Dynamic Data Structures: The Interplay of Invariants and Algorithm Design

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MADALGO
CENTER FOR MASSIVE DATA ALGORITHMICS

Danmarks Grundforskningsfon
Danish National Research Foundation

AARHUS UNIVERSITY

Outline

- Introduction to Algorithms and Data Structures
- Implicit Working-Set Dictionaries
- Skyline Queries
- Catenable Priority Queues with Attrition
- Conclusion

What are Algorithms and Data Structures?

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Algorithm ≋ Recipe

Data ≋ Ingredients

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- Low Space Usage

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- Fast queries and updates
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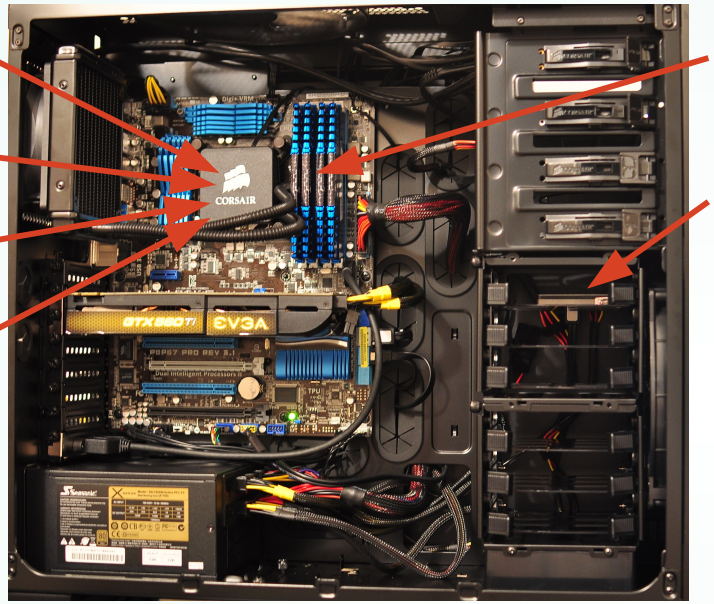

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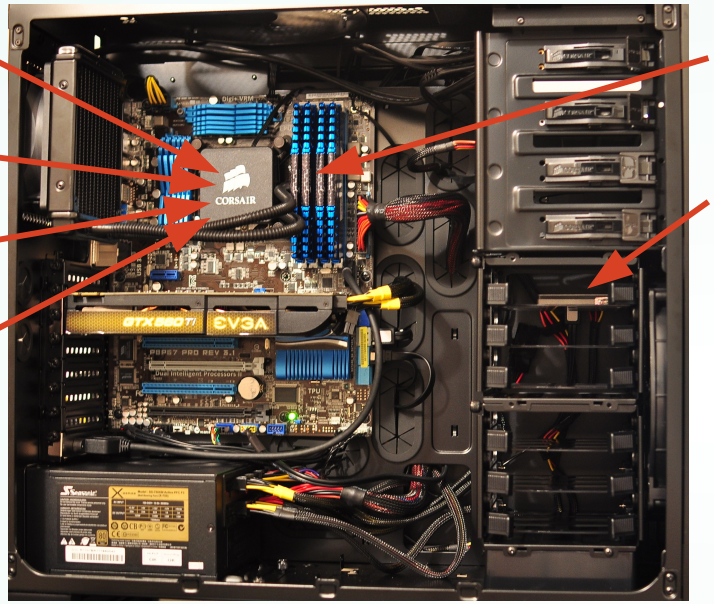

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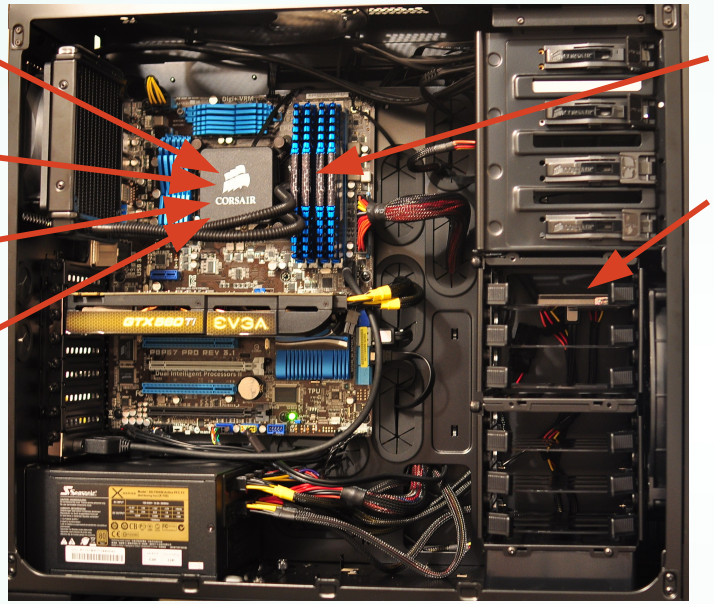

