On the Adaptiveness of Quicksort

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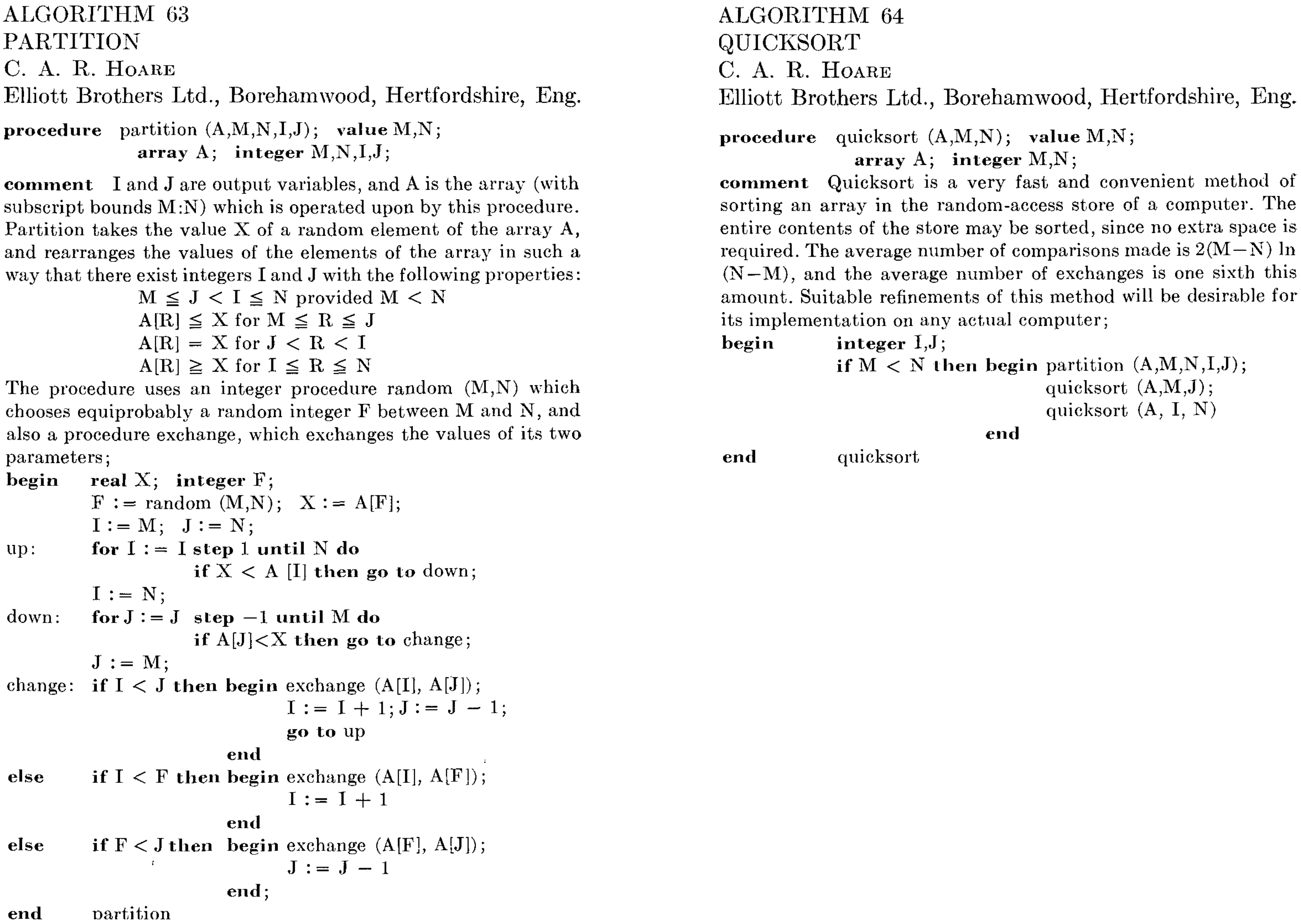


