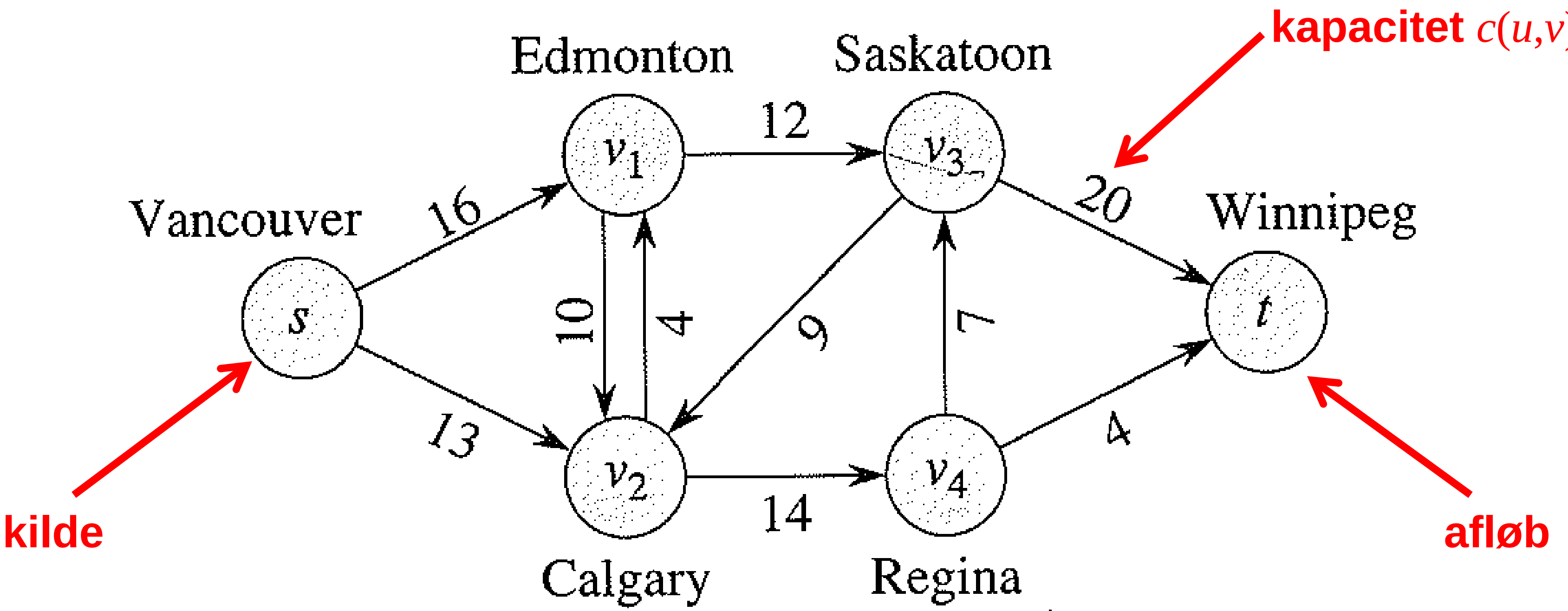
# Algoritmer og Datastrukturer 2

**Gerth Stølting Brodal**

**Maksimale Strømninger [CLRS, kapitel 26.1-26.3]**

AARHUS UNIVERSITET

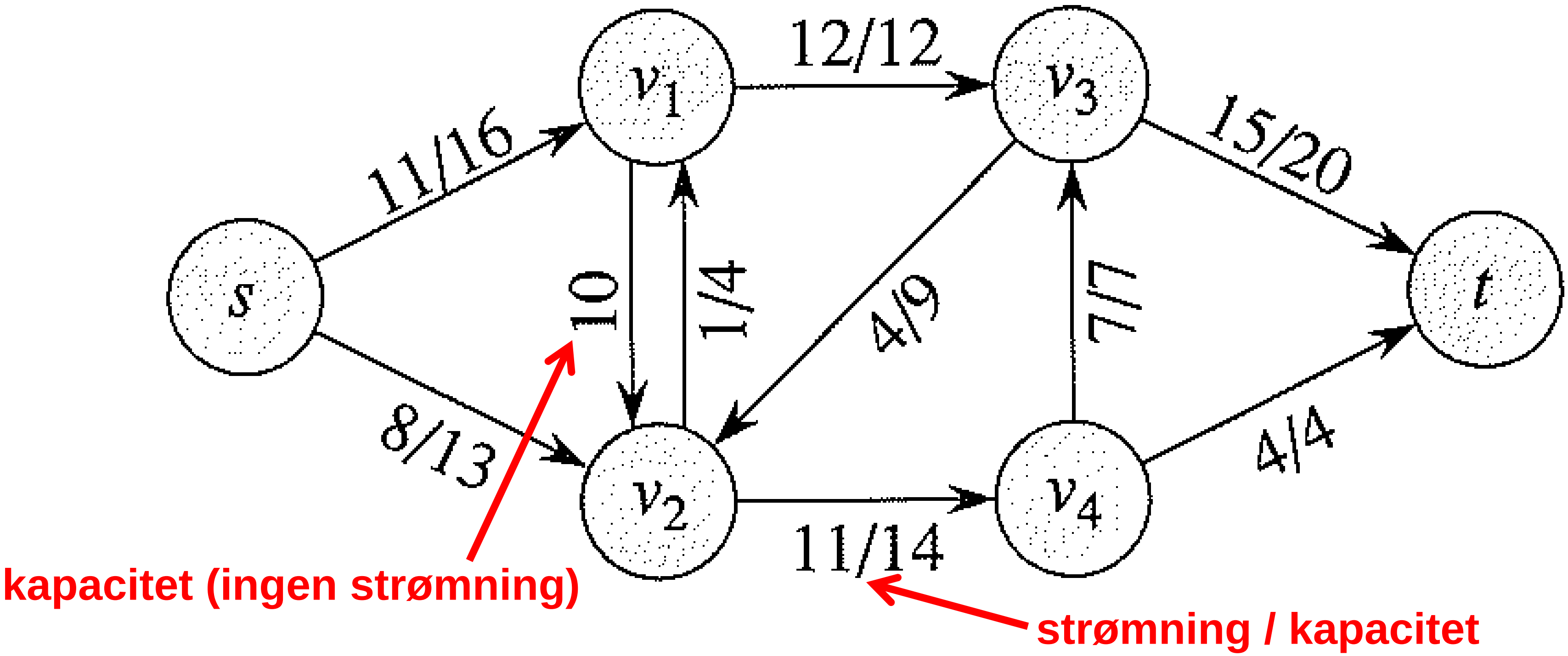
# Maximale Strømninger i Netværk




**Input:** En orienteret graf $G=(V,E)$, hvor alle kanter $(u,v)$ har en kapacitet $c(u,v)$, og to knuder kilde $s \in V$ (source), og afløbet $t \in V$ (sink).