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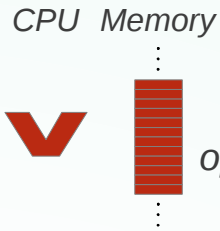

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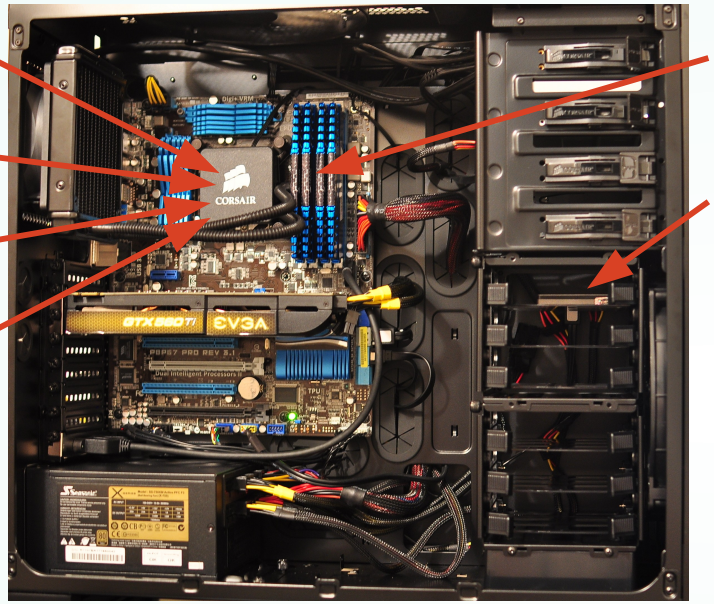

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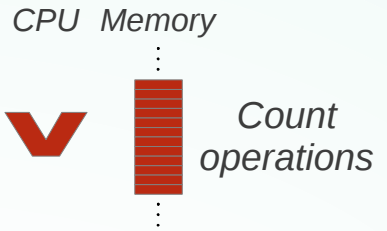

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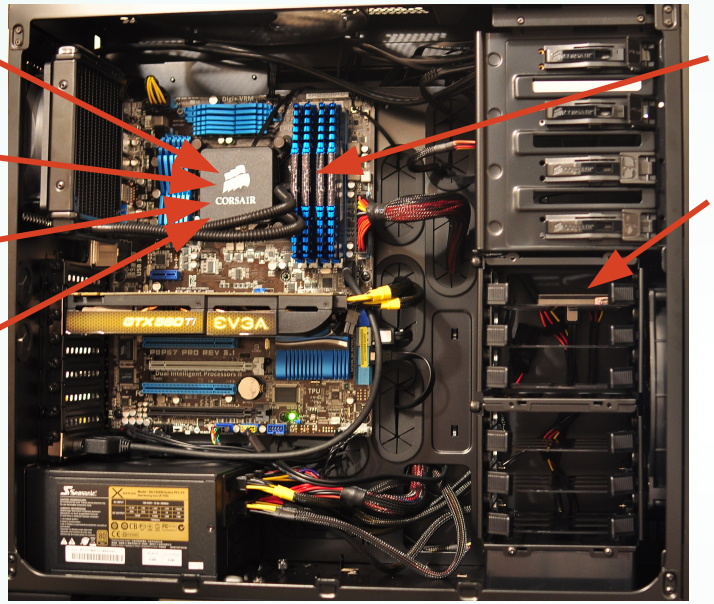

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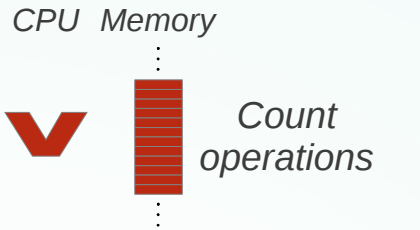

