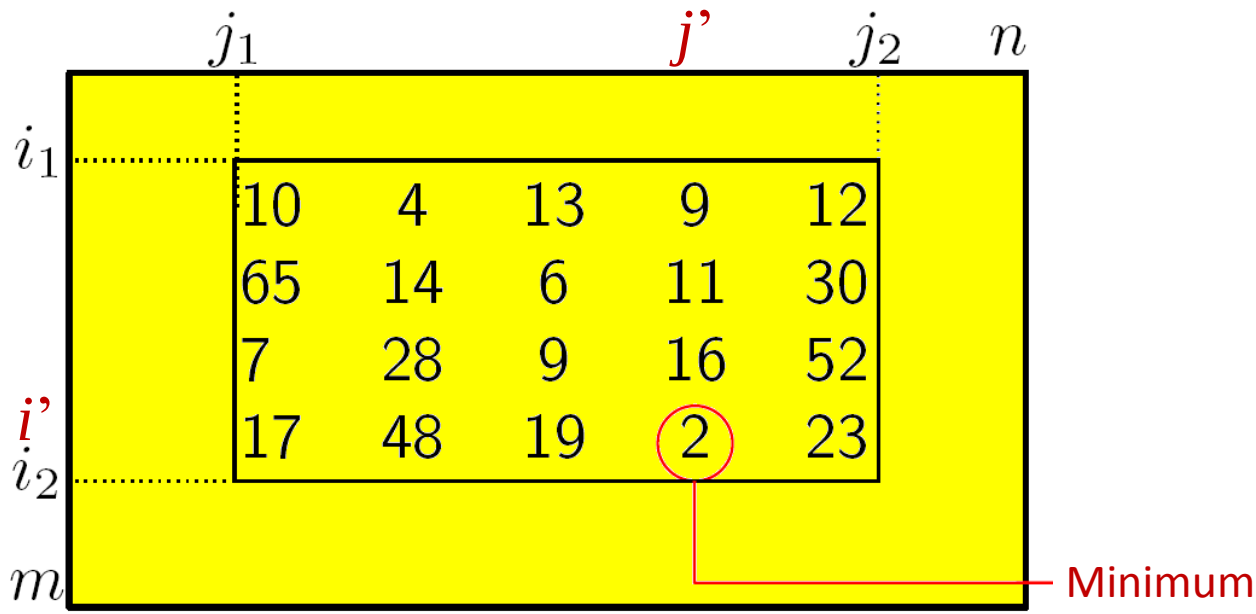
The 2D Range Minimum Problem



Preprocess an m x n-matrix of size $N = n \cdot m$, $m \leq n$, to efficiently support range minimum queries

$$\text{RMQ}([i_1, i_2] \text{x} [j_1, j_2]) = (i', j')$$

$$A_{i',j'} = \min\{ A_{i'',j''} \mid (i'', j'') \in [i_1,i_2] \text{ x } [j_1,j_2] \}, \ (i', j') \in [i_1,i_2] \text{x} [j_1,j_2]$$