**Output:** En maximal strømning $f$ i netværket fra $s$ til $t$.

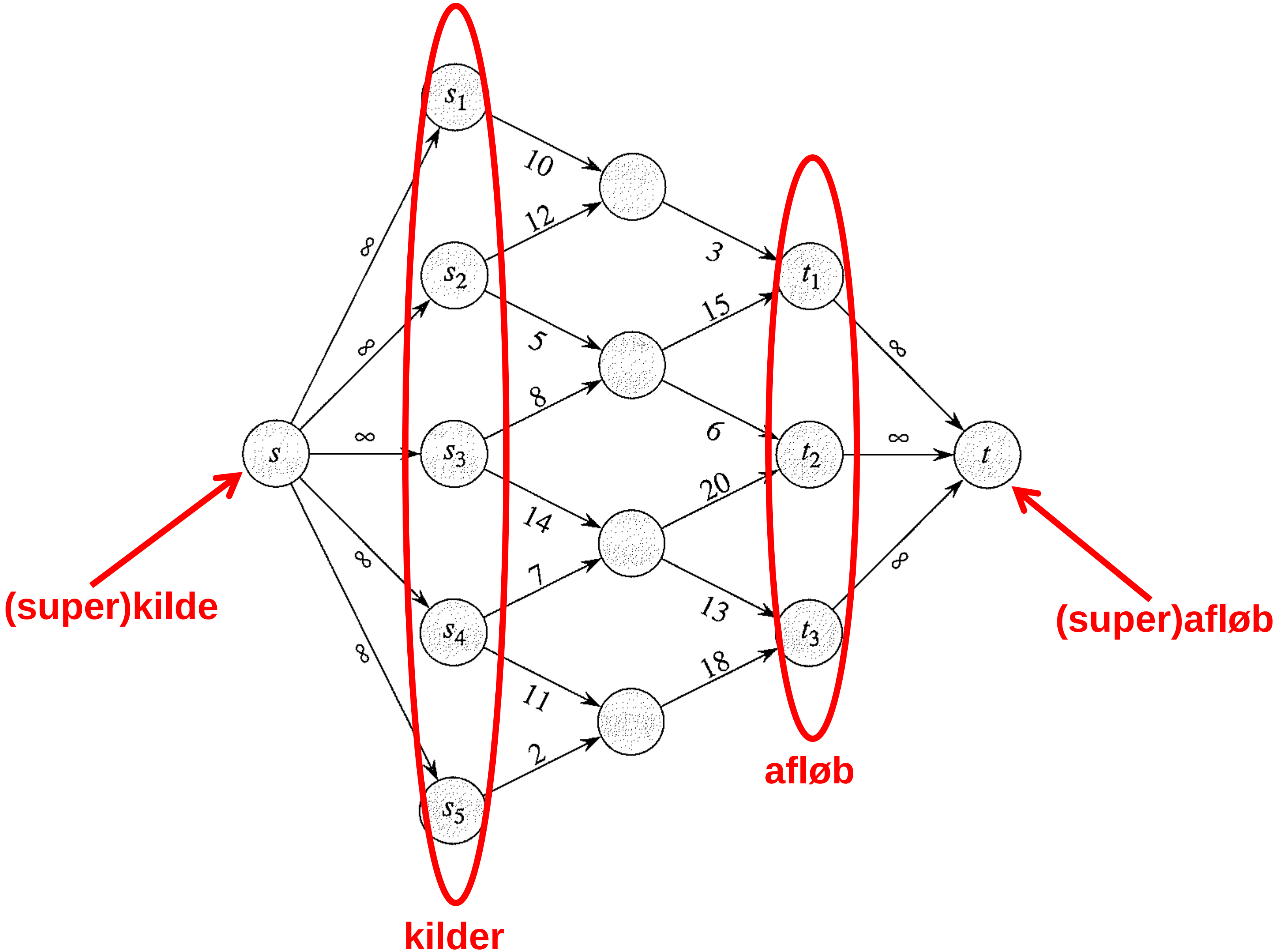
# Maximale Strømninger i Netværk




**Krav til *f* :**

| | |
|---|---|
| $f(u,v) = -f(v,u)$ | for alle kanter (strømbevaring) |
| $\Sigma_{v \in V} f(u,v) = 0$ | for alle knuder ≠ $s,t$ (strøm in=strøm ud) |
| $f(u,v) \leq c(u,v)$ | for alle kanter (strømbevaring) |