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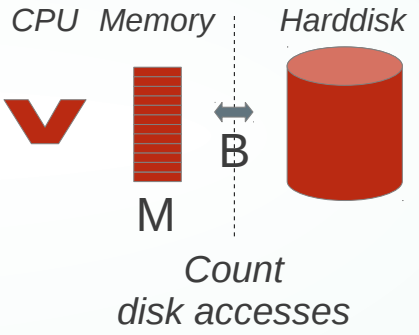

EM Model

Many more

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Static and Dynamic Problems

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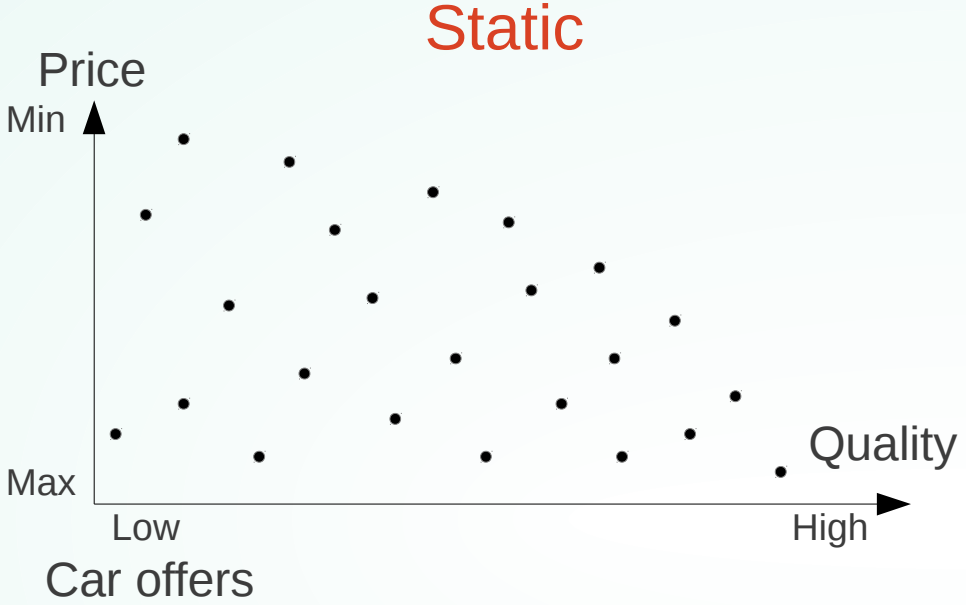

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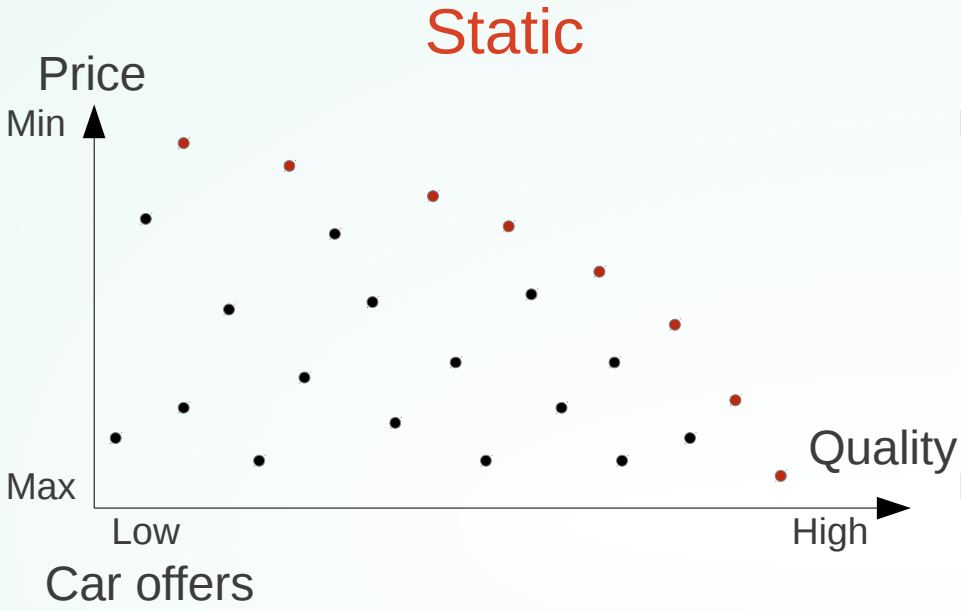