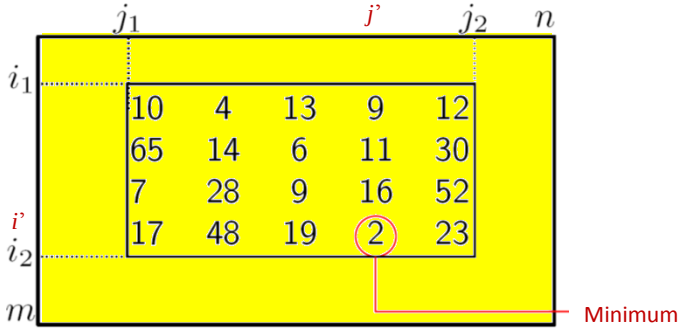
Models

Encoding model

- Queries can access **data structure** but not input matrix

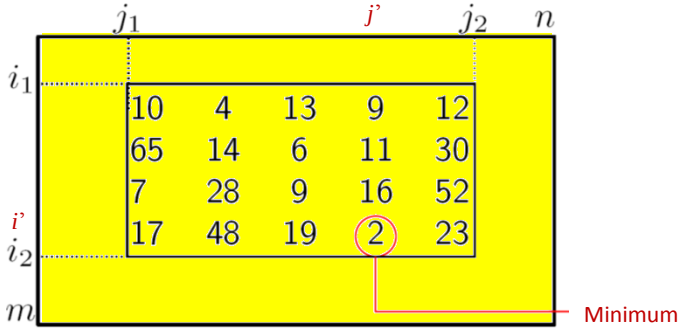
Indexing model

- Queries can access **data structure** and read **input matrix**



Some Trivial Examples...

Solution	Additional space (bits)	Query time	Model
No data structure	0	$O(N)$	Indexing
Tabulate answers	$O(N^2 \log N)$	$O(1)$	Encoding
Store permutation	$O(N \log N)$	$O(N)$	Encoding



Results

1D Range Minimum Queries

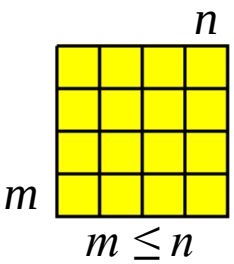
Indexing

- **Upper Bound** (Fischer and Heun (2007))
  - Time $= O(1)$
  - Space $= 2n + o(n) + |A|$ bits
- **NEW** **Lower Bound** (matching upper bound)
  - Time $= \Omega(c)$
  - Space $= O(n/c) + |A|$ bits

Encoding

- **Upper Bound** (Fischer (Latin 2010))
  - Time $= O(1)$
  - Space $= 2n + o(n)$ bits
- **Lower Bound:** Space $= 2n - \Theta(\log n)$ bits

2D Range Minimum Queries



Indexing

- **NEW** Upper Bound
  - Time $= O(1)$
  - Space $= O(N) + |A|$ bits
  - Time $= O(c \log^2 c)$
  - Space $= O(N/c) + |A|$ bits
- **NEW** Lower Bound
  - Time $= \Omega(c)$
  - Space $= O(N/c) + |A|$ bits

Encoding

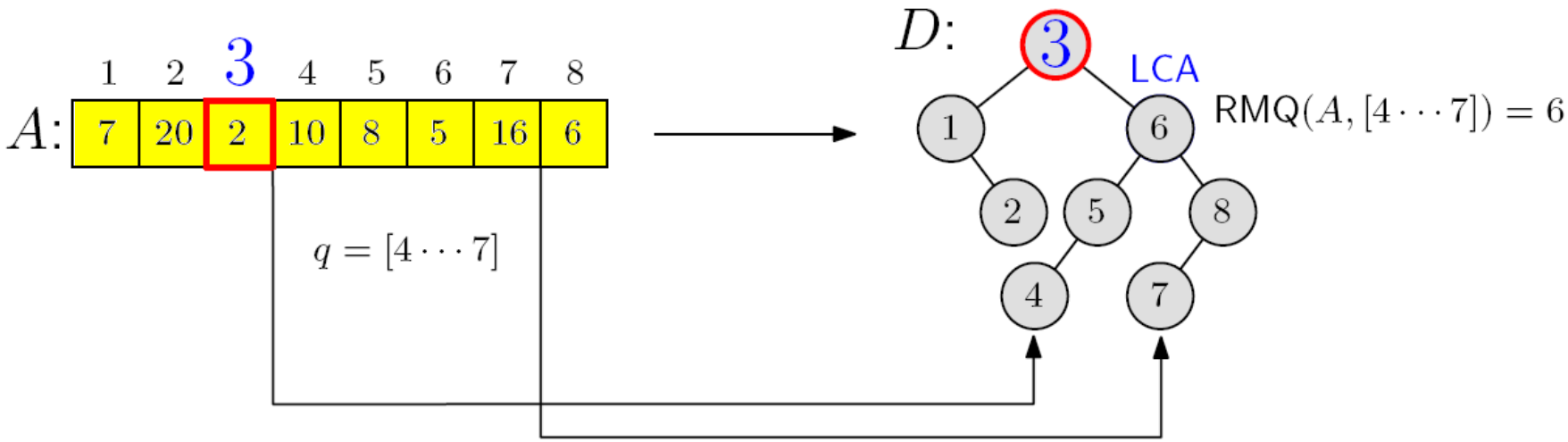
- Upper Bound
  - Time $= O(1)$
  - Space $= O(N \log n)$ bits
- **NEW Proof** Lower Bound: Space $= \Omega(N \log m)$ bits