# Flere kilder og afløb

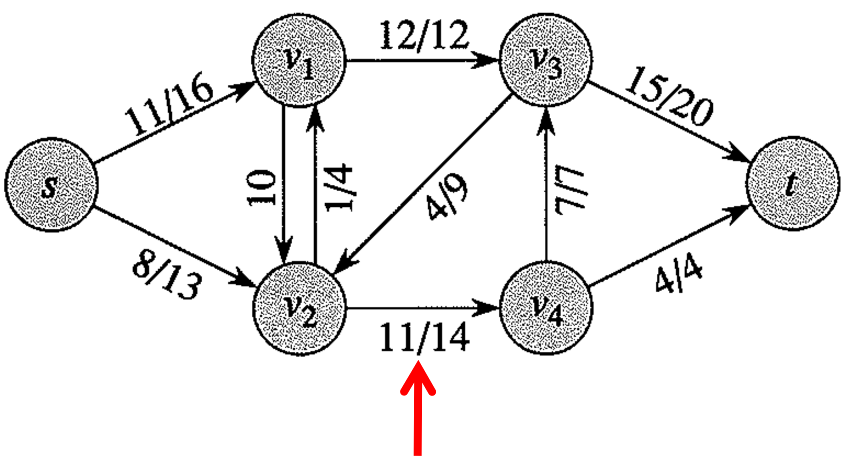


# Ford-Fulkerson

FORD-FULKERSON-METHOD$(G, s, t)$

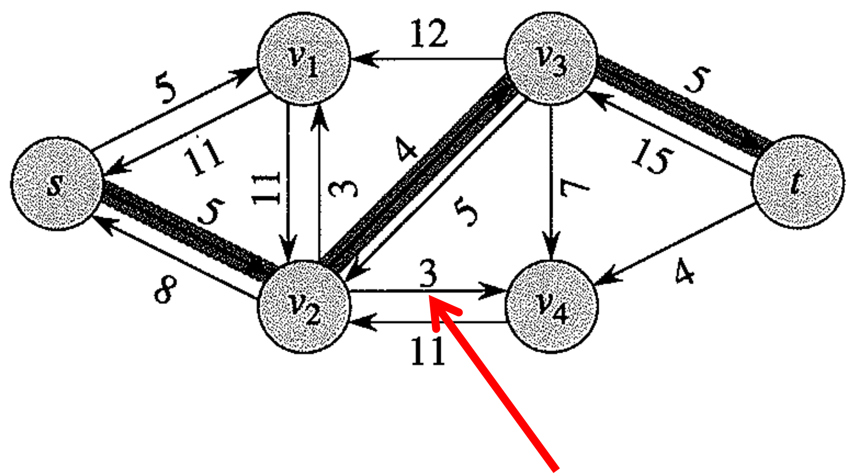
```
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
      augment flow f along p
return f
```

**Strømning / Kapacitet**




$f(u,v)$ / $c(u,v)$

**Rest-netværk & forbedrende sti**




$c_f(u,v)=c(u,v)-f(u,v)$

**Observation:** $s$-$t$ sti i rest-netværket ≡ forbedrende sti i netværket

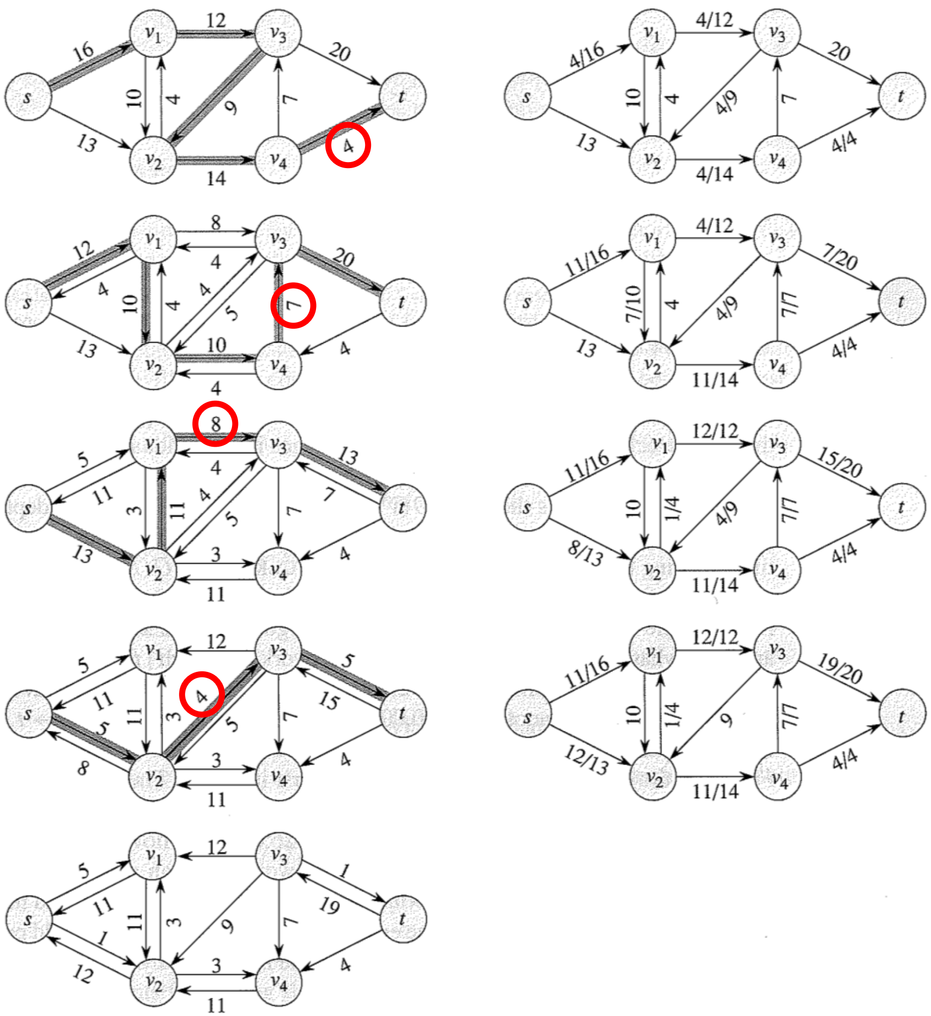
# Ford-Fulkerson

FORD-FULKERSON$(G, s, t)$

**for** each edge $(u, v) \in G.E$
    $(u, v).f = 0$
**while** there exists a path $p$ from $s$ to $t$ in the residual network $G_f$
    $c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
    **for** each edge $(u, v)$ in $p$
        $(u, v).f = (u, v).f + c_f(p)$
        $(v, u).f = (v, u).f - c_f(p)$

