Cache-Oblivious Algorithms

Cache-Oblivious Model

The Unknown Machine

Algorithm	Algorithm
↓	↓
C program	Java program
↓ gcc	↓ javac
Object code	Java bytecode
↓ linux	↓ java
Execution	Interpretation

Can be executed on machines with a specific class of CPUs

Can be executed on any machine with a Java interpreter

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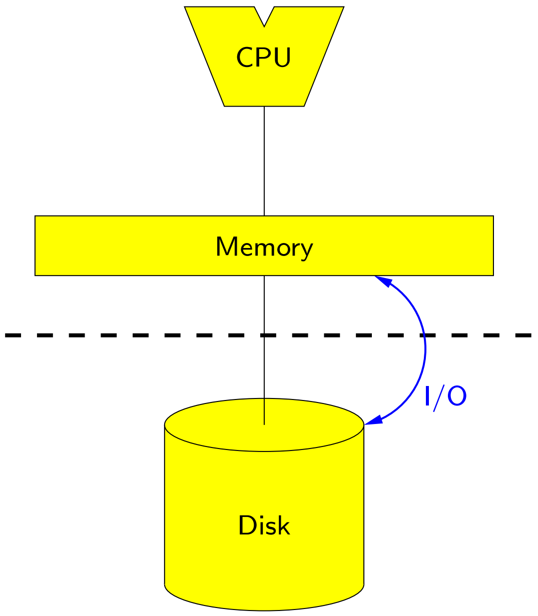
Goal Develop algorithms that are optimized w.r.t. memory hierarchies without knowing the parameters

Cache-Oblivious Model



- I/O model
- Algorithms do not know the parameters B and M
- Optimal off-line cache replacement strategy

Frigo et al. 1999

Justification of the ideal-cache model

Optimal replacement

LRU + 2 $\times$ cache size $\Rightarrow$ at most 2 $\times$ cache misses Sleator an Tarjan, 1985

Corollary

$T_{M,B}(N) = O(T_{2M,B}(N)) \Rightarrow$ #cache misses using LRU is $O(T_{M,B}(N))$

Two memory levels

Optimal cache-oblivious algorithm satisfying $T_{M,B}(N) = O(T_{2M,B}(N))$
$\Rightarrow$ optimal #cache misses on each level of a multilevel cache using LRU

Fully associativity cache

Simulation of LRU

- Direct mapped cache
- Explicit memory management
- Dictionary (2-universal hash functions) of cache lines in memory
- Expected $O(1)$ access time to a cache line in memory

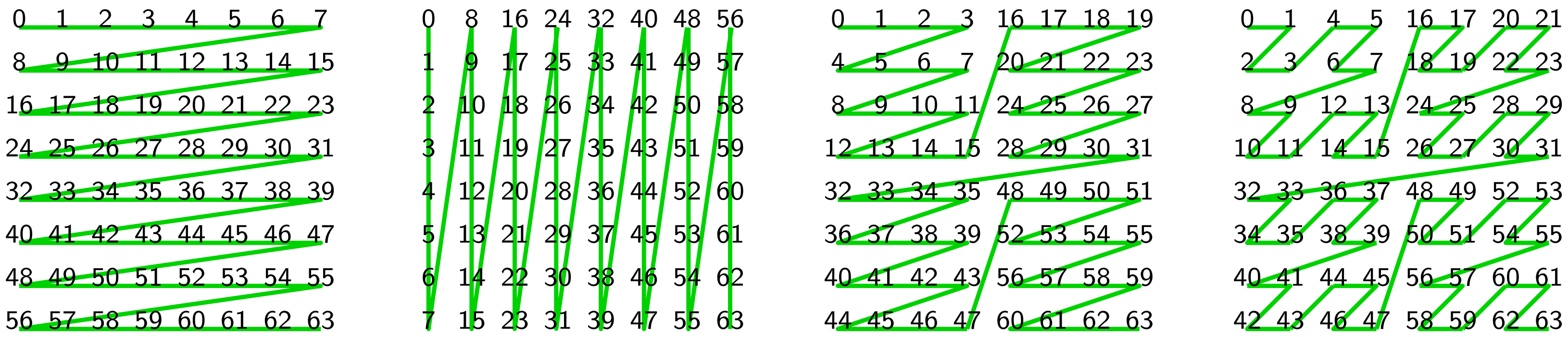
Matrix Multiplication

Matrix Multiplication

Problem

$$C = A \cdot B \ , \quad c_{ij} = \sum_{k=1..N} a_{ik} \cdot b_{kj}$$

Layout of matrices



Row major | Column major | 4×4-blocked | Bit interleaved

Matrix Multiplication

Algorithm 1: Nested loops

- Row major
- Reading a column of B uses N I/Os
- Total $O(N^3)$ I/Os

for $i = 1$ to N
  for $j = 1$ to N
    $c_{ij} = 0$
    for $k = 1$ to N
      $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$

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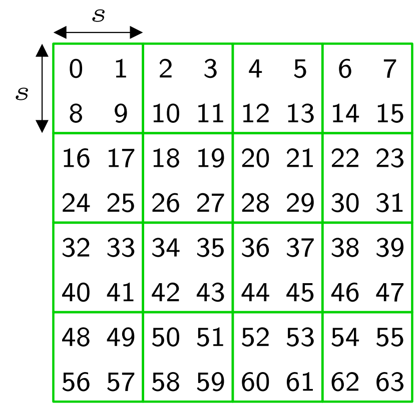
Algorithm 2: Blocked algorithm (cache-aware)

- Partition A and B into blocks of size $s \times s$ where $s = \Theta(\sqrt{M})$
- Apply Algorithm 1 to the $\frac{N}{s} \times \frac{N}{s}$ matrices where elements are $s \times s$ matrices



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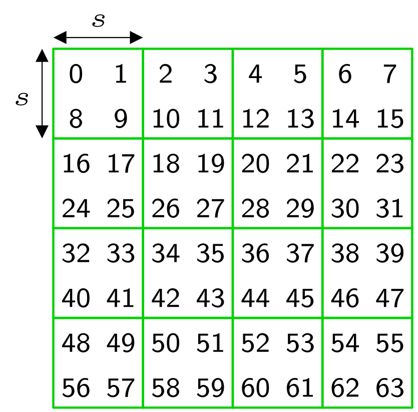

$$O\left(\left(\frac{N}{s}\right)^3 \cdot \frac{s^2}{B}\right) = O\left(\frac{N^3}{s \cdot B}\right) = O\left(\frac{N^3}{B\sqrt{M}}\right) \text{ I/Os}$$

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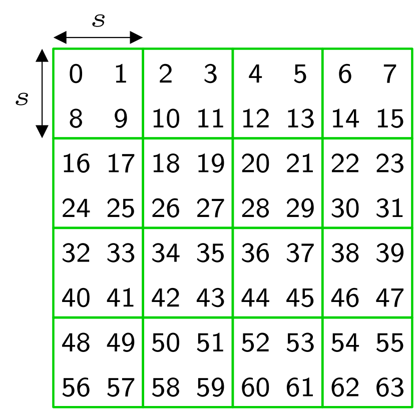

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- Optimal

Hong & Kung, 1981

Matrix Multiplication

Algorithm 3: Recursive algorithm (cache-oblivious)

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

– 8 recursive $\frac{N}{2} \times \frac{N}{2}$ matrix multiplications + 4 $\frac{N}{2} \times \frac{N}{2}$ matrix sums

Matrix Multiplication

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– # I/Os if bit interleaved or (row major and $M = \Omega(B^2)$)

$$T(N) \leq \begin{cases} O(\frac{N^2}{B}) & \text{if } N \leq \varepsilon\sqrt{M} \\ 8 \cdot T\left(\frac{N}{2}\right) + O\left(\frac{N^2}{B}\right) & \text{otherwise} \end{cases}$$

$$T(N) \leq O\left(\frac{N^3}{B\sqrt{M}}\right)$$

Matrix Multiplication

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$$T(N) \leq O\left(\frac{N^3}{B\sqrt{M}}\right)$$

- Optimal — Hong & Kung, 1981
- Non-square matrices — Frigo et al., 1999

Matrix Multiplication

Algorithm 4: Strassen's algorithm (cache-oblivious)

– 7 recursive $\frac{N}{2} \times \frac{N}{2}$ matrix multiplications + $O(1)$ matrix sums

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned}
m_1 &:= (a_{21} + a_{22} - a_{11})(b_{22} - b_{12} + b_{11}) \\
m_2 &:= a_{11} b_{11} \\
m_3 &:= a_{12} b_{21} \\
m_4 &:= (a_{11} - a_{21})(b_{22} - b_{12}) \\
m_5 &:= (a_{21} + a_{22})(b_{12} - b_{11}) \\
m_6 &:= (a_{12} - a_{21} + a_{11} - a_{22}) b_{22} \\
m_7 &:= a_{22}(b_{11} + b_{22} - b_{12} - b_{21})
\end{aligned}$$

$$\begin{aligned}
c_{11} &:= m_2 + m_3 \\
c_{12} &:= m_1 + m_2 + m_5 + m_6 \\
c_{21} &:= m_1 + m_2 + m_4 - m_7 \\
c_{22} &:= m_1 + m_2 + m_4 + m_5
\end{aligned}$$

Matrix Multiplication

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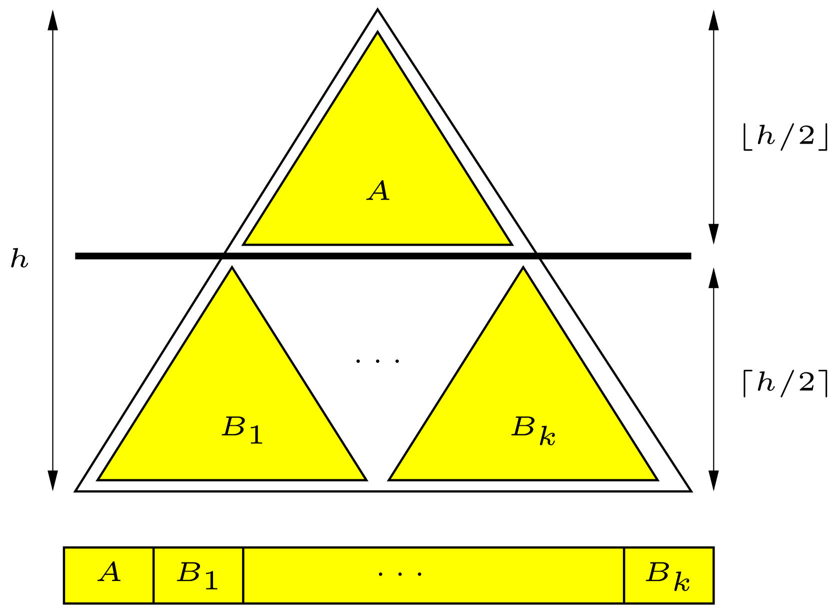
$$T(N) \leq \begin{cases} O(\frac{N^2}{B}) & \text{if } N \leq \varepsilon\sqrt{M} \\ 7 \cdot T\left(\frac{N}{2}\right) + O\left(\frac{N^2}{B}\right) & \text{otherwise} \end{cases}$$

$$T(N) \leq O\left(\frac{N^{\log_2 7}}{B\sqrt{M}}\right) \qquad \log_2 7 \approx 2.81$$