ALGORITHM 63
PARTITION
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure  partition (A,M,N,I,J);  value M,N;
              array A;  integer M,N,I,J;
comment  I and J are output variables, and A is the array (with
subscript bounds M:N) which is operated upon by this procedure.
Partition takes the value X of a random element of the array A,
and rearranges the values of the elements of the array in such a
way that there exist integers I and J with the following properties:
              M ≦ J < I ≦ N provided M < N
              A[R] ≦ X for M ≦ R ≦ J
              A[R] = X for J < R < I
              A[R] ≧ X for I ≦ R ≦ N
The procedure uses an integer procedure random (M,N) which
chooses equiprobably a random integer F between M and N, and
also a procedure exchange, which exchanges the values of its two
parameters;
begin    real X;  integer F;
         F := random (M,N);  X := A[F];
         I := M;  J := N;
up:      for I := I step 1 until N do
                   if X < A [I] then go to down;
         I := N;
down:    for J := J  step −1 until M do
                   if A[J]<X then go to change;
         J := M;
change:  if I < J then begin exchange (A[I], A[J]);
                             I := I + 1; J := J − 1;
                             go to up
                       end
else     if I < F then begin exchange (A[I], A[F]);
                             I := I + 1
                       end
else     if F < J then  begin exchange (A[F], A[J]);
                             J := J − 1
                       end;
end      partition
```

ALGORITHM 64
QUICKSORT
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure  quicksort (A,M,N);  value M,N;
              array A;  integer M,N;
comment  Quicksort is a very fast and convenient method of
sorting an array in the random-access store of a computer. The
entire contents of the store may be sorted, since no extra space is
required. The average number of comparisons made is 2(M−N) ln
(N−M), and the average number of exchanges is one sixth this
amount. Suitable refinements of this method will be desirable for
its implementation on any actual computer;
begin       integer I,J;
            if M < N then begin partition (A,M,N,I,J);
                                quicksort (A,M,J);
                                quicksort (A, I, N)
                          end
end         quicksort
```

Quicksort

- Introduced by Hoare in 1961
- Simple, randomized sorting algorithm
- Elements are compared and swapped within the input array "implicit sorting algorithm" (except for the runtime stack)
- Expected number of comparisons $\sim 1.4n\log_2 n$ [Hoare'62]
- Average number of swaps for a random input is $1/6$ the expected number of comparisons [Hoare'62]

This talk

- Characterize the expected number of swaps performed by Quicksort by the amount of disorder (inversions) in the input

$$O(n(1+\log(1+Inv/n)))$$

Adaptiveness

- What if the input is nearly sorted ?
- Adaptive sorting - the running time depends both on the input size and the presortedness in the input
- A common measure of presortedness:

$$Inv(x_1 \ldots x_n) = |\{(i,j) \mid i < j \wedge x_i > x_j\}|$$

- An optimal sorting algorithm with respect to Inv performs $\Theta(n(1+\log(1+\frac{Inv}{n})))$ comparisons