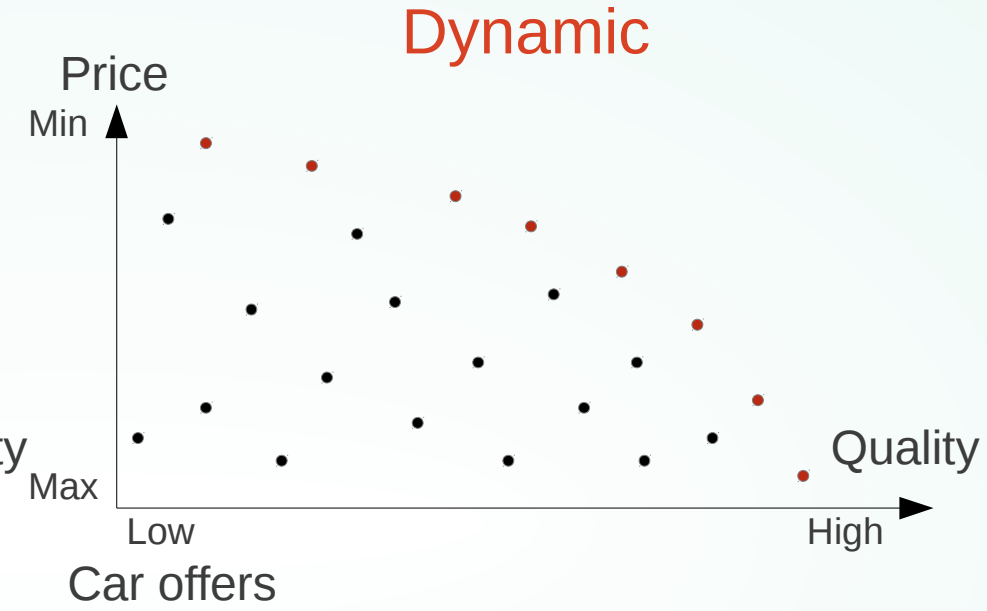

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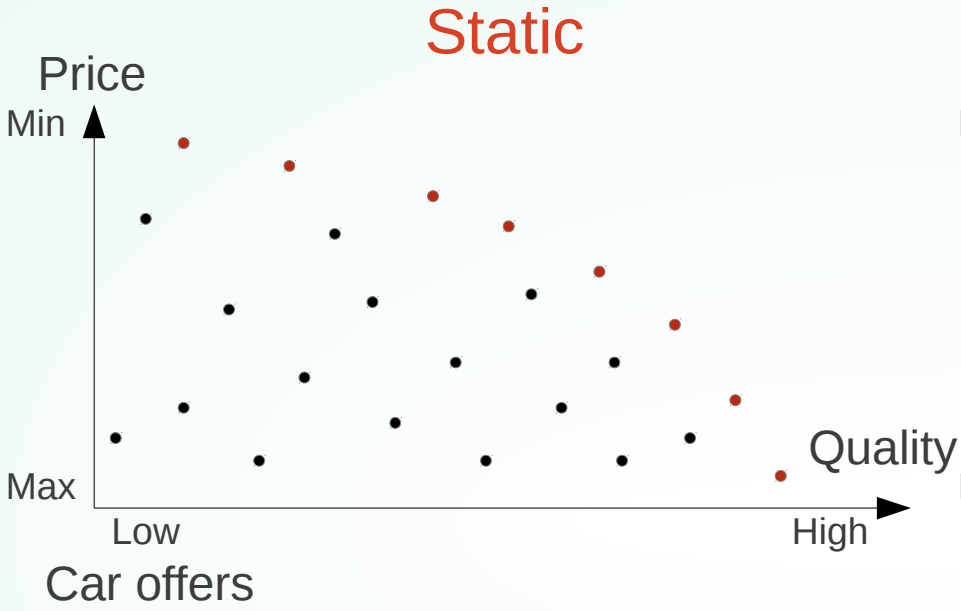