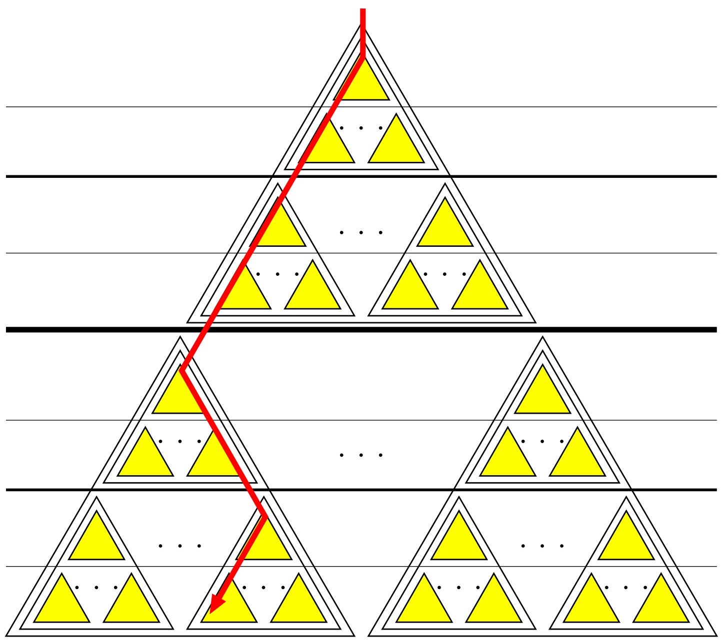
Cache-Oblivious Search Trees

Static Cache-Oblivious Trees

Recursive memory layout $\equiv$ van Emde Boas layout



Degree $O(1)$



Searches use $O(\log_B N)$ I/Os

Prokop 1999

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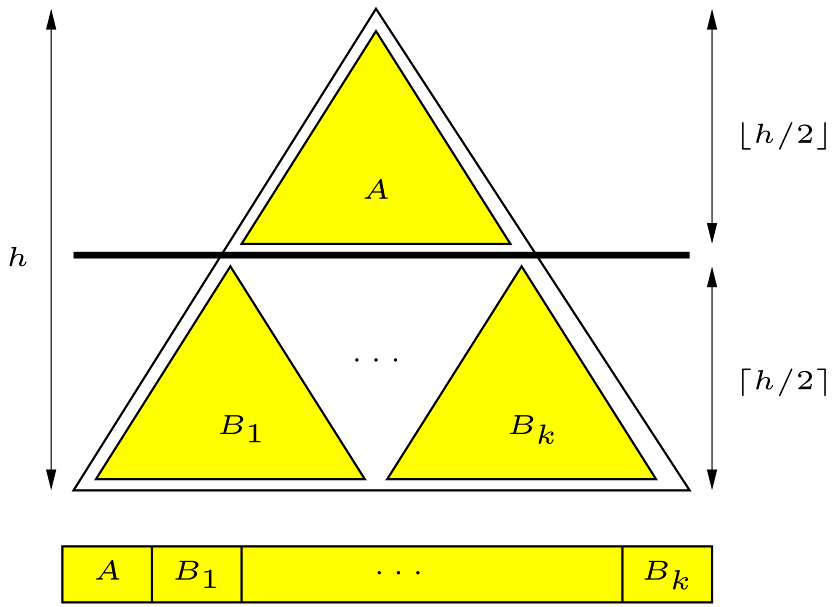

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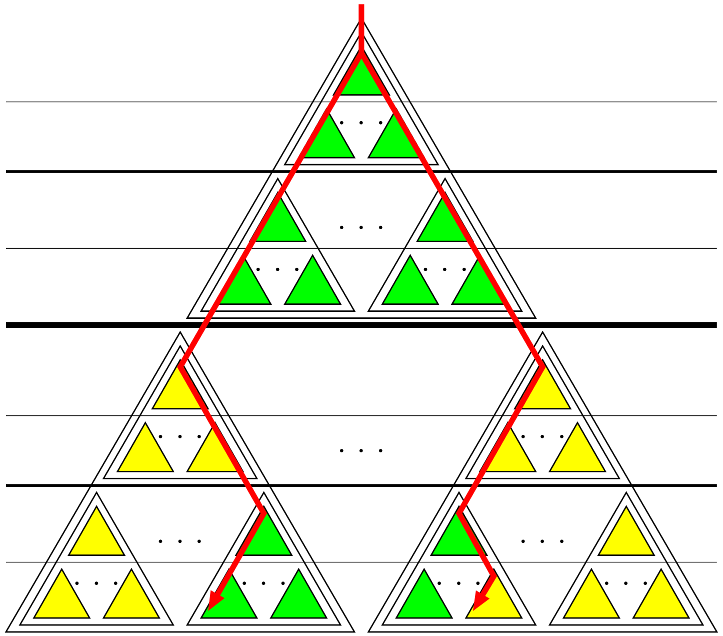

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Range reportings use $O\left(\log_B N + \frac{k}{B}\right)$ I/Os

Prokop 1999

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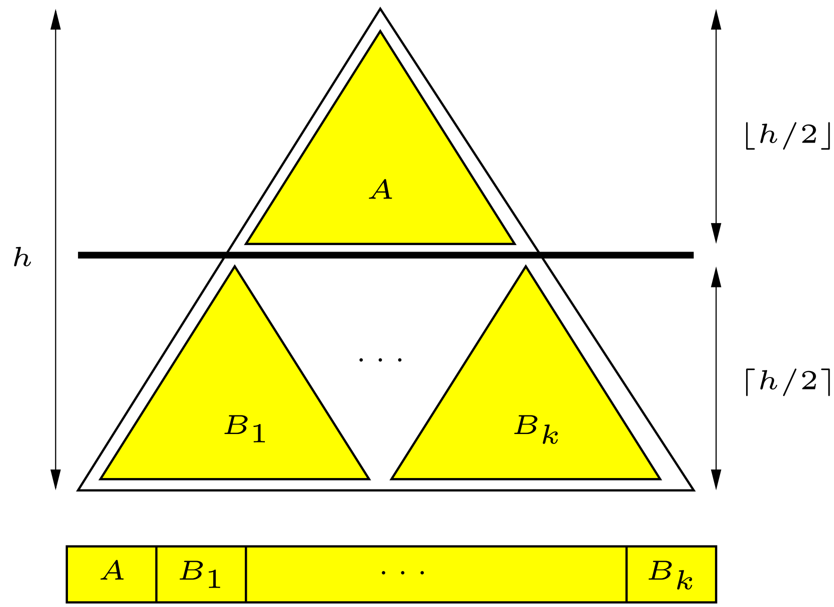

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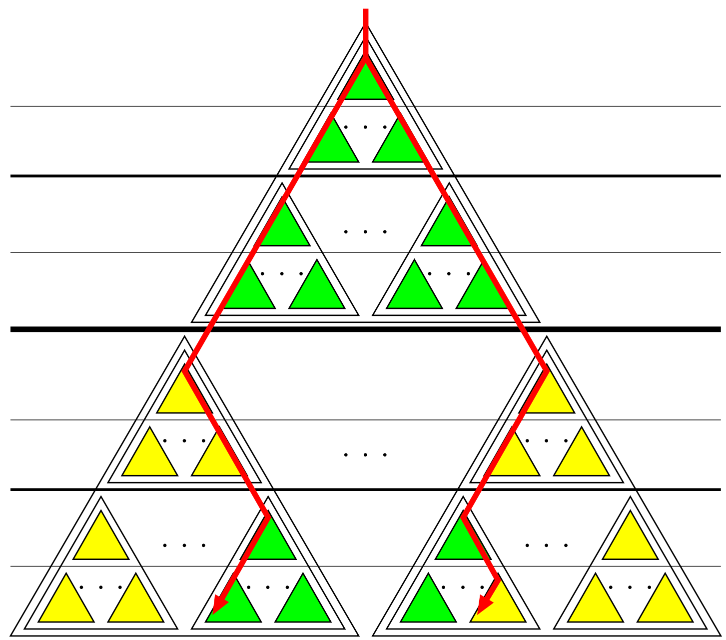


Searches use $\mathrm{O}(\log_B N)$ I/Os

Range reportings use $\mathrm{O}\left(\log_B N + \frac{k}{B}\right)$ I/Os

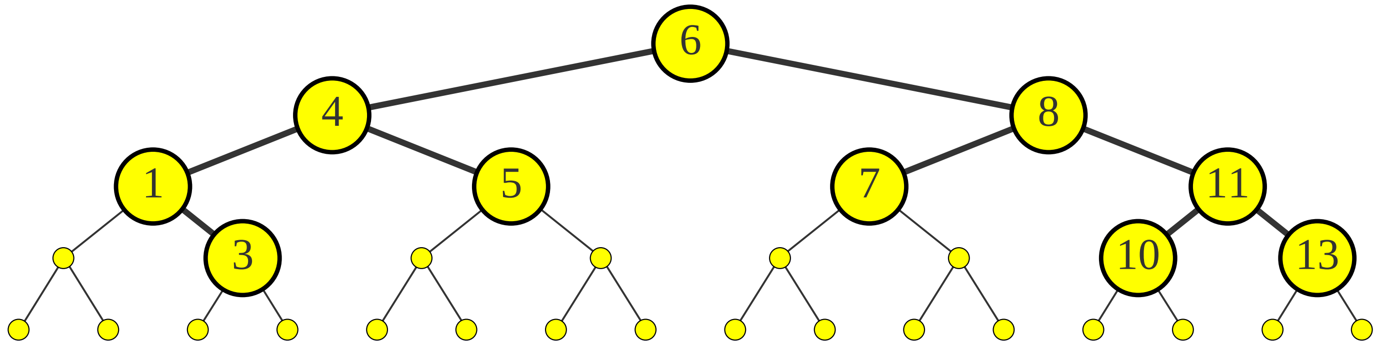
Prokop 1999

Best possible $(\log_2 e + o(1)) \log_B N$

Bender, Brodal, Fagerberg, Ge, He, Hu
Iacono, López-Ortiz 2003

Dynamic Cache-Oblivious Trees

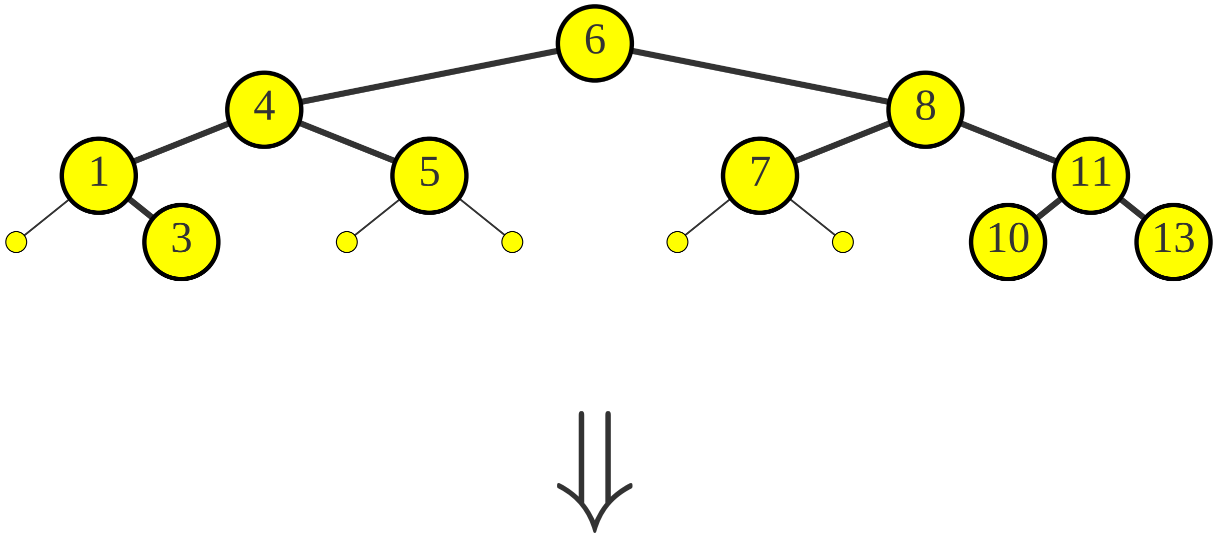
- Embed a dynamic tree of small height into a complete tree
- Static van Emde Boas layout
- Rebuild data structure whenever N doubles of halves



Search	$\mathrm{O}(\log_B N)$
Range Reporting	$\mathrm{O}\left(\log_B N + \frac{k}{B}\right)$
Updates	$\mathrm{O}\left(\log_B N + \frac{\log^2 N}{B}\right)$