Demain et al. (2009)

1D

1D	Encoding model	Index model
Upper bound		
Lower bound		

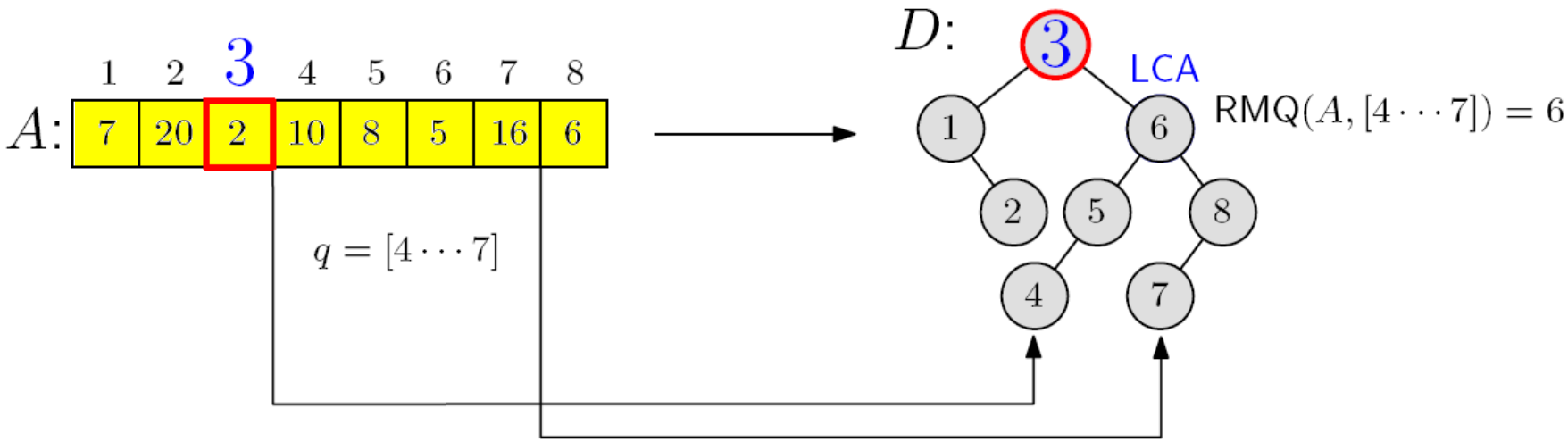
Lower Bound (1D, Encoding)



- For each input array consider the Cartesian tree
- Each binary tree is a possible Cartesian tree
- RMQ queries can reconstruct the Cartesian tree
- # Cartesian trees is $\binom{2n}{n}/(n+1)$
- # bits ≥ $\log\binom{2n}{n}/(n+1) = 2n - \Theta(\log n)$