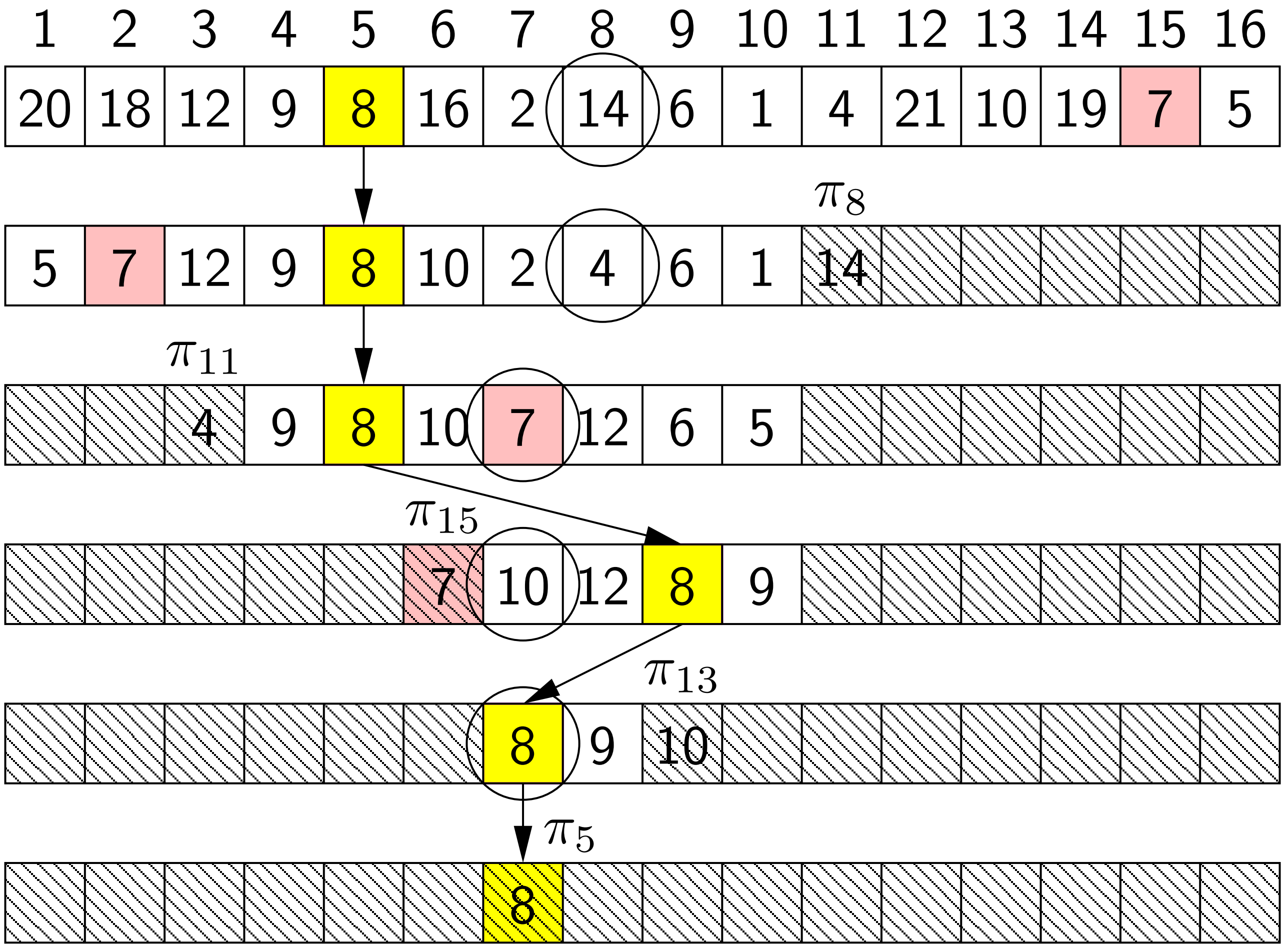
Quicksort

- The expected number of comparisons performed is independent of the order of the input (expected $O(n \log n)$)
- The number of swaps can be significantly smaller for nearly sorted inputs (we prove $O(n(1+\log(1+\frac{Inv}{n})))$)

Quicksort C source

```c
#define Item int
#define random(l,r) (l+rand() % (r-l+1))
#define swap(A, B) { Item t = A; A = B; B = t; }
void quicksort(Item a[], int l, int r)
   { int i;
     if (r <= l) return;
     i = partition(a, l, r);
     quicksort(a, l, i-1);
     quicksort(a, i+1, r);
   }
int partition(Item a[], int l, int r)
  { int i = l-1, j = r+1, p = random(l,r);
    Item v = a[p];
    for (;;) {
        while (++i < j && a[i] <= v);
        while (--j > i && v <= a[j]);
        if (j <= i) break;
        swap(a[i], a[j]);
      }
    if (p < i) i--;
    swap(a[i], a[p]);
    return i;
  }
```

Pivots vs Swaps



The first pivot causing $x_5 = 8$ to be swapped is $x_{15} = 7$
($\pi_5 = 7$, $\pi_{15} = 6$, and $5 \leq \pi_{15} < \pi_5$)

Main Theorem (I)

Theorem

Quicksort performs expected $\leq n + n \ln\left(\frac{4Inv}{n} + 1\right)$ swaps.