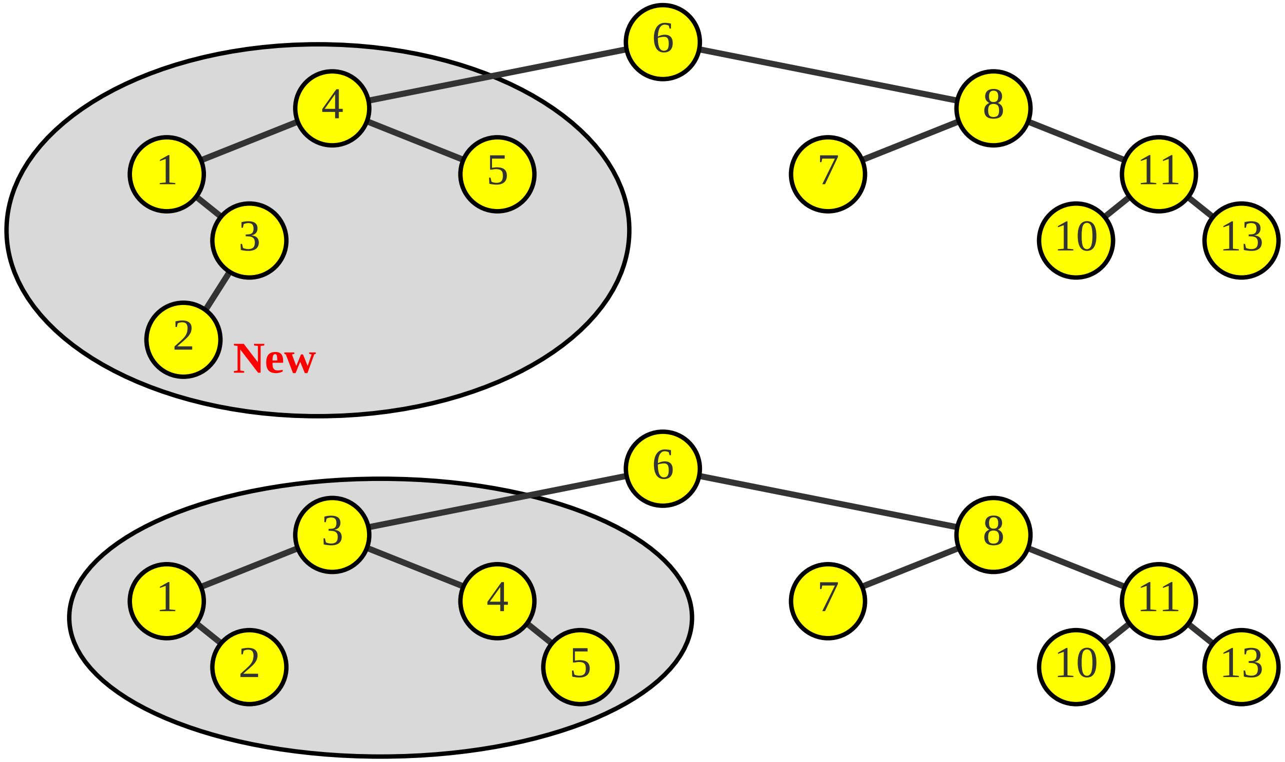
Brodal, Fagerberg, Jacob 2001

Example



6	4	8	1	–	3	5	–	–	7	–	–	11	10	13

Binary Trees of Small Height



- If an insertion causes non-small height then rebuild subtree at nearest ancestor with sufficient few descendents
- Insertions require amortized time $O(\log^2 N)$

Andersson and Lai 1990

Binary Trees of Small Height

- For each level i there is a threshold $\tau_i = \tau_L + i\Delta$, such that $0 < \tau_L = \tau_0 < \tau_1 < \cdots < \tau_H = \tau_U < 1$
- For a node v_i on level i defi nethe density

$$\rho(v_i) = \frac{\#\text{ nodes below } v_i}{m_i}$$

where m_i = # possible nodes below v_i with depth at most H

Insertion

- Insert new element
- If depth $> H$ then locate neirest ancestor v_i with $\rho(v_i) \leq \tau_i$ and rebuild subtree at v_i to have minimum height and elements evenly distributed between left and right subtrees

Andersson and Lai 1990

Binary Trees of Small Height

Theorem Insertions require amortized time $O(\log^2 N)$

Proof Consider two redistributions of v_i

- After the first redistribution $\rho(v_i) \leq \tau_i$
- Before second redistribution a child v_{i+1} of v_i has $\rho(v_{i+1}) > \tau_{i+1}$
- Insertions below v_i : $m(v_{i+1}) \cdot (\tau_{i+1} - \tau_i) = m(v_{i+1}) \cdot \Delta$
- Redistribution of v_i costs $m(v_i)$, i.e. per insertion below v_i

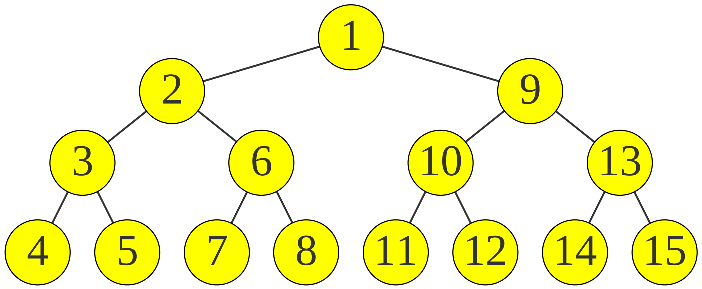
$$\frac{m(v_i)}{m(v_{i+1}) \cdot \Delta} \leq \frac{2}{\Delta}$$

- Total insertion cost per element

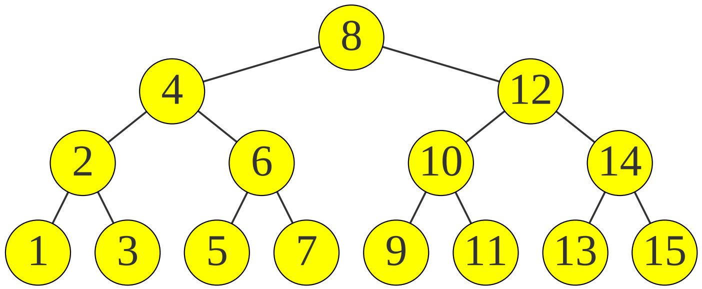
$$\sum_{i=0}^{H} \frac{2}{\Delta} = O(\log^2 N) \qquad \square$$