1D	Encoding model	Index model
Upper bound		
Lower bound	●	

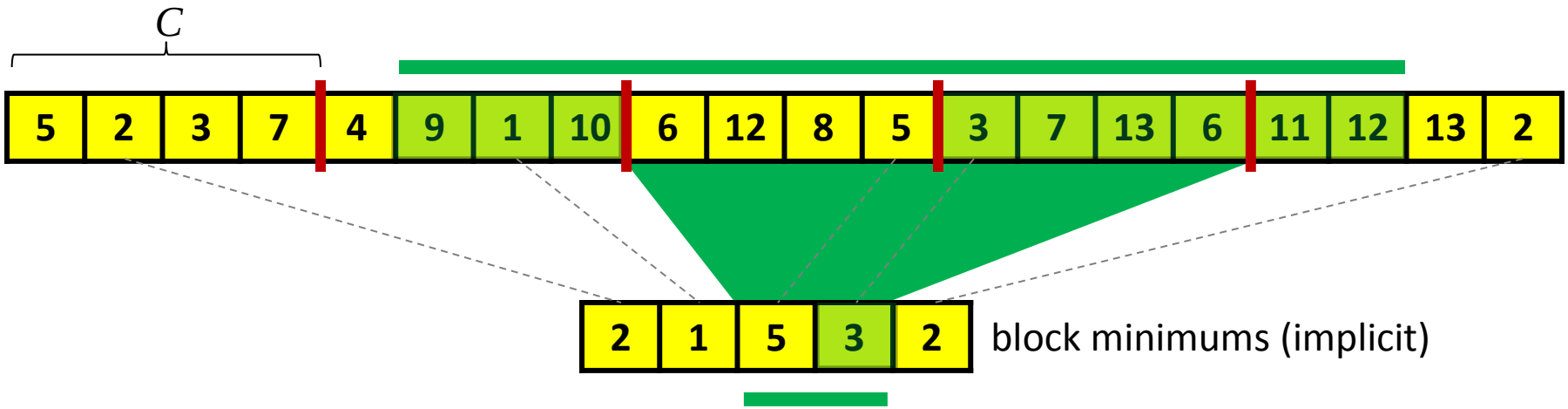
Upper Bound (1D, Encoding)



- For an input array consider the Cartesian tree
- Succint representation using $4n+o(n)$ bits and O(1) query time (Sadakane 2007)
- Improved to $2n+o(n)$ (Fischer 2010)

1D	Encoding model	Index model
Upper bound	●	
Lower bound	●	

Upper Bounds (1D, Indexing)

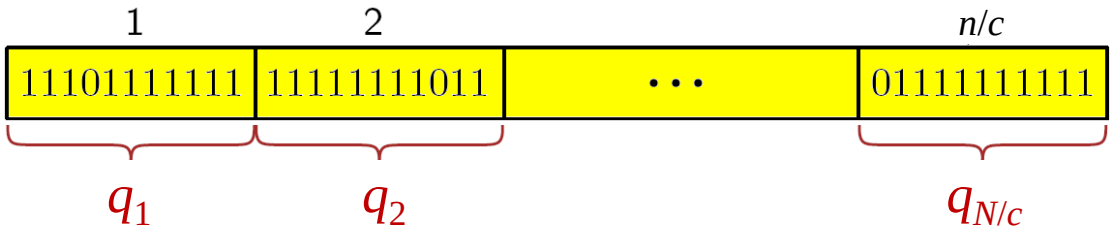


- Build encoding $O(n/c)$ bit structure for block minimums
- RMQ = query to encoding structure + $3c$ elements, i.e. query time $O(c)$

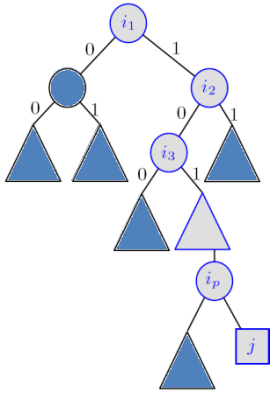
1D	Encoding model	Index model
Upper bound	●	●
Lower bound	●	

Lower Bounds (1D, Indexing)

Thm Space n/c bits implies $\Omega(c)$ query time



- Consider n/C queries for $c^{n/c}$ different {0,1} inputs with exactly one zero in each block
- $c^{n/c} / 2^{n/c}$ inputs share some data structure
- Every query is a decision tree of height $\leq d$