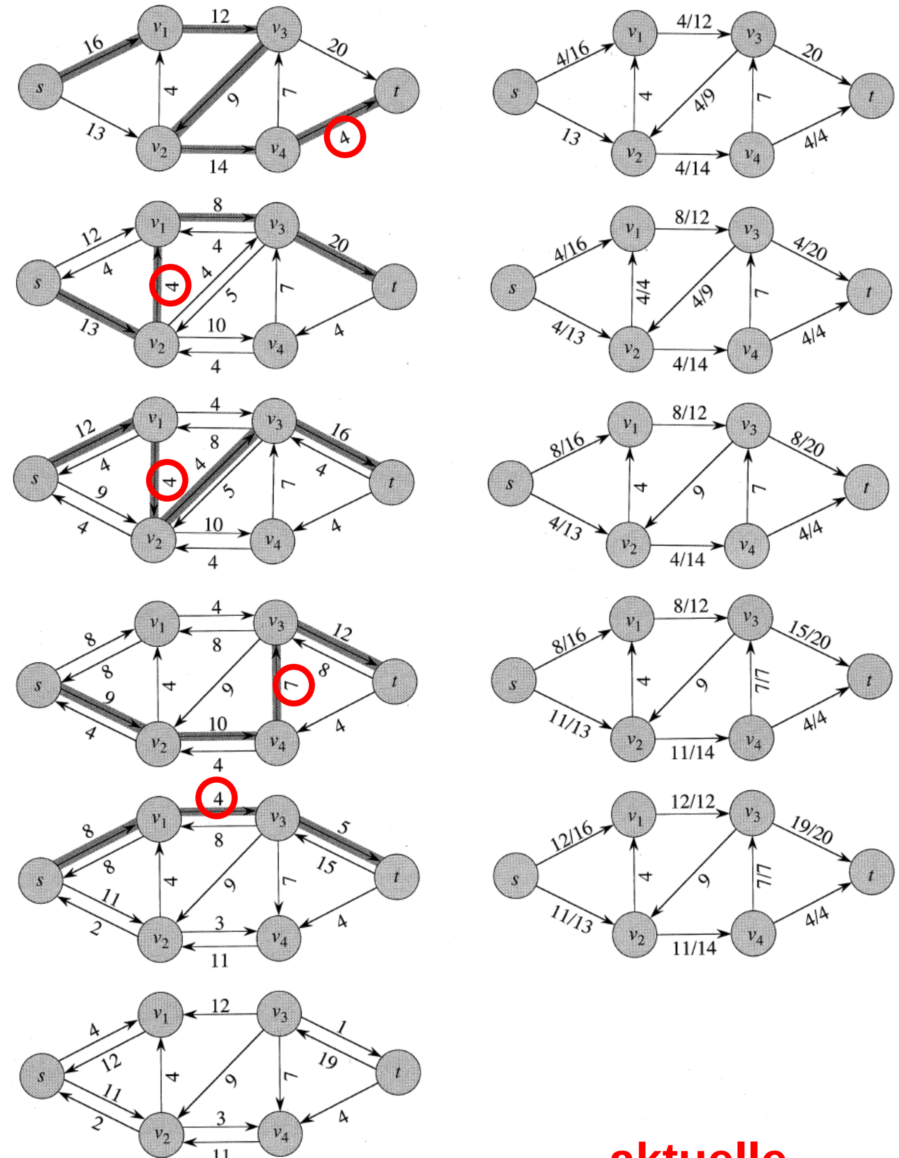
**maximale forøgelse langs stien *p***

# Ford-Fulkerson : eksempel



rest-netværk

aktuelle strømning



rest-netværk

aktuelle strømning

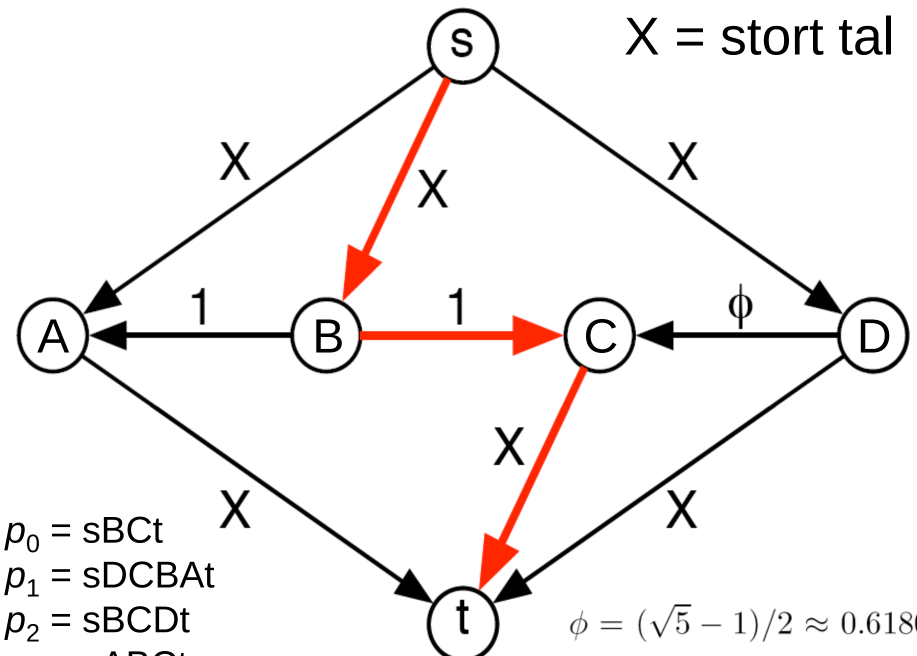
# Ford-Fulkerson : Analyse

**Sætning**

Hvis alle kapaciteterne i et netværk er heltallige og $f^*$ er en maximal strømning, så tager Ford-Fulkerson algoritmen tid $O(E \cdot |f^*|)$.