Andersson and Lai 1990

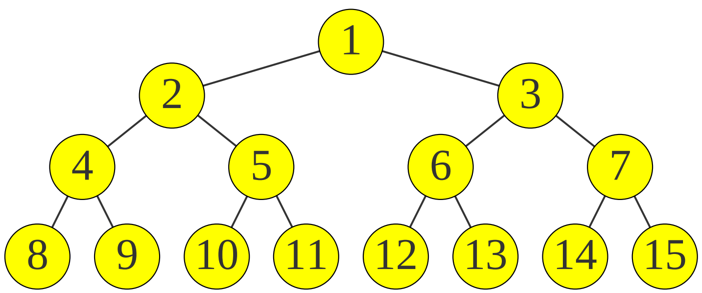
Memory Layouts of Trees



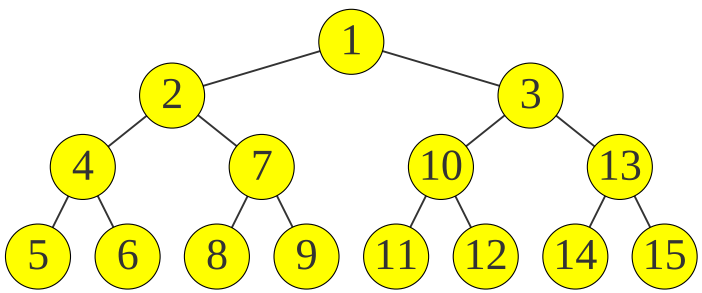
DFS



inorder

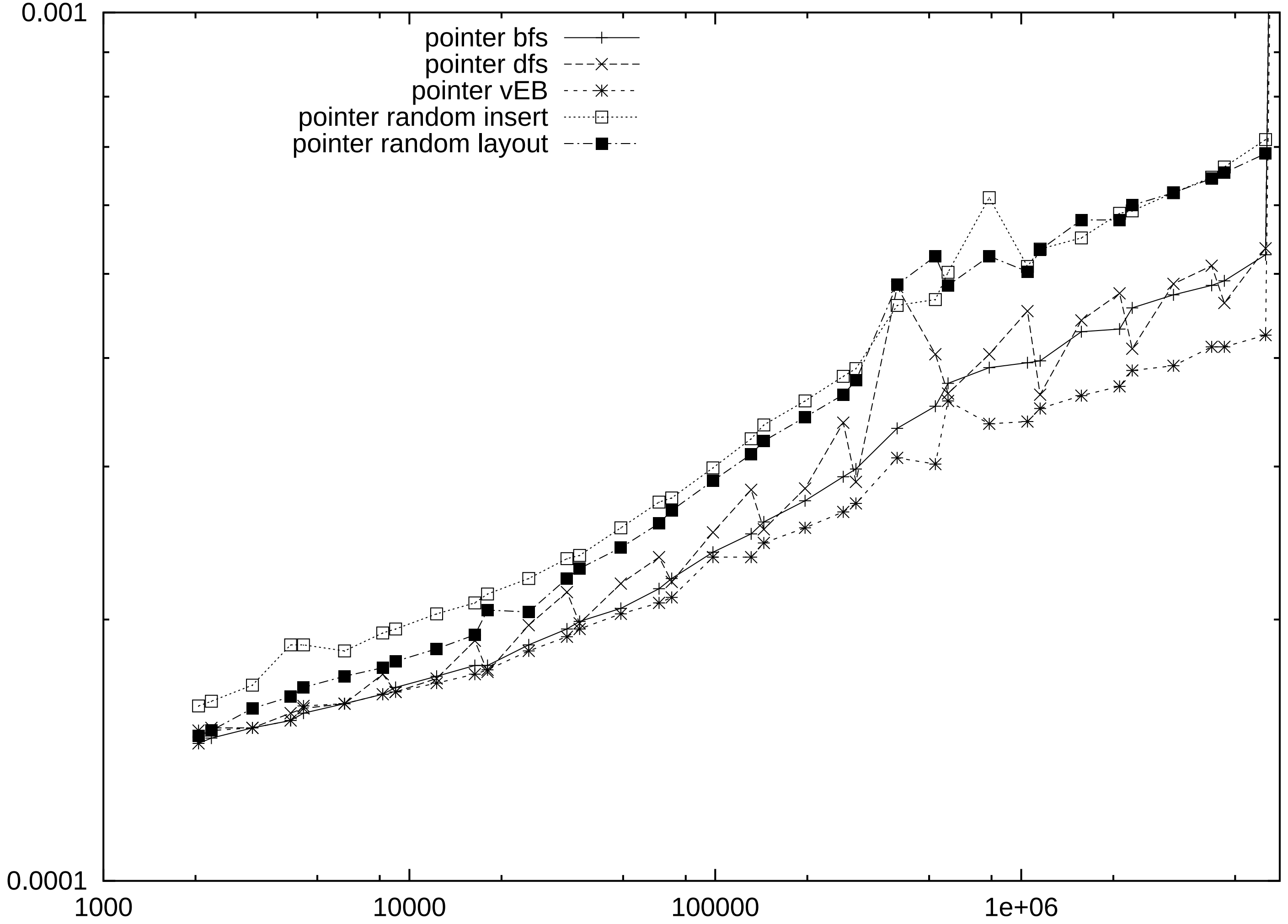


BFS



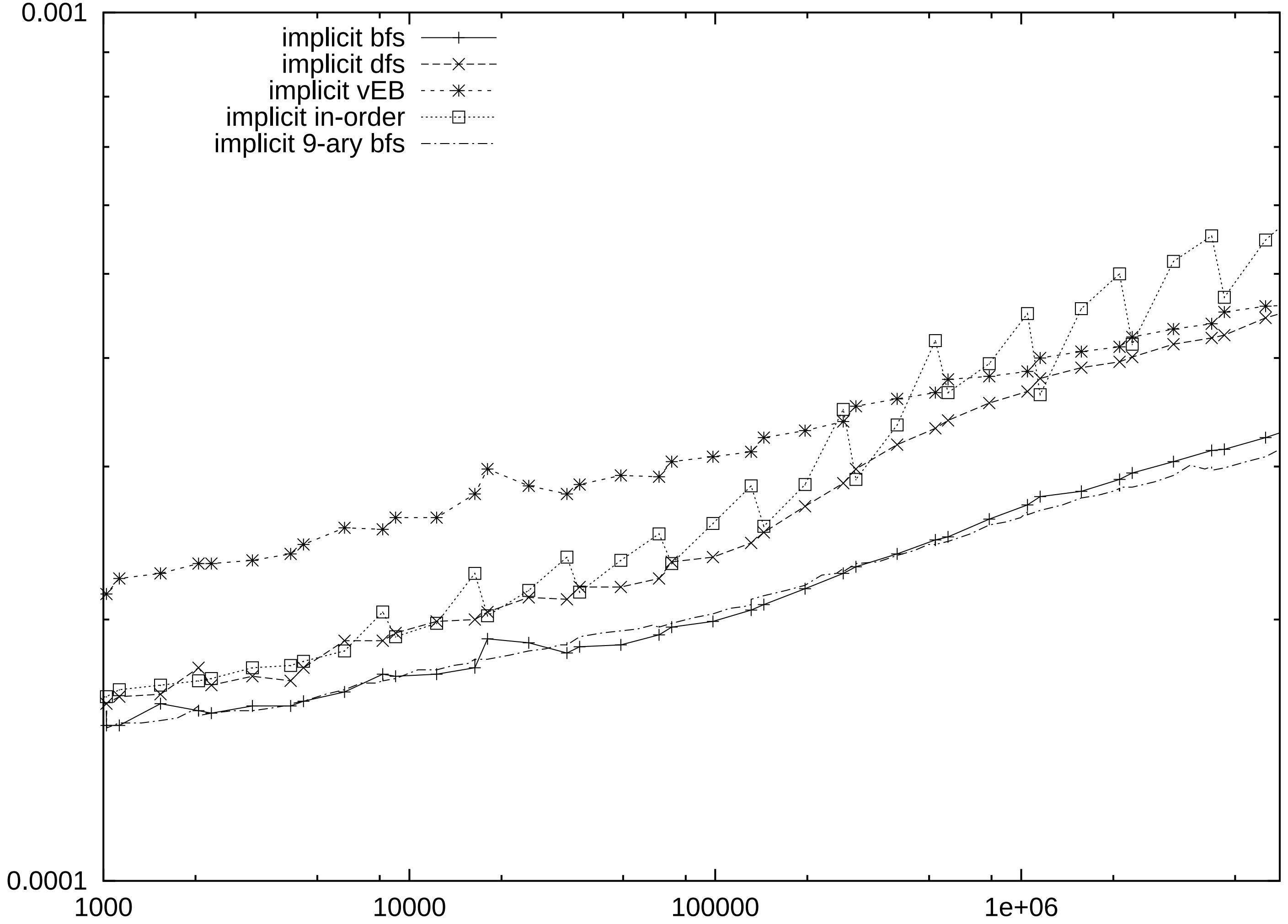
van Emde Boas
(in theory best)

Searches in Pointer Based Layouts



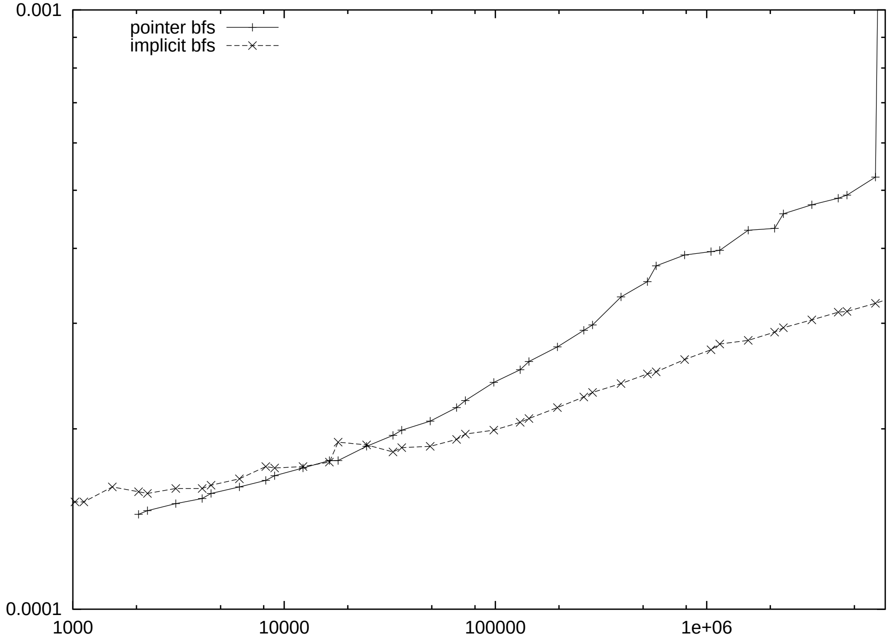
- van Emde Boas layout wins, followed by the BFS layout

Searches with Implicit Layouts

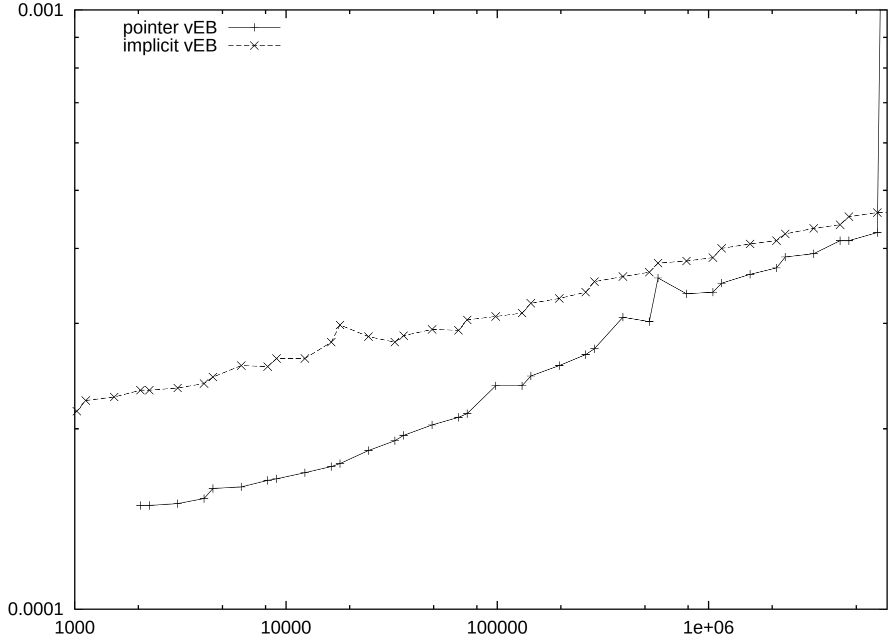


- BFS layout wins due to simplicity and caching of topmost levels
- van Emde Boas layout requires quite complex index computations

Implicit vs Pointer Based Layouts



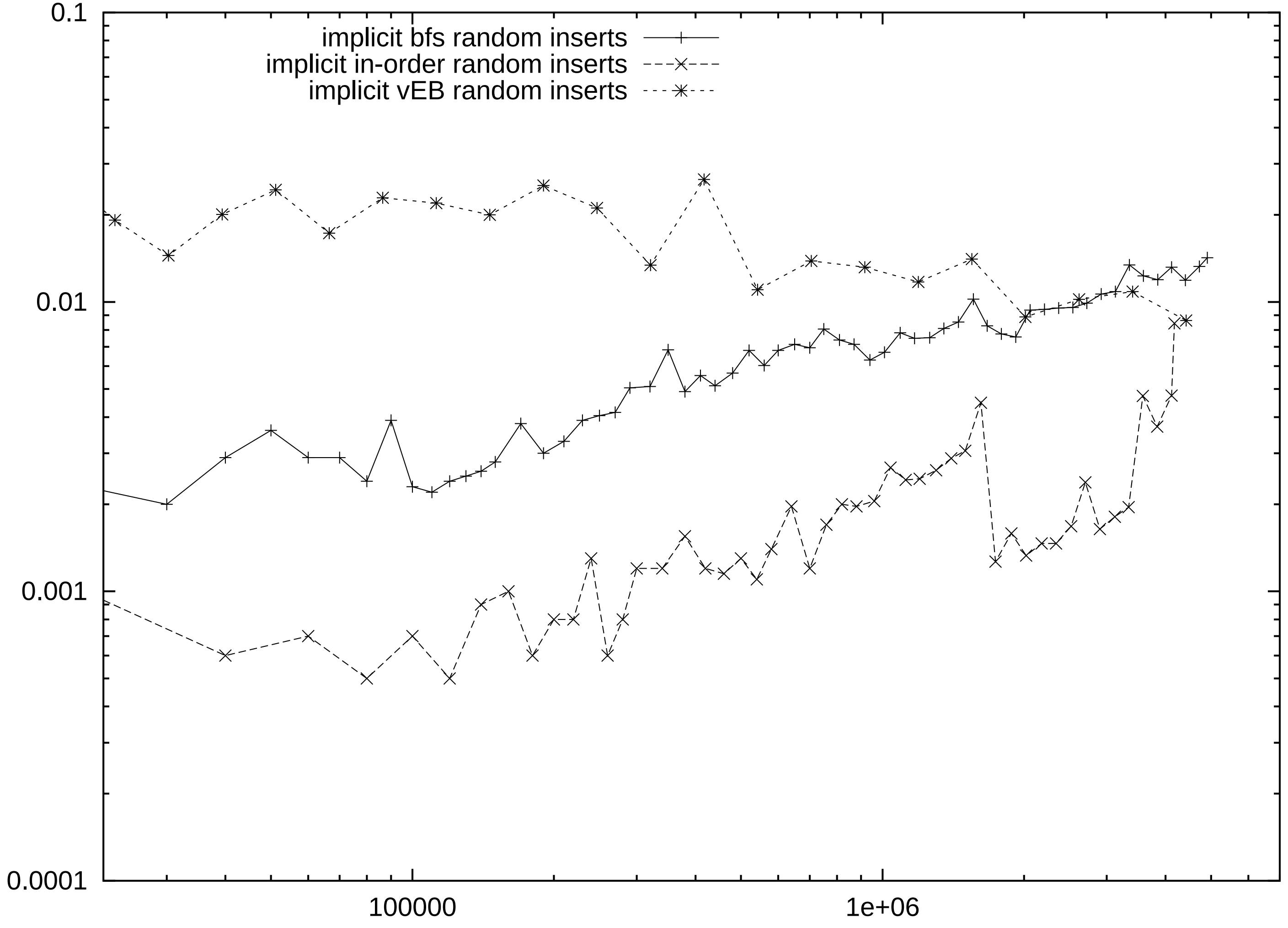
BFS layout



van Emde Boas layout

- Implicit layouts become competitive as n grows

Insertions in Implicit Layouts



- Insertions are rather slow (factor 10-100 over searches)

Summary

- Dynamic cache-oblivious search trees

Search	$O(\log_B N)$
Range Reporting	$O\left(\log_B N + \frac{k}{B}\right)$
Updates	$O\left(\log_B N + \frac{\log^2 N}{B}\right)$

- Update time $O(\log_B N)$ by one level of indirection (implies sub-optimal range reporting)
- Importance of memory layouts
- van Emde Boas layout gives good cache performance
- Computation time is important when considering caches