1D	Encoding model	Index model
Upper bound	●	●
Lower bound	●	●

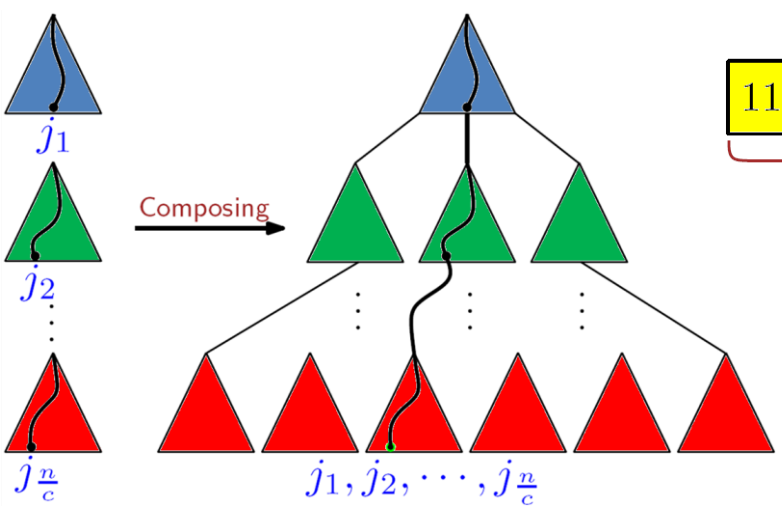
Lower Bounds (1D, Indexing) *cont.*

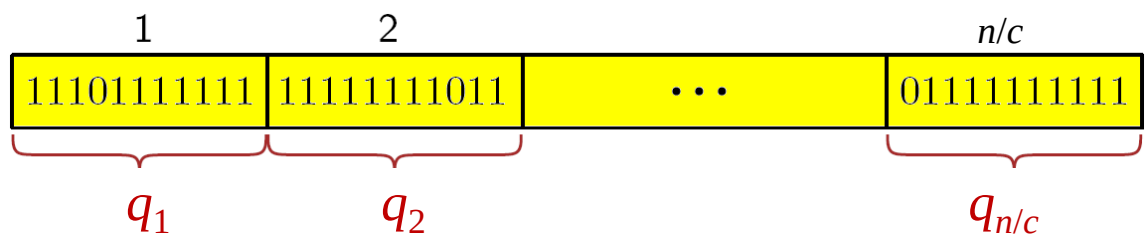
- Combine queries to decision tree identifying input
- Prune non-reachable branches

➡ # zeroes on any path ≤ n/c

➡ $$\frac{c^{n/c}}{2^{n/c}} \leq \#\text{inputs} = \#\text{leaves} \leq \binom{d \cdot n/c}{n/c}$$

➡ query time $d = \Omega(c)$





1D	Encoding model	Index model
Upper bound	●	●
Lower bound	●	●

2D

2D	Encoding model	Index model
Upper bound		
Lower bound		

Upper Bounds (2D, Indexing)

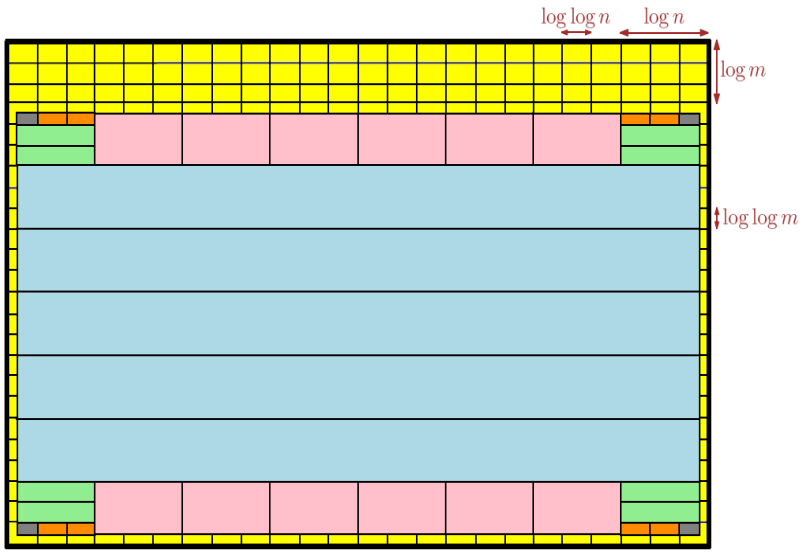
$O(1)$ time using $O(N)$ words

Atallah and Yuan (SODA 2010)