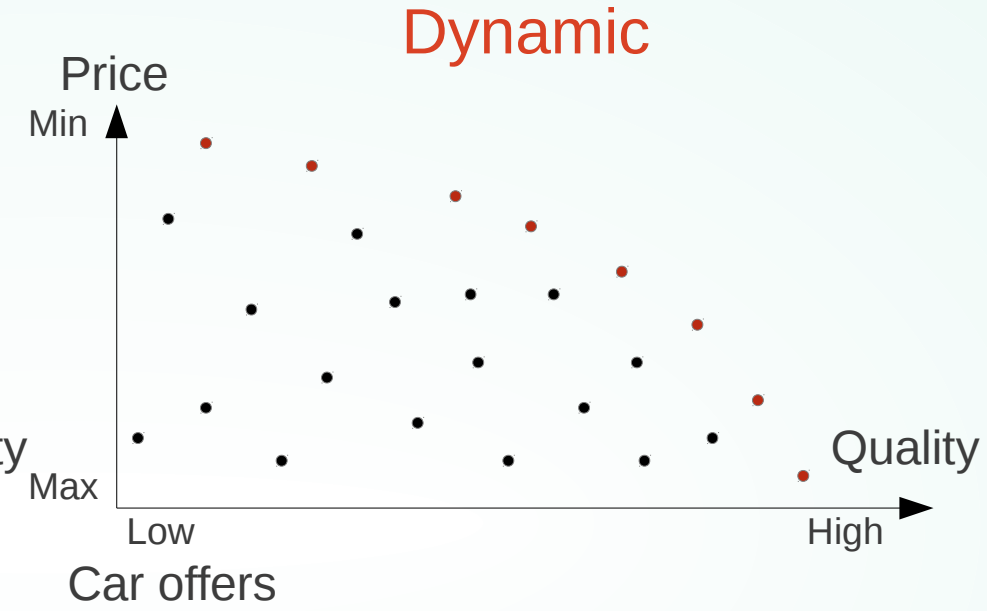

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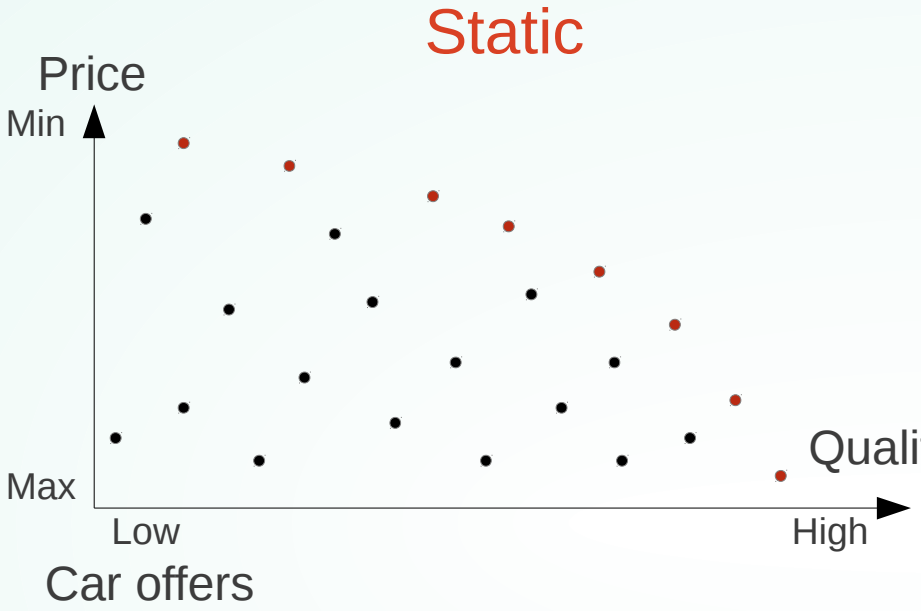