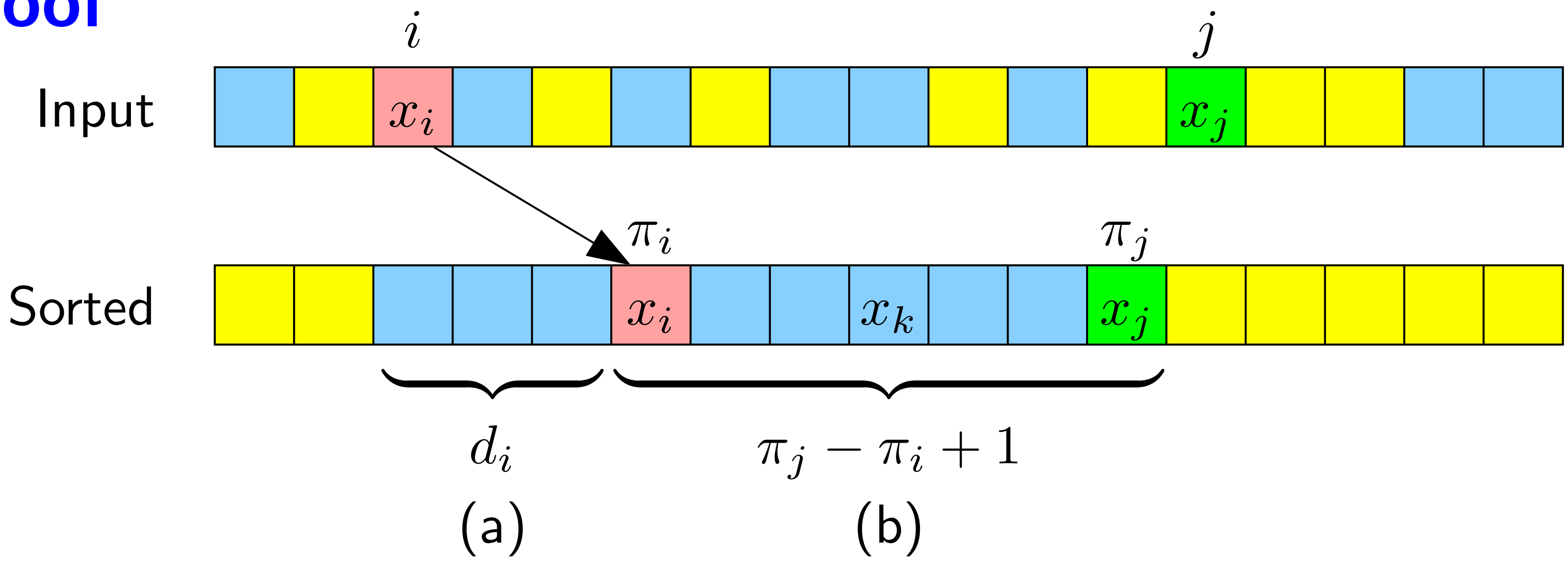
- $(x_1, \ldots, x_n)$ – input sequence of distinct elements
- π_i – rank of x_i in the sorted sequence
- $d_i = |\pi_i - i|$
- $X_{ij} = 1$ if when x_j becomes a pivot then x_i and x_j are in the same set and x_i is swapped

Main Theorem (II)

Lemma

$$\Pr[X_{ij}=1] \le \begin{cases} \frac{1}{|\pi_i-\pi_j|+1} & \text{for } i \le \pi_j < \pi_i \text{ or } \pi_i < \pi_j \le i, \\ \frac{1}{|\pi_i-\pi_j|+1} - \frac{1}{|\pi_i-\pi_j|+1+d_i} & \text{otherwise.} \end{cases}$$

Proof



(a) Pivots forcing x_i to be swapped

(b) Pivots separating x_i and x_j

Main Theorem (III)

Theorem

Quicksort performs expected $\leq n + n \ln\left(\frac{4Inv}{n} + 1\right)$ swaps.

Proof

$$\begin{aligned}
\mathrm{E}\left[\sum_{j=1}^{n}\left(1 + \frac{1}{2}\sum_{i=1,i\neq j}^{n} X_{ij}\right)\right] &= n + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n} \Pr(X_{ij} = 1) \\
&\leq \sum_{i=1}^{n}\sum_{k=1}^{2d_i+1}\frac{1}{k} \\
&\leq n + n\ln\left(\frac{4Inv}{n} + 1\right)
\end{aligned}$$