6	4	8	1	–	3	5	–	–	7	–	–	11	10	13

Cache-Oblivious Sorting

Sorting Problem

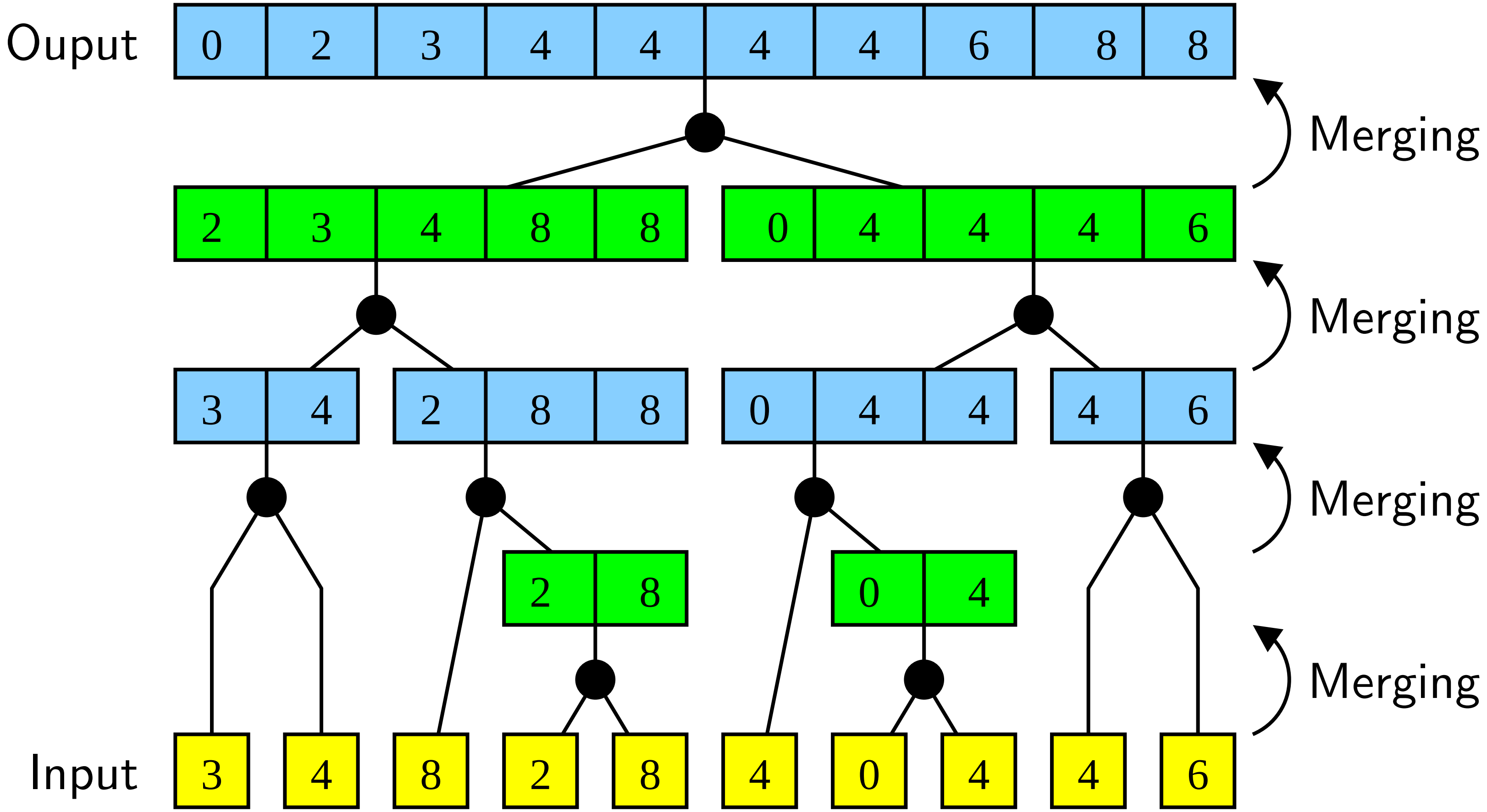
- Input : array containing $x_1, \ldots, x_N$
- Output : array with $x_1, \ldots, x_N$ in sorted order
- Elements can be compared and copied

3	4	8	2	8	4	0	4	4	6

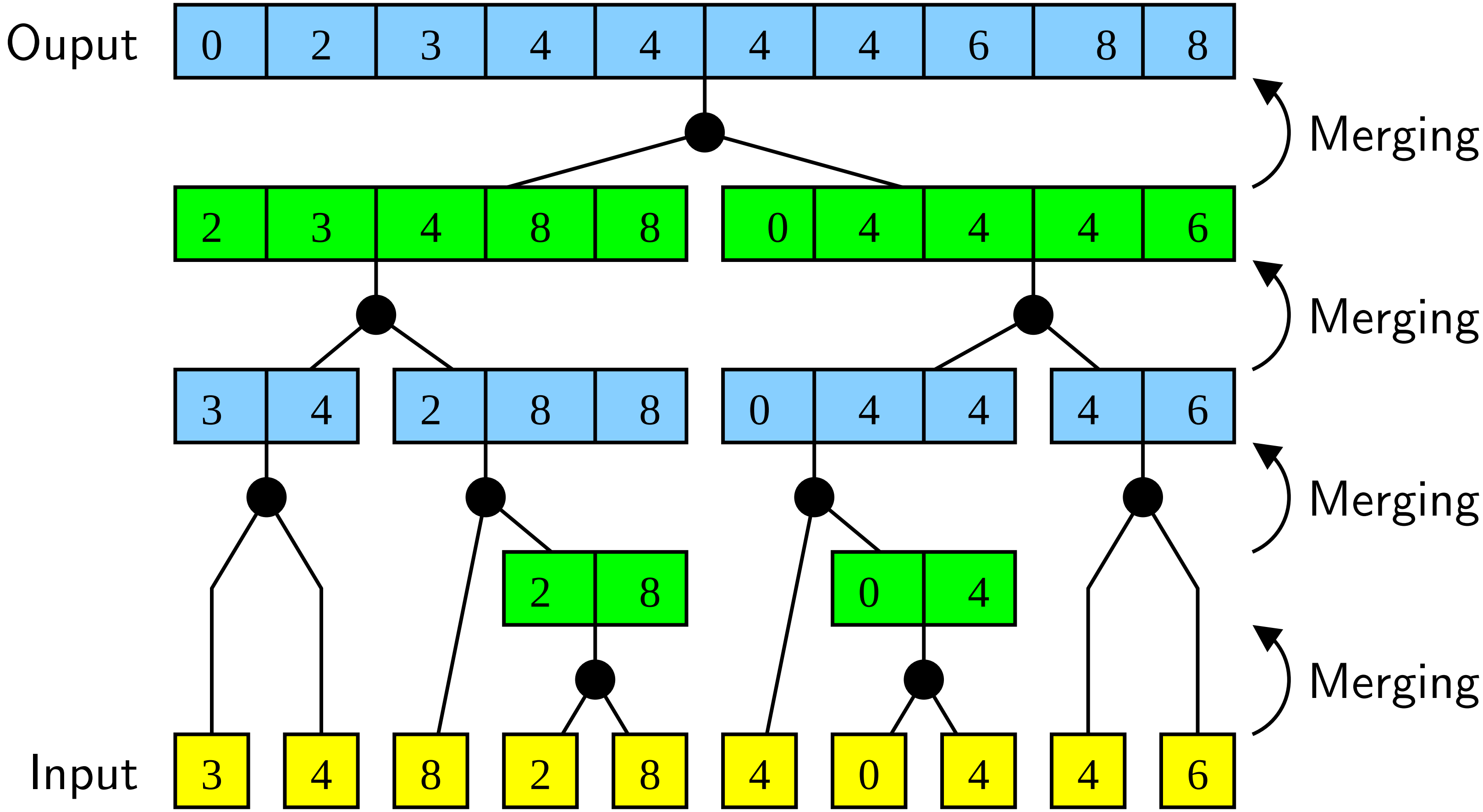
$$\Downarrow$$

0	2	3	4	4	4	4	6	8	8

Binary Merge-Sort



Binary Merge-Sort



- Recursive; two arrays; size $O(M)$ internally in cache
- $O(N \log N)$ comparisons
- $O\left(\frac{N}{B} \log_2 \frac{N}{M}\right)$ I/Os

Merge-Sort

Degree	I/O
2	$O\left(\frac{N}{B}\log_2\frac{N}{M}\right)$
d $(d \leq \frac{M}{B} - 1)$	$O\left(\frac{N}{B}\log_d\frac{N}{M}\right)$
$\Theta\left(\frac{M}{B}\right)$	$O\left(\frac{N}{B}\log_{M/B}\frac{N}{M}\right) = O(\mathrm{Sort}_{M,B}(N))$

Aggarwal and Vitter 1988

Funnel-Sort

2 $(M \geq B^{1+\varepsilon})$	$O(\frac{1}{\varepsilon}\,\mathrm{Sort}_{M,B}(N))$

Frigo, Leiserson, Prokop and Ramachandran 1999

Brodal and Fagerberg 2002

Lower Bound

Brodal and Fagerberg 2003

	Block Size	Memory	I/Os
Machine 1	B_1	M	t_1
Machine 2	B_2	M	t_2

One algorithm, two machines, $B_1 \leq B_2$

Trade-off

$$8t_1B_1 + 3t_1B_1 \log \frac{8Mt_2}{t_1B_1} \geq N \log \frac{N}{M} - 1.45N$$

Lower Bound

	Assumption	I/Os	
Lazy Funnel-sort	$B \leq M^{1-\varepsilon}$	$(a)\ B_2 = M^{1-\varepsilon}$:	$\mathrm{Sort}_{B_2,M}(N)$
		$(b)\ B_1 = 1$:	$\mathrm{Sort}_{B_1,M}(N) \cdot \frac{1}{\varepsilon}$
Binary Merge-sort	$B \leq M/2$	$(a)\ B_2 = M/2$:	$\mathrm{Sort}_{B_2,M}(N)$
		$(b)\ B_1 = 1$:	$\mathrm{Sort}_{B_1,M}(N) \cdot \log M$