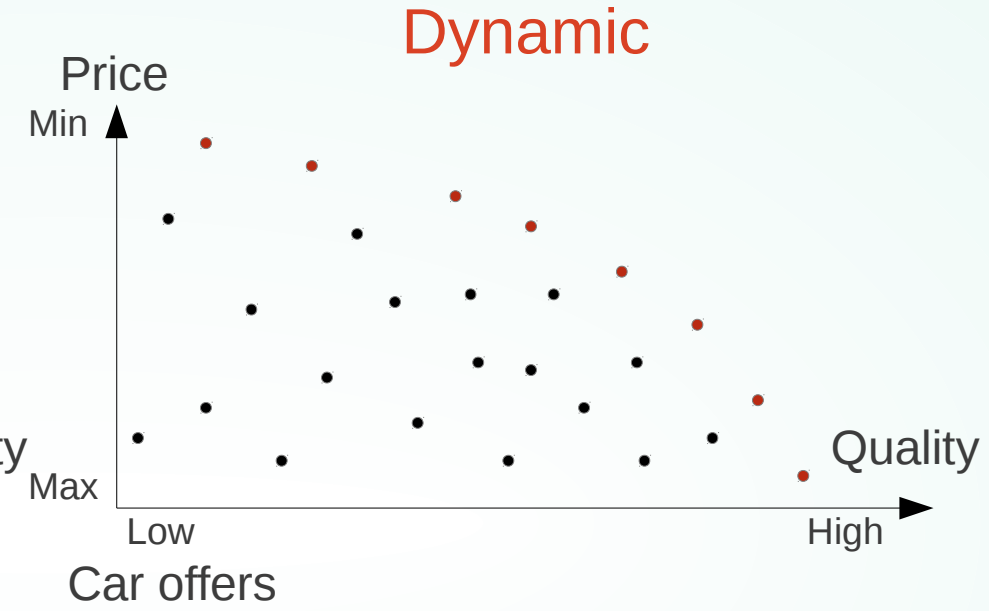

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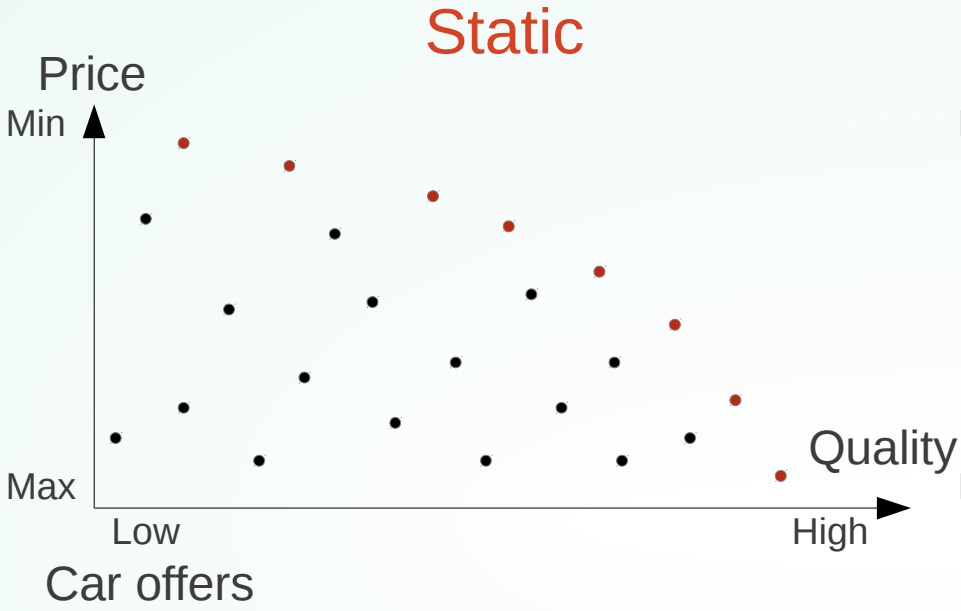