- Using two-levels of recursion, tabulating micro-blocks of size

  $\log\log m$ x $\log\log n$

  $O(1)$ time using $O(N)$ bits

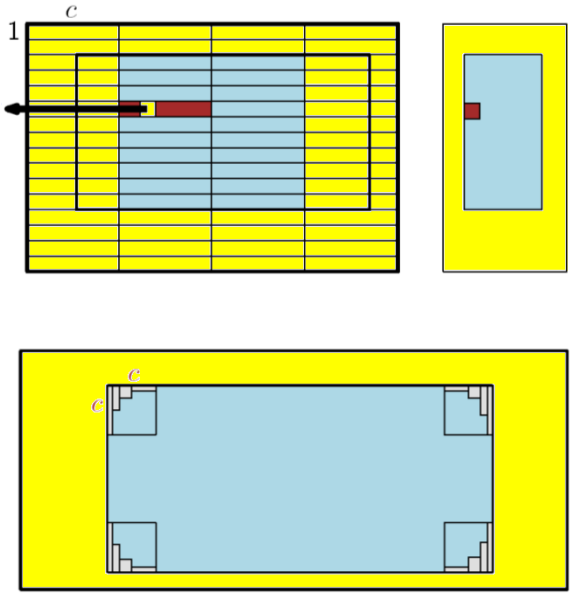


2D	Encoding model	Index model
Upper bound		●
Lower bound		

Upper Bounds (2D, Indexing) *cont.*

Thm $O(N/c \cdot \log c)$ bits and $O(c \log c)$ query time

- Build $\log c$ indexing structures for compressed matrices for block sizes $2^i \times c/2^i$, each using $O(N/c)$ bits and can locate $O(1)$ blocks with minimum key in $O(1)$ time
- Query: $O(1)$ blocks for each block size in time $O(c)$ + elements not covered by blocks in time $O(c \log c)$



2D	Encoding model	Index model
Upper bound		●
Lower bound		

Lower Bounds (2D, Indexing)

c

1	1	0	1	1	1	1	1	1	0	1	0	1	1	1	1	1	1	0	1	0	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1	0	1	1	1
1	1	1	0	1	1	0	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1	0	1	1
1	1	1	1	0	1	1	0	1	1	0	1	1	1	1	1	1	0	1	1	1	1	0	1	1
0	1	1	1	1	1	1	0	1	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1	0
1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	0	1	1	1	1	1	0	1	1	1

- As for 1D consider {0,1} matrices and partition the array into blocks of c elements each containing exactly one zero
- As for 1D an algorithm being able to identify the zero in each block using N/c bits will require time $\Omega(c)$

2D	Encoding model	Index model
Upper bound		●
Lower bound		●

Upper Bounds (2D, Encoding)

29	-14	10	15
2	7	0	13
-4	-5	-1	21
5	20	-17	32

input matrix

15	2	10	12
7	9	6	11
4	3	5	14
8	13	1	16

rank matrix

- Translate input matrix into rank matrix using $O(N \log N)$ bits
- Apply index structure to rank matrix using $O(N)$ bits achieving $O(1)$ query time

2D	Encoding model	Index model
Upper bound	●	●
Lower bound		●

NEW Proof

Lower Bound (2D, Encoding)