# Ford-Fulkerson
## ∞ mange forbedrende stier

X = stort tal




$\phi = (\sqrt{5}-1)/2 \approx 0.618034$

$p_0$ = sBCt
$p_1$ = sDCBAt
$p_2$ = sBCDt
$p_3$ = sABCt

De forbedrende stier $p_0,\{p_1,p_2,p_1,p_3\}^*$ konvergerer mod en strømning af størrelse

$$1 + 2\sum_{i=1}^{\infty}\phi^i = 1 + \frac{2}{1-\phi} = 4 + \sqrt{5} < 7$$

hvilket er langt fra det optimale på 2X+1.

| Aktuel strømning | | | Forbedrende sti | Forbedring | Strømning |
|---|---|---|---|---|---|
| BA | BC | DC | | | |
| 0,0000 | 0,0000 | 0,0000 | | | 0,0000 |
| 0,0000 | 1,0000 | 0,0000 | sBCt | 1,0000 | 1,0000 |
| 0,6180 | 0,3820 | 0,6180 | sDCBAt | 0,6180 | 1,6180 |
| 0,6180 | 1,0000 | 0,0000 | sBCDt | 0,6180 | 2,2361 |
| 1,0000 | 0,6180 | 0,3820 | sDCBAt | 0,3820 | 2,6180 |
| 0,6180 | 1,0000 | 0,3820 | sABCt | 0,3820 | 3,0000 |
| 0,8541 | 0,7639 | 0,6180 | sDCBAt | 0,2361 | 3,2361 |
| 0,8541 | 1,0000 | 0,3820 | sBCDt | 0,2361 | 3,4721 |
| 1,0000 | 0,8541 | 0,5279 | sDCBAt | 0,1459 | 3,6180 |
| 0,8541 | 1,0000 | 0,5279 | sABCt | 0,1459 | 3,7639 |
| 0,9443 | 0,9098 | 0,6180 | sDCBAt | 0,0902 | 3,8541 |
| 0,9443 | 1,0000 | 0,5279 | sBCDt | 0,0902 | 3,9443 |
| 1,0000 | 0,9443 | 0,5836 | sDCBAt | 0,0557 | 4,0000 |
| 0,9443 | 1,0000 | 0,5836 | sABCt | 0,0557 | 4,0557 |
| 0,9787 | 0,9656 | 0,6180 | sDCBAt | 0,0344 | 4,0902 |
| 0,9787 | 1,0000 | 0,5836 | sBCDt | 0,0344 | 4,1246 |
| 1,0000 | 0,9787 | 0,6049 | sDCBAt | 0,0213 | 4,1459 |
| 0,9787 | 1,0000 | 0,6049 | sABCt | 0,0213 | 4,1672 |
| 0,9919 | 0,9868 | 0,6180 | sDCBAt | 0,0132 | 4,1803 |
| 0,9919 | 1,0000 | 0,6049 | sBCDt | 0,0132 | 4,1935 |
| 1,0000 | 0,9919 | 0,6130 | sDCBAt | 0,0081 | 4,2016 |
| 0,9919 | 1,0000 | 0,6130 | sABCt | 0,0081 | 4,2098 |
| 0,9969 | 0,9950 | 0,6180 | sDCBAt | 0,0050 | 4,2148 |
| 0,9969 | 1,0000 | 0,6130 | sBCDt | 0,0050 | 4,2198 |
| 1,0000 | 0,9969 | 0,6161 | sDCBAt | 0,0031 | 4,2229 |
| 0,9969 | 1,0000 | 0,6161 | sABCt | 0,0031 | 4,2260 |
| 0,9988 | 0,9981 | 0,6180 | sDCBAt | 0,0019 | 4,2279 |
| 0,9988 | 1,0000 | 0,6161 | sBCDt | 0,0019 | 4,2299 |
| 1,0000 | 0,9988 | 0,6173 | sDCBAt | 0,0012 | 4,2310 |
| 0,9988 | 1,0000 | 0,6173 | sABCt | 0,0012 | 4,2322 |
| 0,9995 | 0,9993 | 0,6180 | sDCBAt | 0,0007 | 4,2330 |
| 0,9995 | 1,0000 | 0,6173 | sBCDt | 0,0007 | 4,2337 |
| 1,0000 | 0,9995 | 0,6178 | sDCBAt | 0,0005 | 4,2341 |
| 0,9995 | 1,0000 | 0,6178 | sABCt | 0,0005 | 4,2346 |
| 0,9998 | 0,9997 | 0,6180 | sDCBAt | 0,0003 | 4,2349 |
| 0,9998 | 1,0000 | 0,6178 | sBCDt | 0,0003 | 4,2352 |
| 1,0000 | 0,9998 | 0,6179 | sDCBAt | 0,0002 | 4,2353 |
| 0,9998 | 1,0000 | 0,6179 | sABCt | 0,0002 | 4,2355 |
| ... | ... | ... | ... | ... | ... |

# Edmonds-Karp Algoritmen

= Ford-Fulkerson hvor forbedrende stier har færrest mulige kanter (BFS)