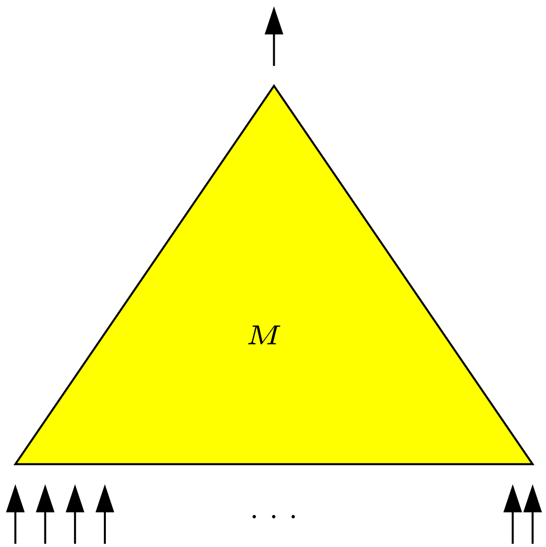
Corollary $(a) \Rightarrow (b)$

Funnel-Sort

k-merger

Frigo et al., FOCS'99

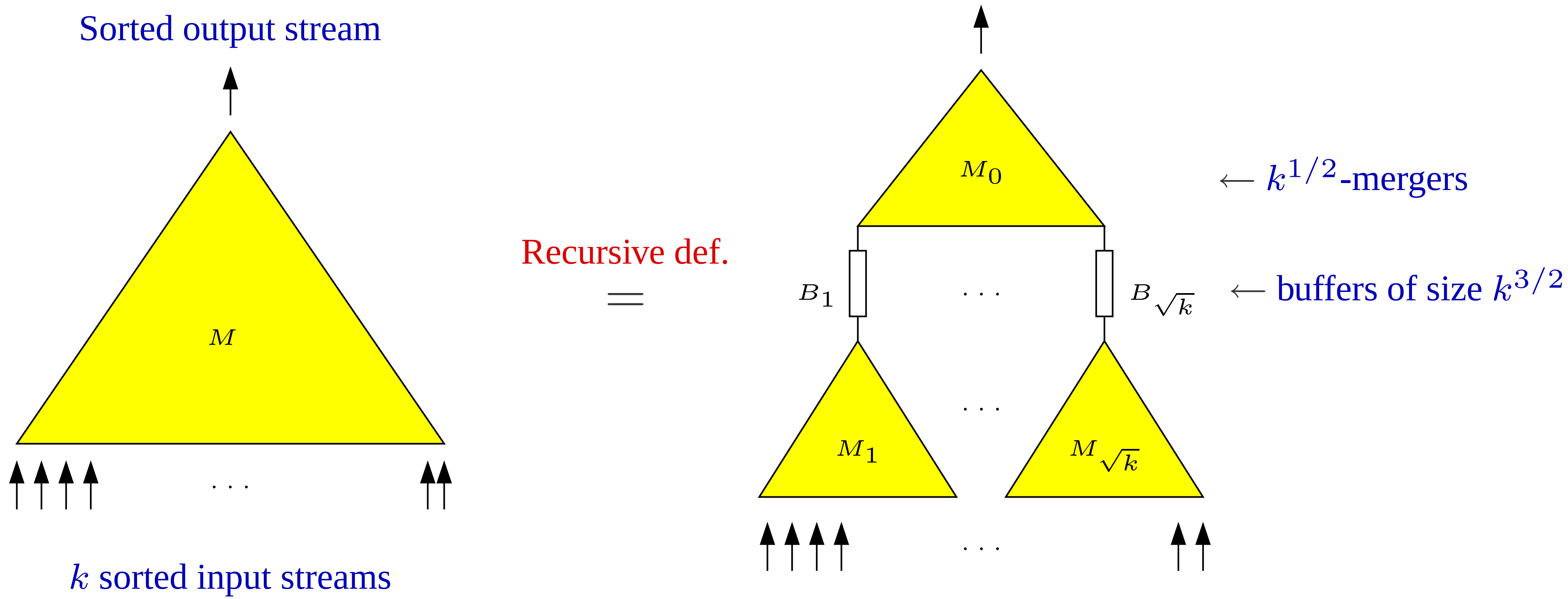
Sorted output stream



k sorted input streams

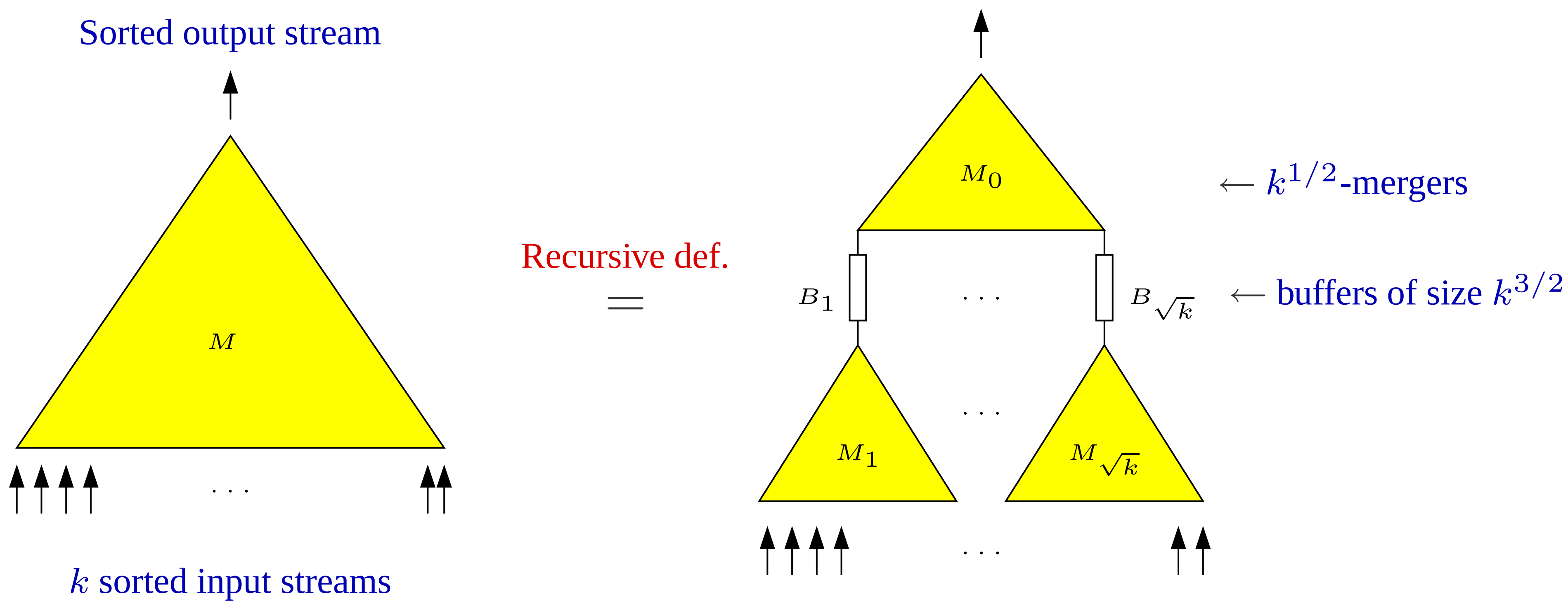
k-merger

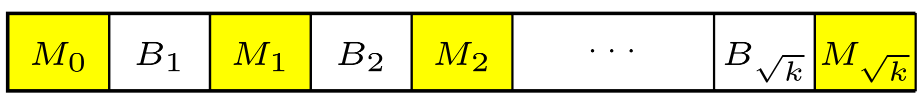
Frigo et al., FOCS'99



k-merger

Frigo et al., FOCS'99

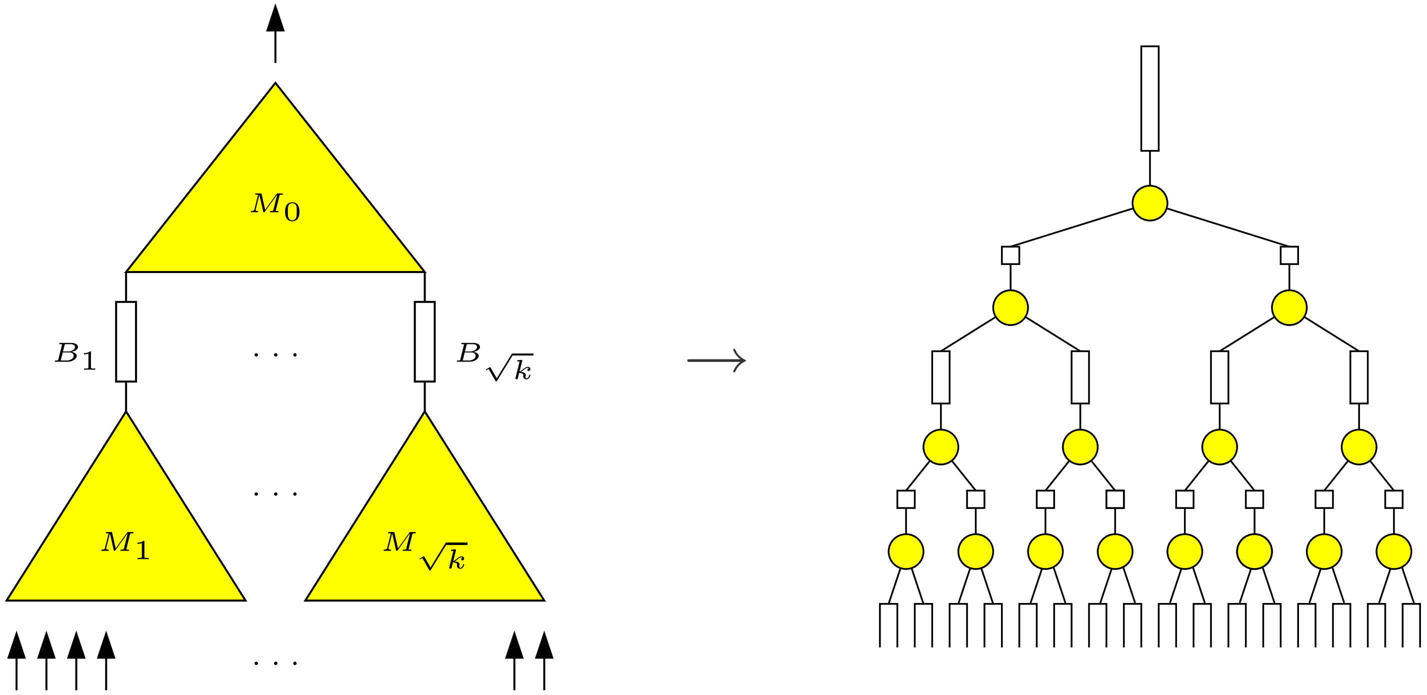




Recursive Layout

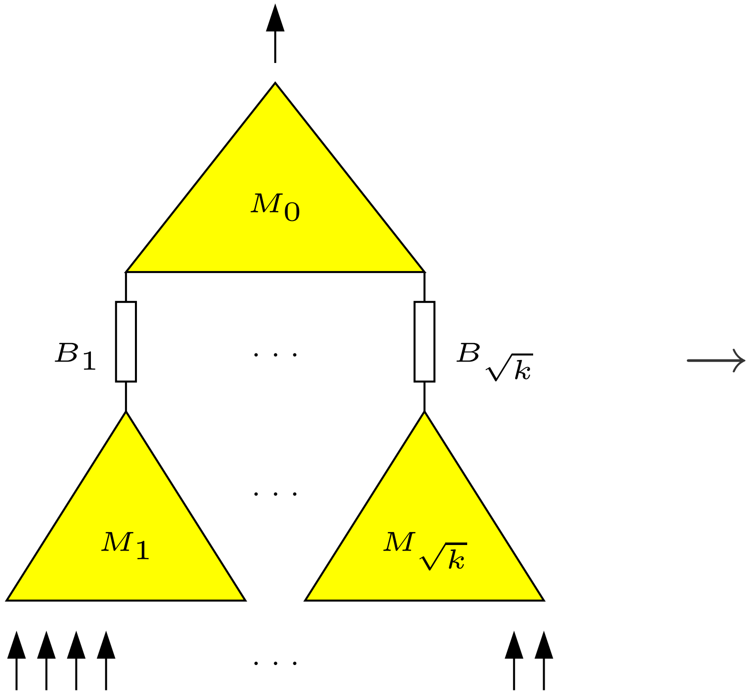
Lazy k-merger

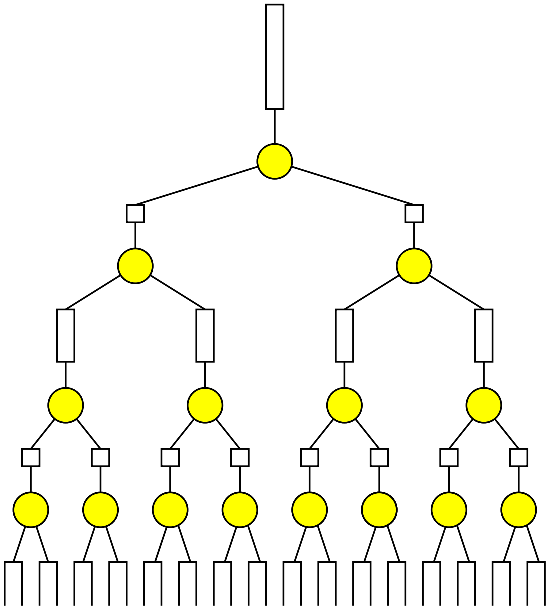
Brodal and Fagerberg 2002



Lazy k-merger

Brodal and Fagerberg 2002

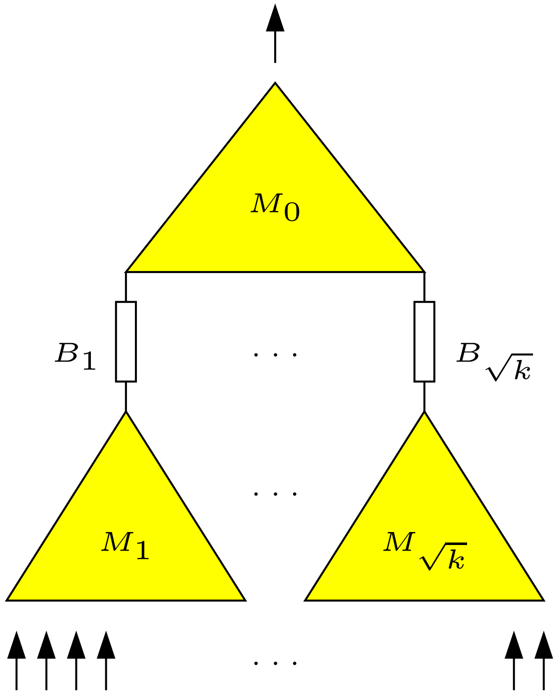




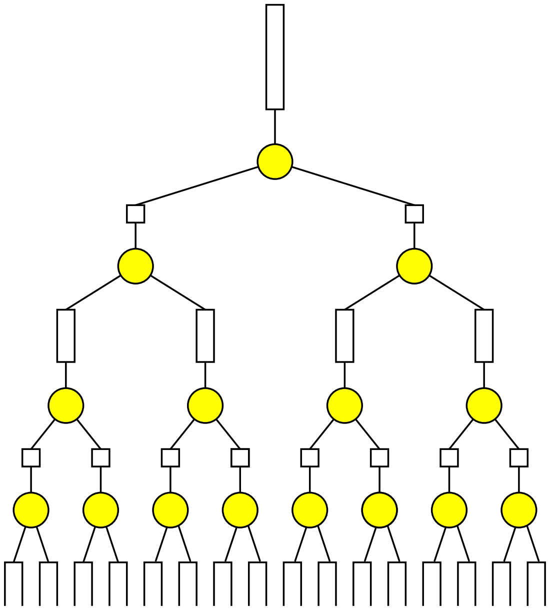
Procedure **Fill**(v)
    while out-buffer not full
        if left in-buffer empty
            Fill(left child)
        if right in-buffer empty
            Fill(right child)
        perform one merge step

Lazy k-merger

Brodal and Fagerberg 2002



$\longrightarrow$



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- **while** out-buffer not full
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    - **Fill**(left child)
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    - **Fill**(right child)
  - perform one merge step

Lemma