using $\sum_{i=1}^{n} d_i \leq 2Inv$

Experimental Setup

- Two types of input

  (1) x_i uniformly at random in $[i-d..i+d]$ for increasing d, i.e. small d_i

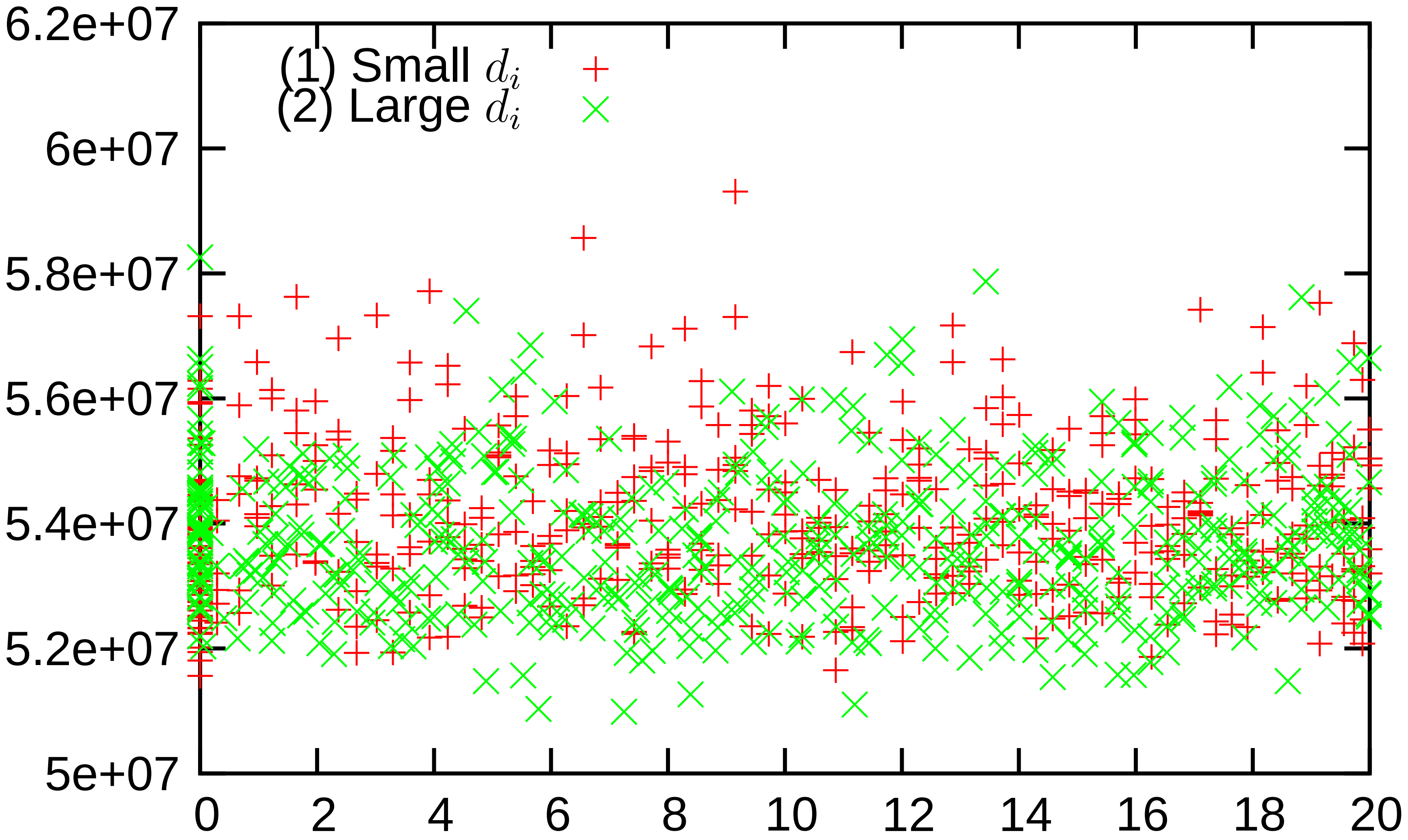
  (2) $x_i = i$ except for some random i where x_i is randomly in $[0..n-1]$, i.e. large d_i

- Compare #comparisons, # swaps, and the running times against $\log \frac{Inv}{n}$