**Sætning** Finder $O(V{\cdot}E)$ forbedrende stier, dvs. tid $O(V{\cdot}E^2)$

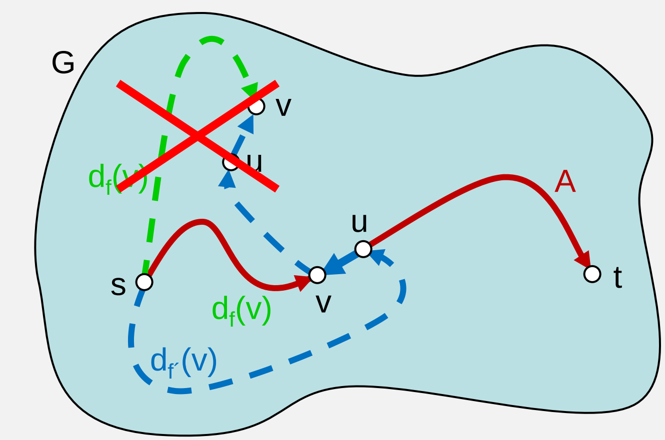
## Bevis

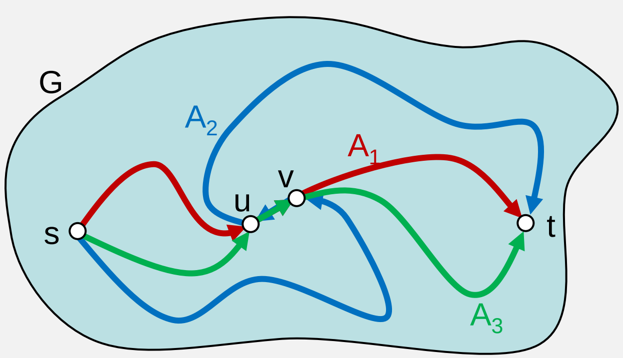
$f$ = flow i G

$A$ = forbedrende sti i $G_f$

$f'$ = flow efter A

$d_f(v)$ = afstand (#kanter) fra s til v i $G_f$




**Lemma 1** $\forall\, v : d_f(v) \le d_{f'}(v)$

**Bevis** Antag modsætnings der findes en knude v med $d_f(v) > d_{f'}(v)$, og v har <u>mindst mulig</u> $d_{f'}(v)$.

$$d_f(v) \underset{\substack{\uparrow\\ \text{BFS (Edmonds-Karp)}}}{=} d_f(u) - 1 \underset{\substack{\uparrow\\ \text{minimalitet af v}}}{\le} d_{f'}(u) - 1 \underset{\substack{\uparrow\\ \text{def. af } d_{f'}}}{\le} d_{f'}(v) - 2 \quad ⚡ \quad \square$$

**Lemma 2** En kant (u,v) kan kun være flaskehalsen på en forbedrende sti ≤ |V|/2 gange

**Bevis** Antag (u,v) flaskehalsen når $A_1$ og senere $A_3$ anvendes; inden $A_3$ må der findes $A_2$ der sender flow langs (v,u). $f_i$ flow før $A_i$.

$$d_{f_3}(v) \underset{\substack{\uparrow\\ \text{BFS}}}{=} d_{f_3}(u)+1 \underset{\substack{\uparrow\\ \text{Lemma 1}}}{\ge} d_{f_2}(u)+1 \underset{\substack{\uparrow\\ \text{BFS}}}{=} d_{f_2}(v)+2 \underset{\substack{\uparrow\\ \text{Lemma 1}}}{\ge} d_{f_1}(v)\underline{+2}$$

Lemma følger fra $0 \le d_{f_i}(v) \le |V|-2$ for $v \notin \{s,t\}$ $\square$

**Korollar** #forbedrende stier er ≤ 2|E|·|V|/2

# Maksimale strømninger – Historisk overblik

| year | discoverer(s) | bound |
|---|---|---|
| 1951 | Dantzig [11] | $O(n^2 mU)$ |
| 1956 | Ford & Fulkerson [17] | $O(nmU)$ |
| 1970 | Dinitz [13]<br>Edmonds & Karp [15] | $O(nm^2)$ |
| 1970 | Dinitz [13] | $O(n^2 m)$ |
| 1972 | Edmonds & Karp [15]<br>Dinitz [14] | $O(m^2 \log U)$ |
| 1973 | Dinitz [14]<br>Gabow [19] | $O(nm \log U)$ |
| 1974 | Karzanov [36] | $O(n^3)$ |
| 1977 | Cherkassky [9] | $O(n^2 m^{1/2})$ |
| 1980 | Galil & Naamad [20] | $O(nm \log^2 n)$ |
| 1983 | Sleator & Tarjan [46] | $O(nm \log n)$ |
| 1986 | Goldberg & Tarjan [26] | $O(nm \log(n^2/m))$ |
| 1987 | Ahuja & Orlin [2] | $O(nm + n^2 \log U)$ |
| 1987 | Ahuja et al. [3] | $O(nm \log(n\sqrt{\log U}/m))$ |
| 1989 | Cheriyan & Hagerup [7] | $E(nm + n^2 \log^2 n)$ |
| 1990 | Cheriyan et al. [8] | $O(n^3 / \log n)$ |
| 1990 | Alon [4] | $O(nm + n^{8/3} \log n)$ |
| 1992 | King et al. [37] | $O(nm + n^{2+\epsilon})$ |
| 1993 | Phillips & Westbrook [44] | $O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$ |
| 1994 | King et al. [38] | $O(nm \log_{m/(n \log n)} n)$ |
| 1997 | Goldberg & Rao [24] | $O(\min(n^{2/3}, m^{1/2}) m \log(n^2/m) \log U)$ |
| 2012 | Orlin | $O(mn)$ |

[CLRS 26.2]

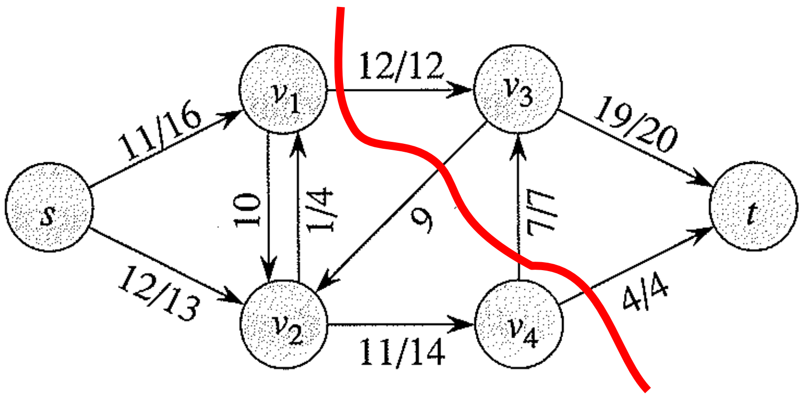
Andrew Goldberg
1998

$n$ = # knuder, $m$ = # kanter, kapaciteter i intervallet $[1..U]$

# Maximale strømninger

**Sætning**

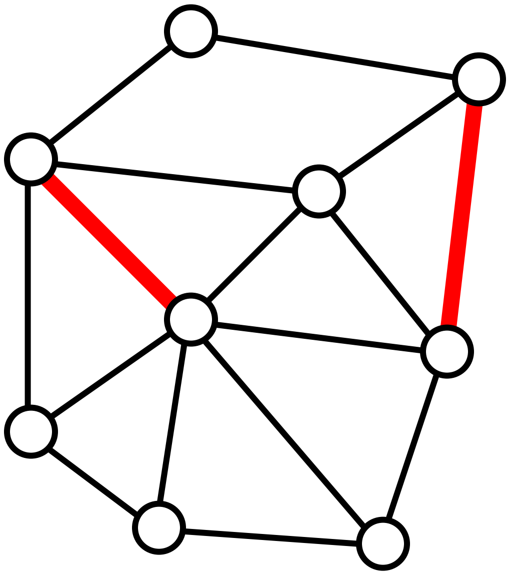
Maximal strømning = kapaciteten af et minimalt ($s$,$t$)-snit