If $M \geq B^2$ and output buffer has size k^3 then $O(\frac{k^3}{B} \log_M(k^3) + k)$ I/Os are done during an invocation of **Fill**(root)

Funnel-Sort

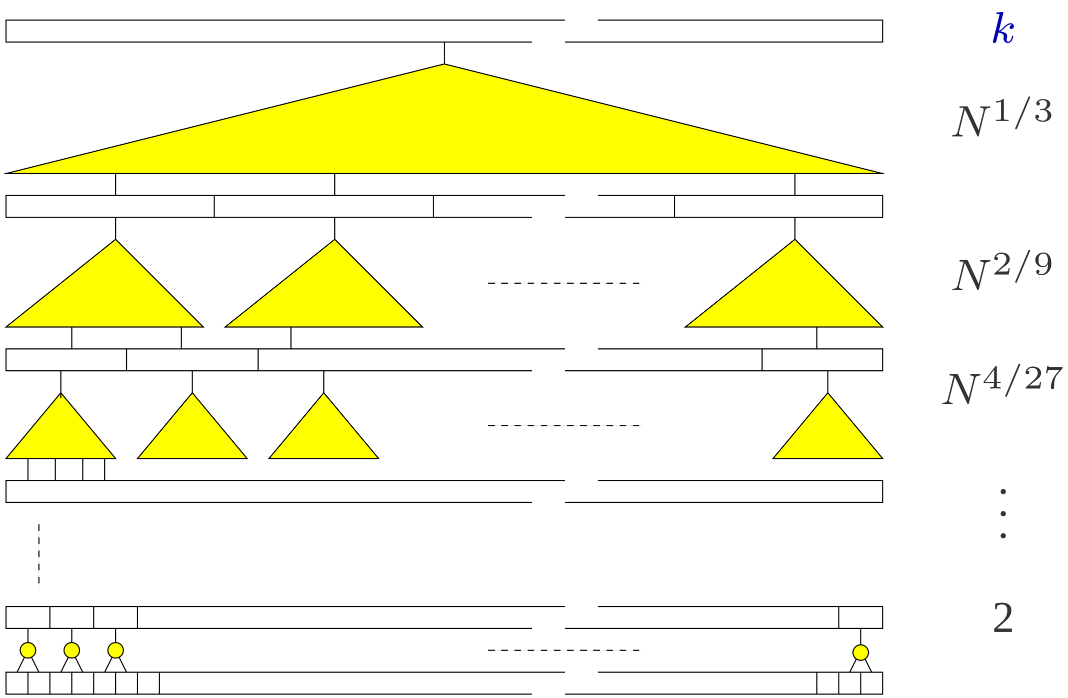
Brodal and Fagerberg 2002

Frigo, Leiserson, Prokop and Ramachandran 1999

Divide input in $N^{1/3}$ segments of size $N^{2/3}$

Recursively **Funnel-Sort** each segment

Merge sorted segments by an $N^{1/3}$-merger



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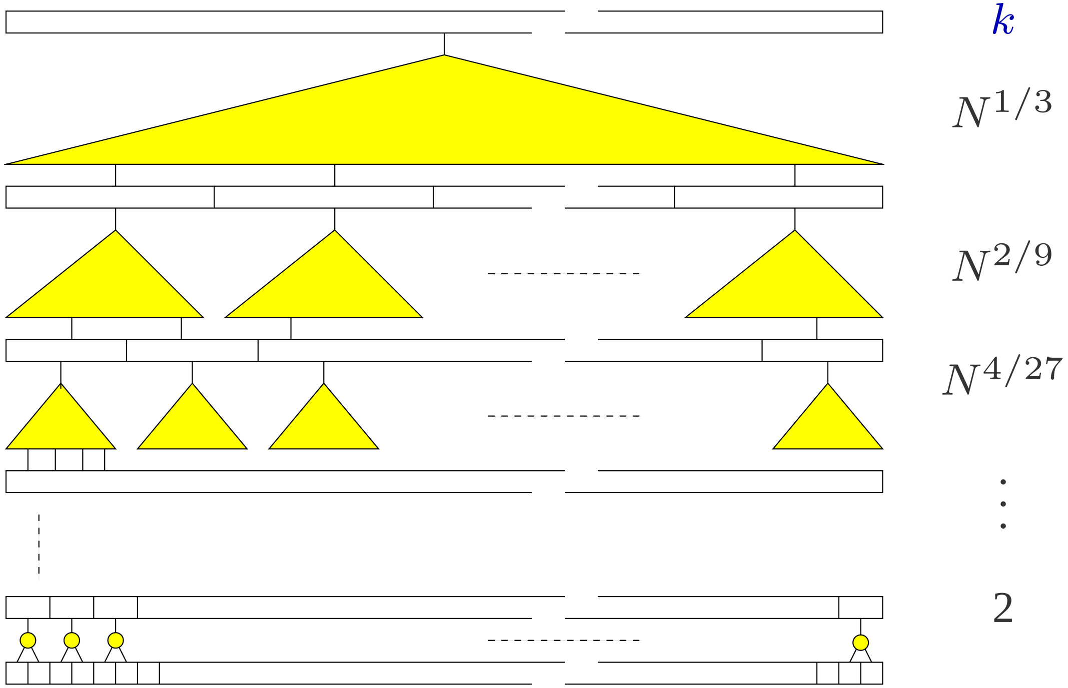
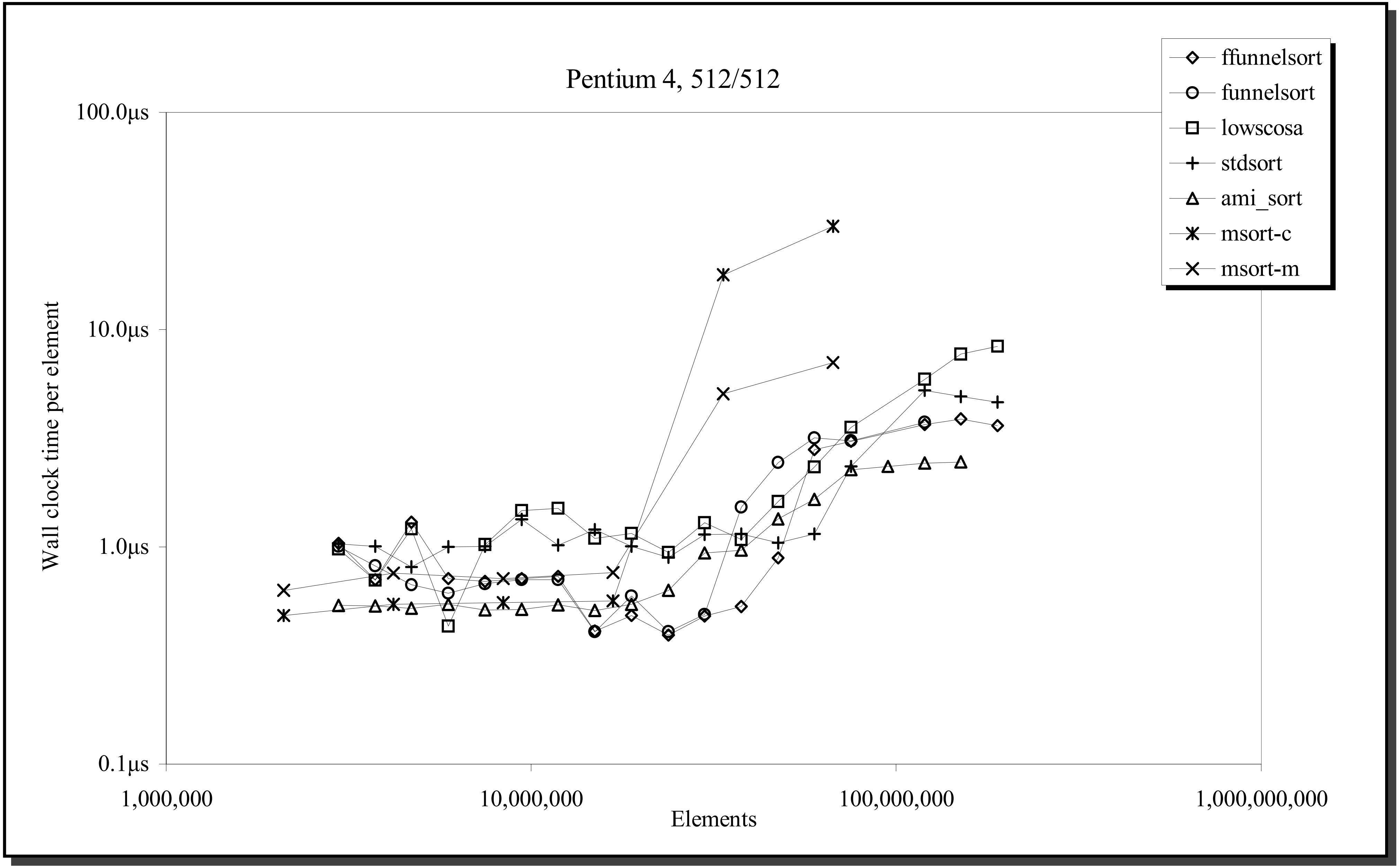


Theorem Funnel-Sort performs $O(\mathrm{Sort}_{M,B}(N))$ I/Os for $M \geq B^2$

Hardware

Processor type	Pentium 4	Pentium 3	MIPS 10000
Workstation	Dell PC	Delta PC	SGI Octane
Operating system	GNU/Linux Kernel version 2.4.18	GNU/Linux Kernel version 2.4.18	IRIX version 6.5
Clock rate	2400 MHz	800 MHz	175 MHz
Address space	32 bit	32 bit	64 bit
Integer pipeline stages	20	12	6
L1 data cache size	8 KB	16 KB	32 KB
L1 line size	128 Bytes	32 Bytes	32 Bytes
L1 associativity	4 way	4 way	2 way
L2 cache size	512 KB	256 KB	1024 KB
L2 line size	128 Bytes	32 Bytes	32 Bytes
L2 associativity	8 way	4 way	2 way
TLB entries	128	64	64
TLB associativity	Full	4 way	64 way
TLB miss handler	Hardware	Hardware	Software
Main memory	512 MB	256 MB	128 MB