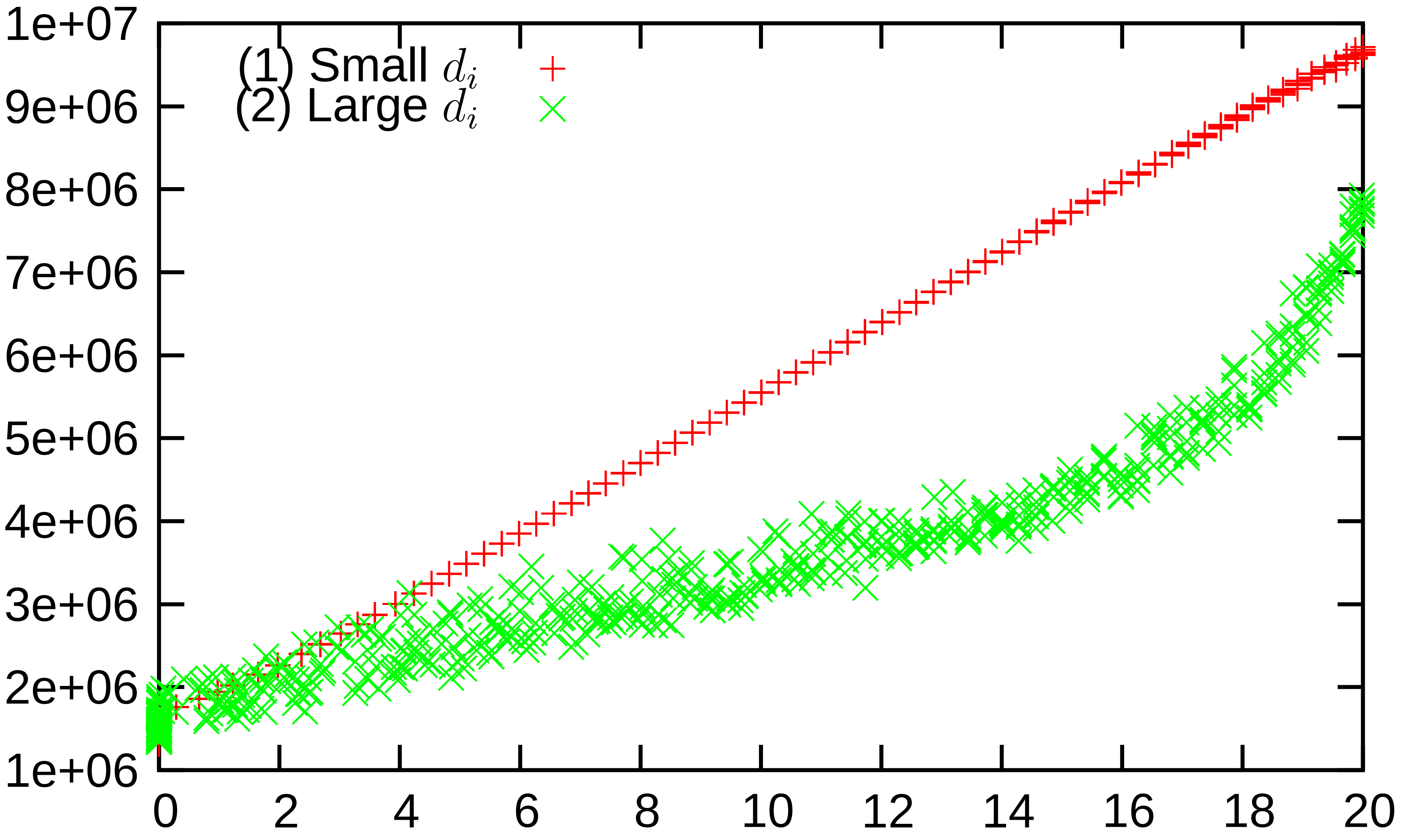
- $n = 2 \times 10^6$

- Intel P4 3.0 GHz, Redhat 9, Linux 2.4.20, gcc 3.3.2 using optimization -O3.