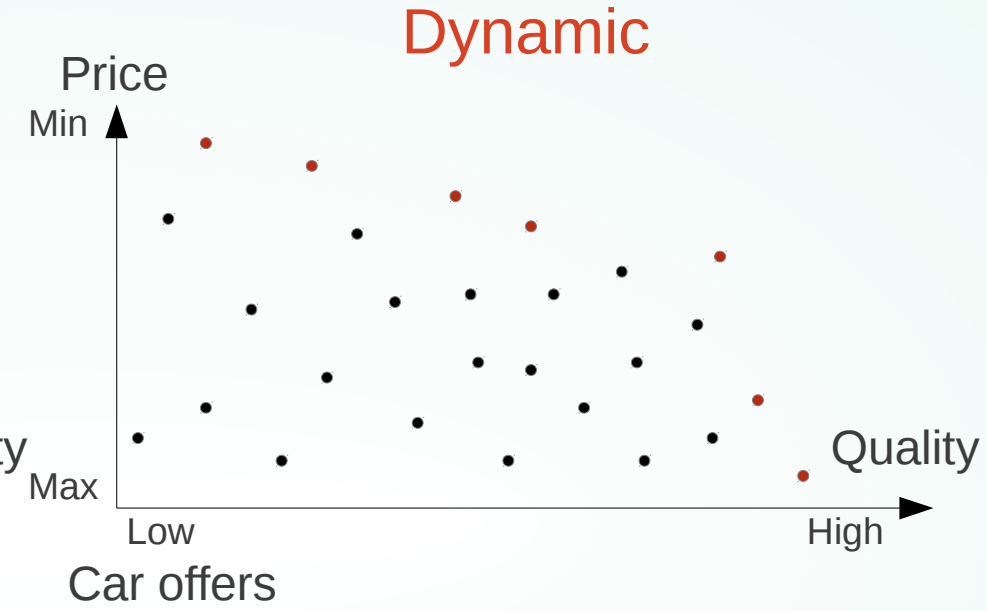

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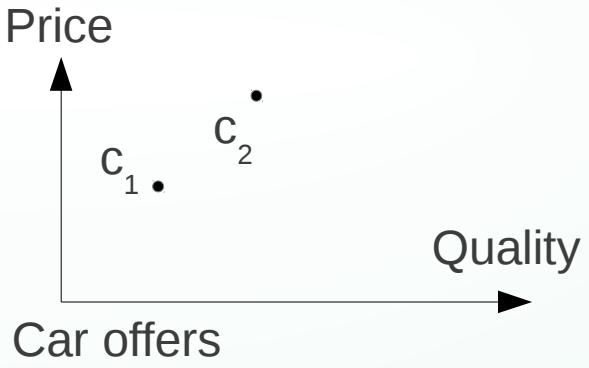

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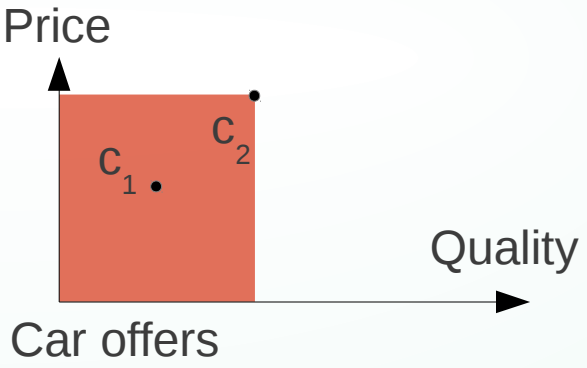

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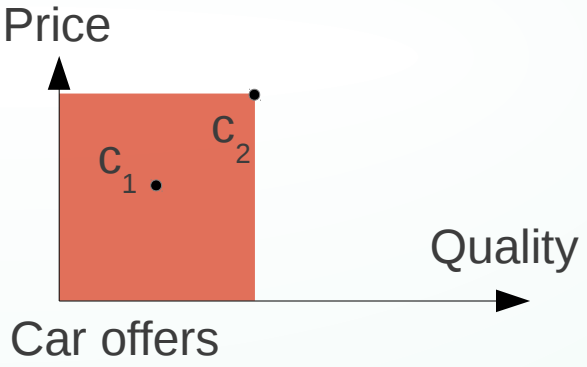

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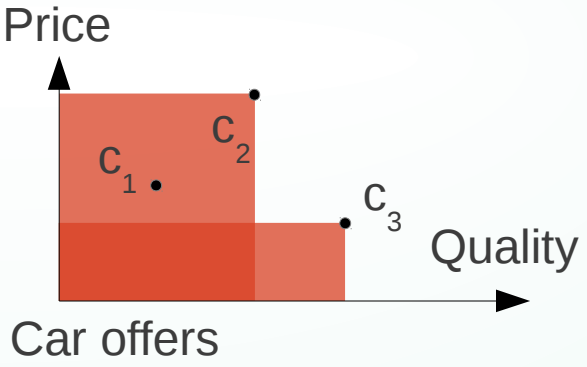