Wall Clock



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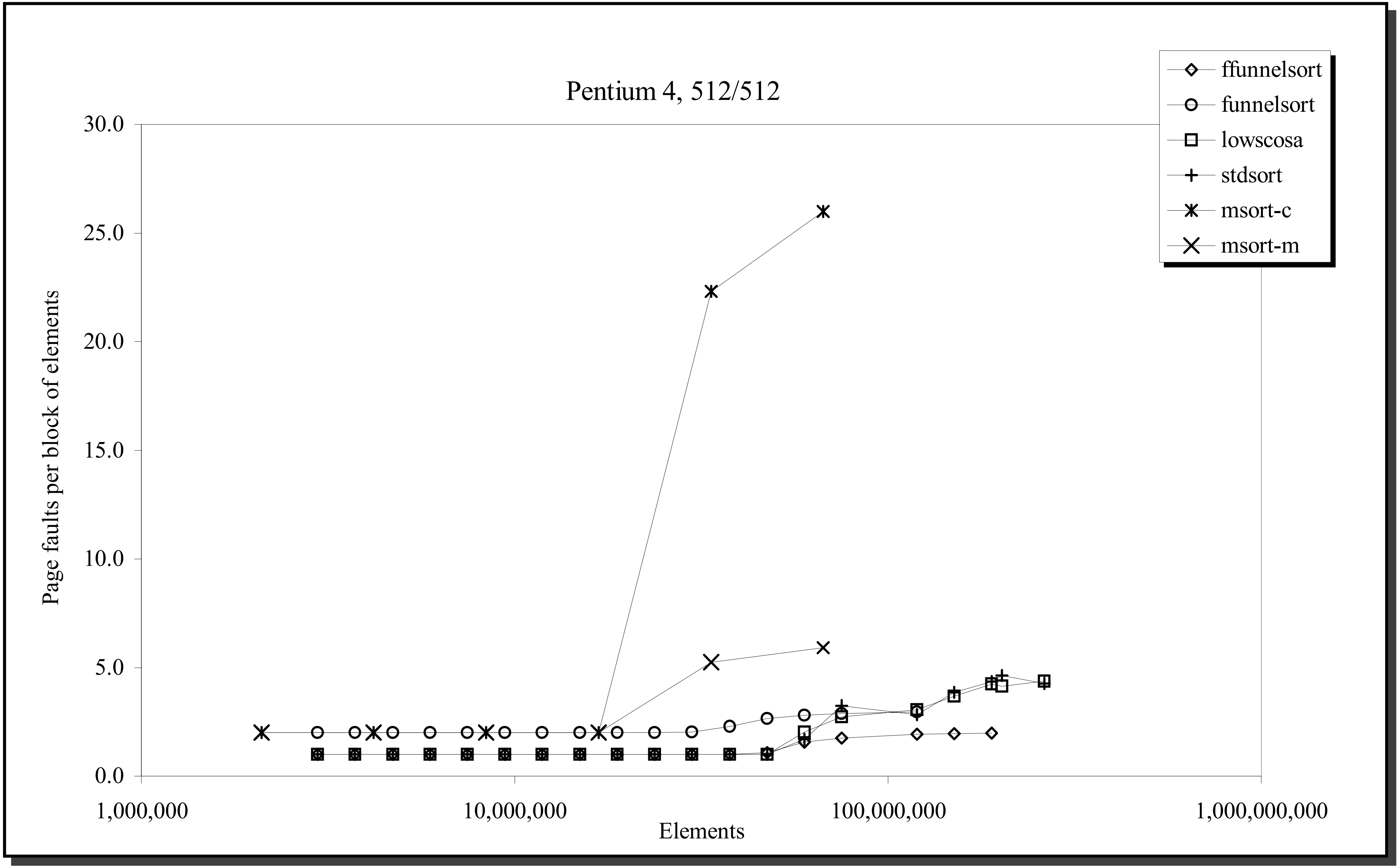
Page Faults



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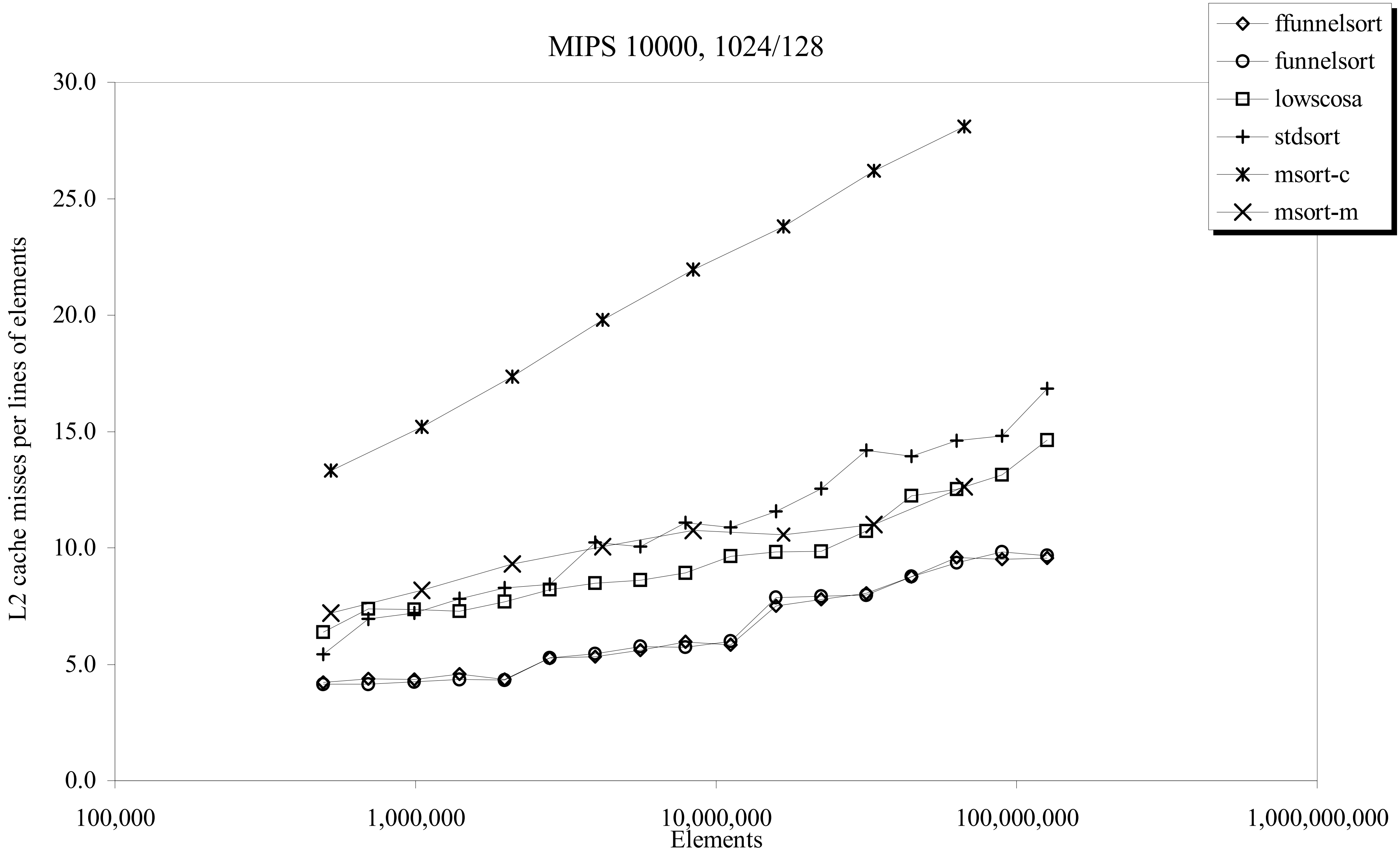
Cache Misses



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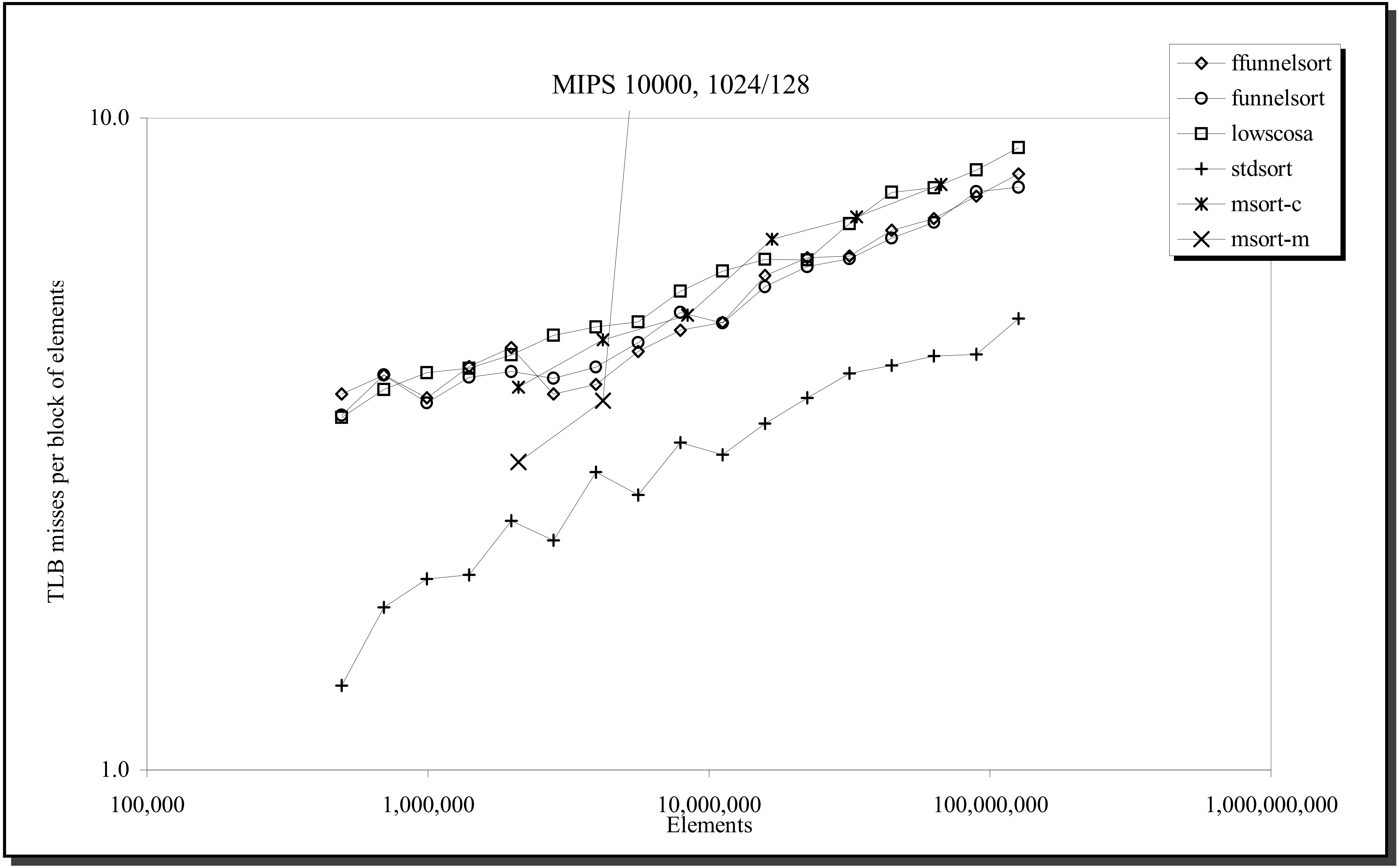
TLB Misses



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Conclusions

Cache oblivious sorting

- is possible
- requires a tall cache assumption $M \geq B^{1+\varepsilon}$
- comparable performance with cache aware algorithms

Future work

- more experimental justifi cation for the cache oblivious model
- limitations of the model — time space trade-offs ?
- tool-box for cache oblivious algorithms