**Sætning**

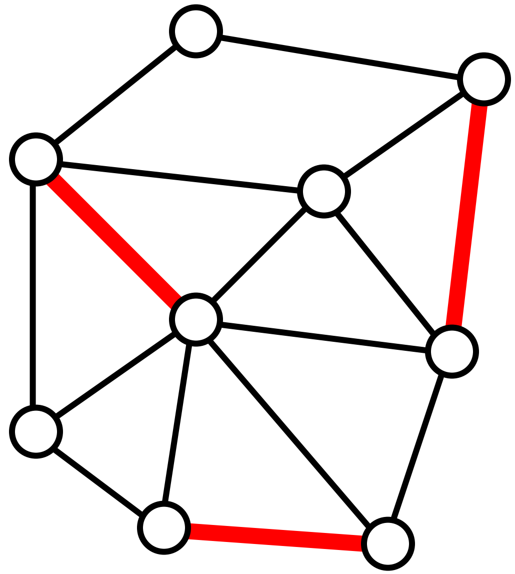
Hvis alle kapaciteter er heltallige så finder Ford-Fulkerson og Edmonds-Karp algoritmerne en strømning hvor *strømmen langs alle kanter er heltalligt*

# Parringer



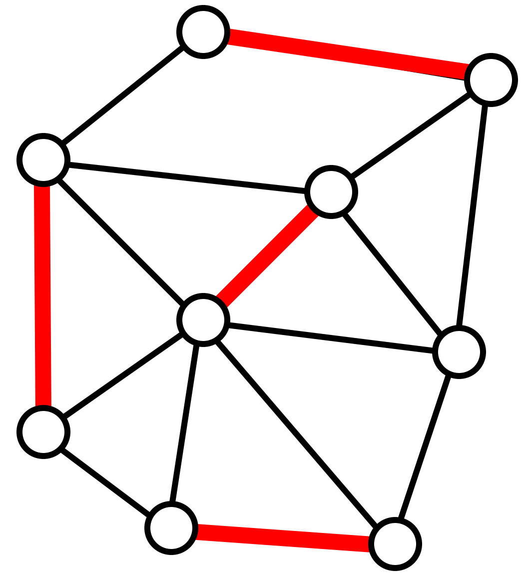
**Parring**

(en delmængde af kanterne hvor hver knude indgår max én gang)



**Maximal Parring**

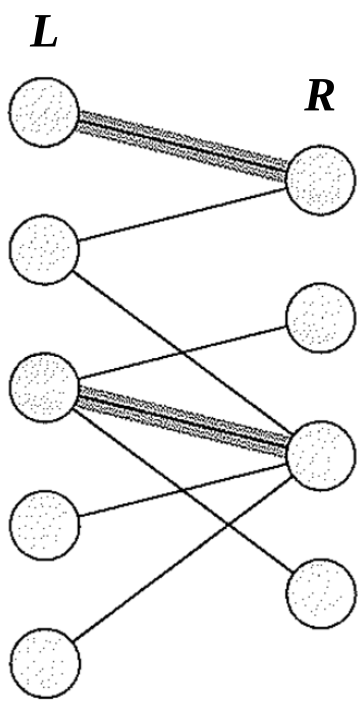
(parring hvor ingen kanter kan tilføjes – kan findes v.h.a grådig algoritme)



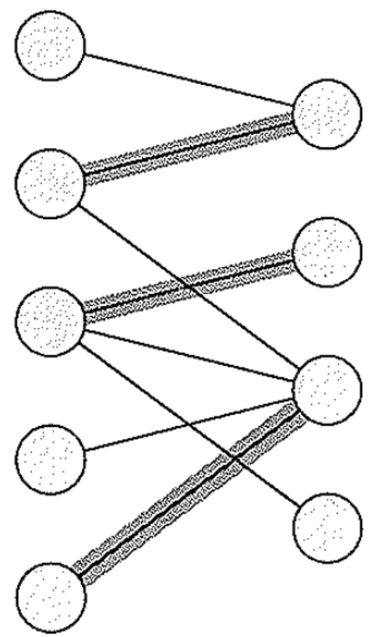
**Maximum Parring**

(findes ingen parringer med flere kanter)