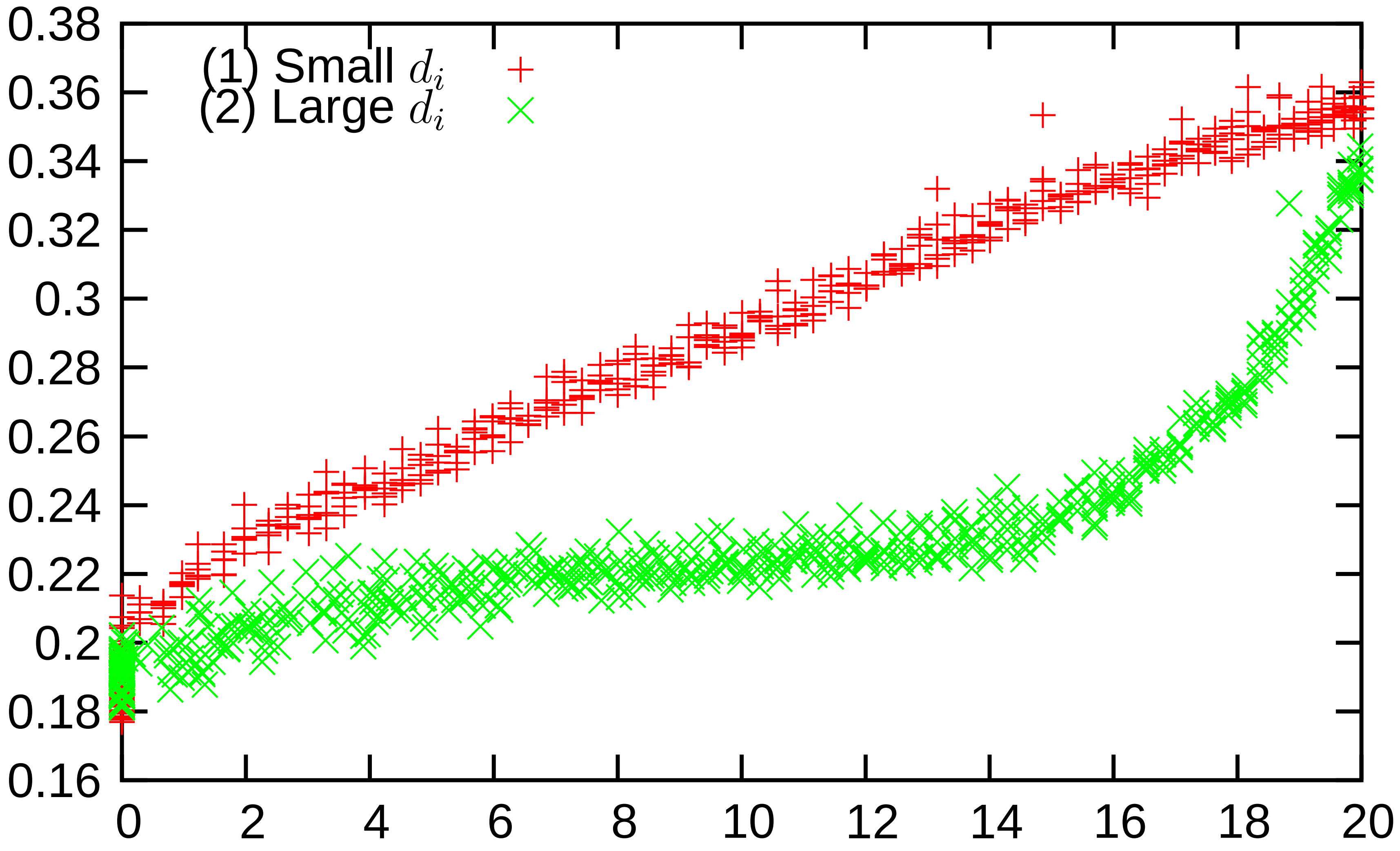
Number of Comparisons



Number of Swaps

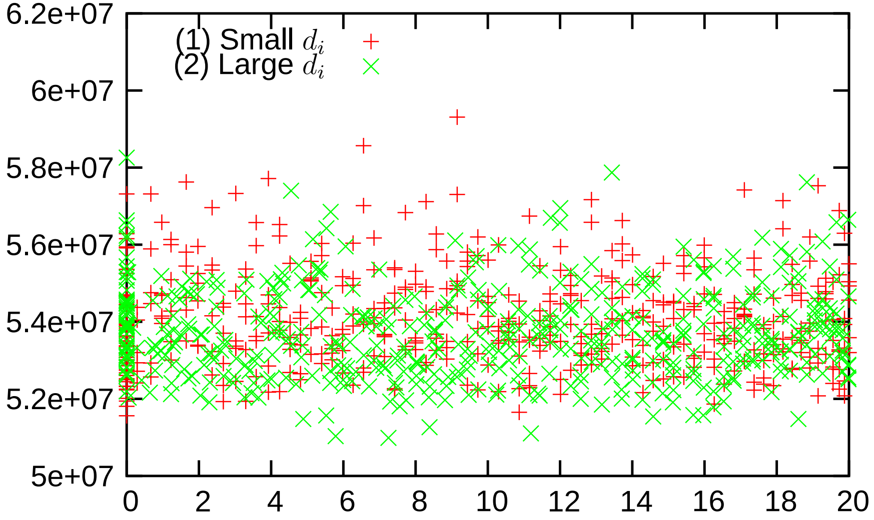


Running Time

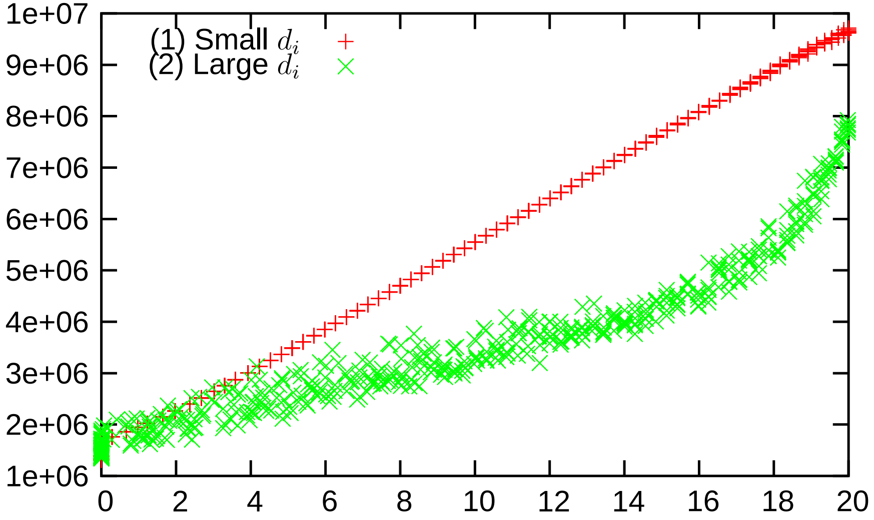


Summary of Experimental results

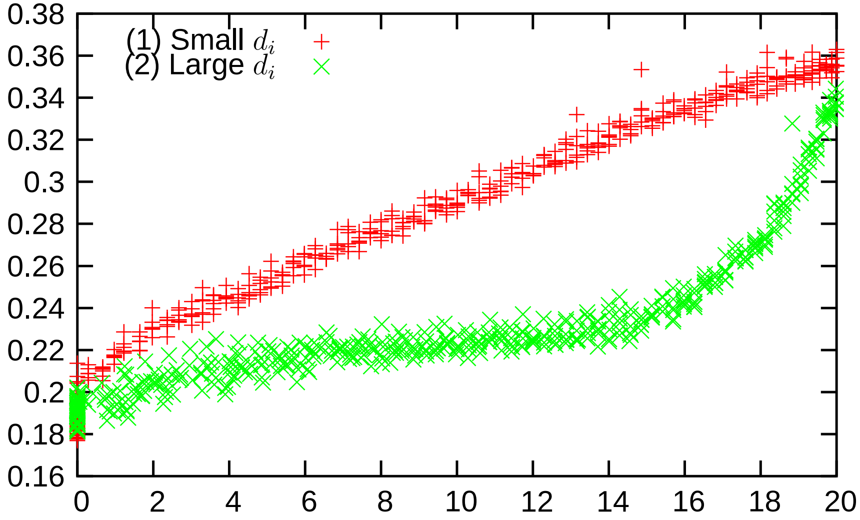
Comparisons



Swaps



Running time



Conclusions

- Quicksort performs expected $O(n(1+\log(1+Inv/n)))$ swaps
- The number of branch mispredictions is given by the number of swaps
- $\#$swaps$/B \leq \#$cache faults$($write$)/2 \leq \#$swaps
- The number of swaps performed can affect the running time of Quicksort by up to a factor of two
- Empirical results confirm the theoretical results