Demaine et al. 2009

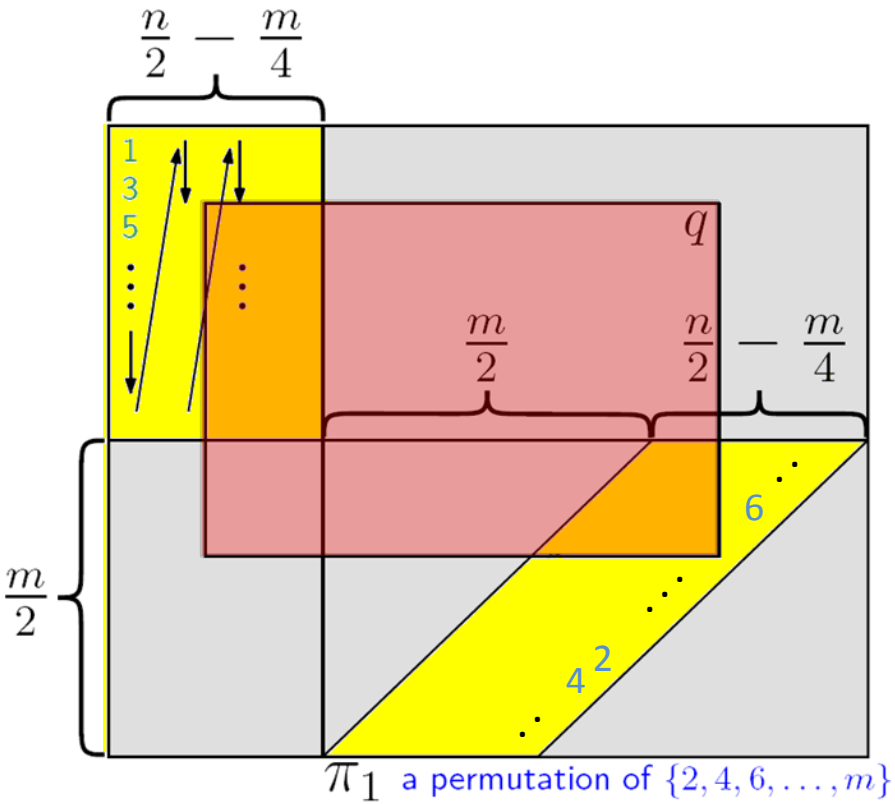
- Define a set of

$$\left(\frac{m}{2}!\right)^{\frac{n}{2}-\frac{m}{4}}$$

matrices where the RMQ answers differ among all matrices

- Bits required is at least

$$\log\left(\frac{m}{2}!\right)^{\frac{n}{2}-\frac{m}{4}} = \Omega(N \log m)$$



2D	Encoding model	Index model
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Conclusion

1D Range Minimum Queries

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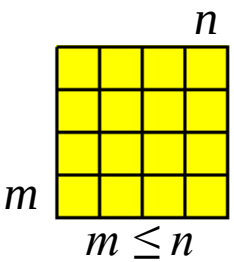

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?

NEW Lower Bound

- Time $= \Omega(c)$
- Space $= O(N/c) + |A|$ bits

Encoding

Upper Bound

- Time $= O(1)$
- Space $= O(N \log n)$ bits

?

NEW Proof

Lower Bound: Space $= \Omega(N \log m)$ bits

Demain et al. (2009)

Tau · Jërë-jëf · Tashakkur · S.aHHa · Sag olun
Giihtu · Djakujo · Dâkujem vám · Thank you
Tesekkür ederim · To-siä · Merci · Tashakur
Taing · Dankon · Efharisto´ · Shukriya · Kiitos
Dhanyabad · Rakhmat · Trugarez · Asante
Köszönöm · Blagodarya · Dziekuje · Eskerrik asko
Grazie · Tak · Bayarlaa · Miigwech · Dank u
Spasibo · Dêkuji vám · Ngiyabonga · Dziakuj
Obrigado · Gracias · A dank aych · Salamat
Takk · Arigatou · Tack · Tänan · Aciu
Korp kun kah · Multumesk · Terima kasih · Danke
Rahmat · Gratias · Mahalo · Dhanyavaad
Paldies · Faleminderit · Diolch · Hvala
Kam-sa-ham-ni-da · Xìe xìe · Mèrcie · Dankie

Thank You