# Parringer i todelte grafer



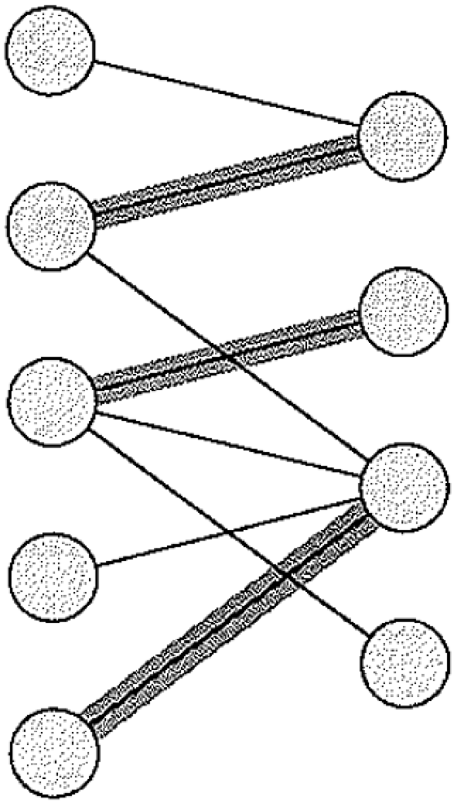

**Maximal Parring**
**af størrelse 2**

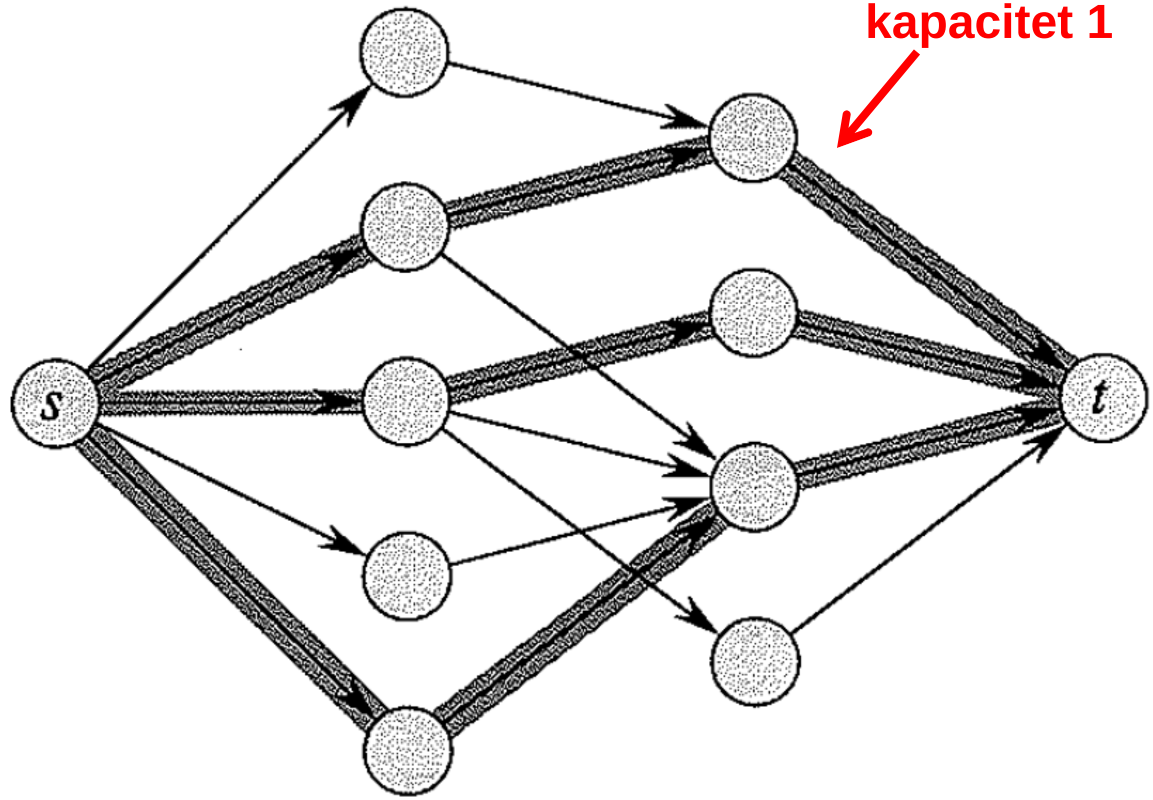


**Maximum Parring**
**af størrelse 3**

# Parring vs. Strømning

alle kanter kapacitet 1






Todelt graf
Maximum matching

≡

Strømnings netværk
Maximum strømning
(i en heltallig løsning,
f.x. Ford-Fulkerson)

# NWERC 2007

*The 2007 ACM Northwestern European Programming Contest*
*Utrecht University, The Netherlands*

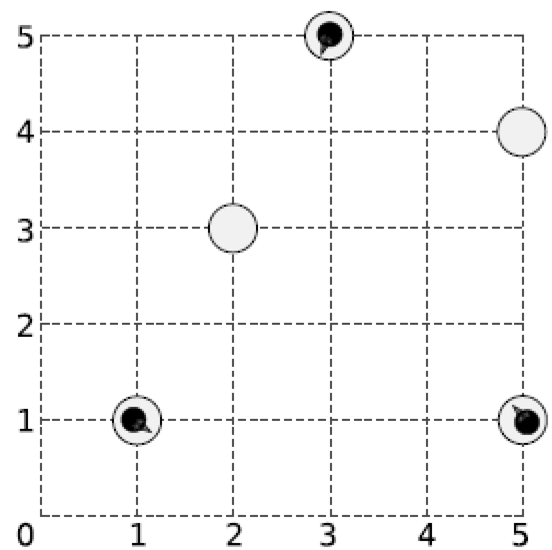
acm International Collegiate Programming Contest

## Problem B: March of the Penguins

2007.nwerc.eu/problems/nwerc07-problemset.pdf

# B March of the Penguins

Somewhere near the south pole, a number of penguins are standing on a number of ice floes. Being social animals, the penguins would like to get together, all on the same floe. The penguins do not want to get wet, so they have use their limited jump distance to get together by jumping from piece to piece. However, temperatures have been high lately, and the floes are showing cracks, and they get damaged further by the force needed to jump to another floe. Fortunately the penguins are real experts on cracking ice floes, and know exactly how many times a penguin can jump off each floe before it disintegrates and disappears. Landing on an ice floe does not damage it. You have to help the penguins find all floes where they can meet.




A sample layout of ice floes with 3 penguins on them.

## Input

On the first line one positive number: the number of testcases, at most 100. After that per testcase:

- One line with the integer $N$ ($1 \leq N \leq 100$) and a floating-point number $D$ ($0 \leq D \leq 100\,000$), denoting the number of ice pieces and the maximum distance a penguin can jump.
- $N$ lines, each line containing $x_i$, $y_i$, $n_i$ and $m_i$, denoting for each ice piece its $X$ and $Y$ coordinate, the number of penguins on it and the maximum number of times a penguin can jump off this piece before it disappears ($-10\,000 \leq x_i, y_i \leq 10\,000$, $0 \leq n_i \leq 10$, $1 \leq m_i \leq 200$).

## Output

Per testcase:

- One line containing a space-separated list of 0-based indices of the pieces on which all penguins can meet. If no such piece exists, output a line with the single number $-1$.

**Sample in- and output**

| Input | Output |
|---|---|
| <pre>2<br>5 3.5<br>1 1 1 1<br>2 3 0 1<br>3 5 1 1<br>5 1 1 1<br>5 4 0 1<br>3 1.1<br>-1 0 5 10<br>0 0 3 9<br>2 0 1 1</pre> | <pre>1 2 4<br>-1</pre> |