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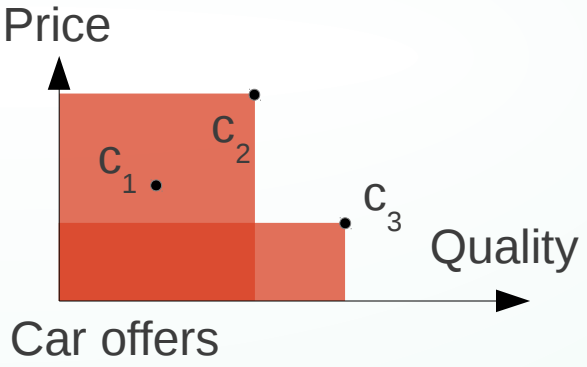
c_1 and c_3 are incomparable



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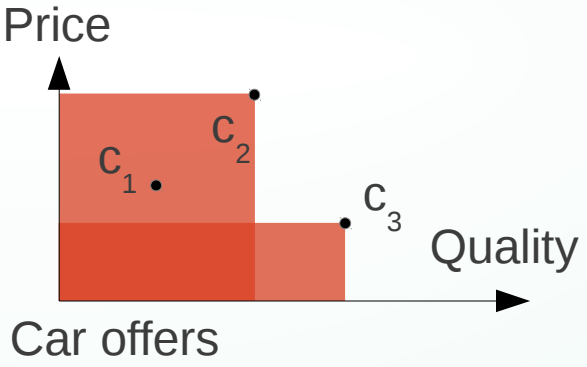

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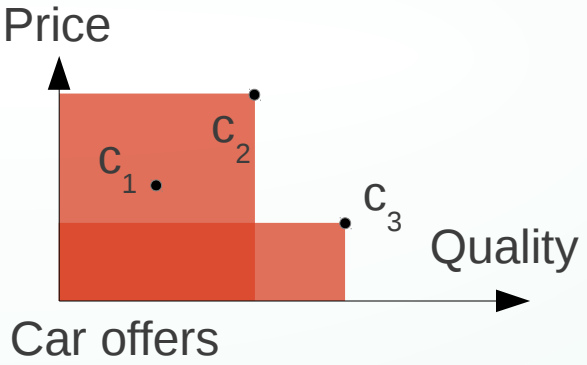

- We notice that the *undominated* (c_2 and c_3) offers are sorted both according to price and quality simultaneously

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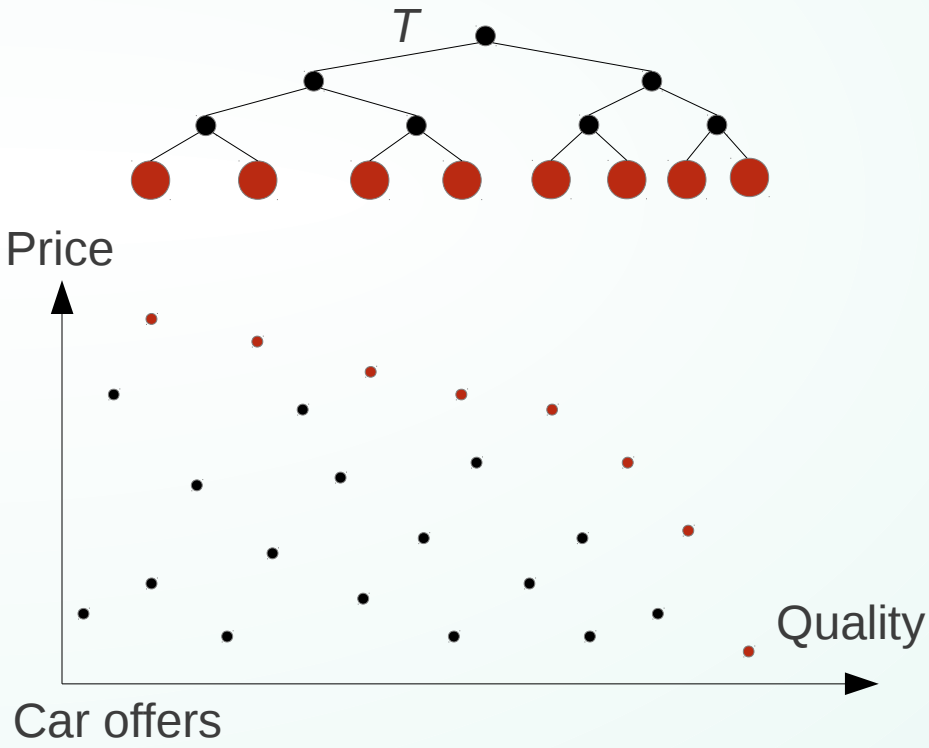

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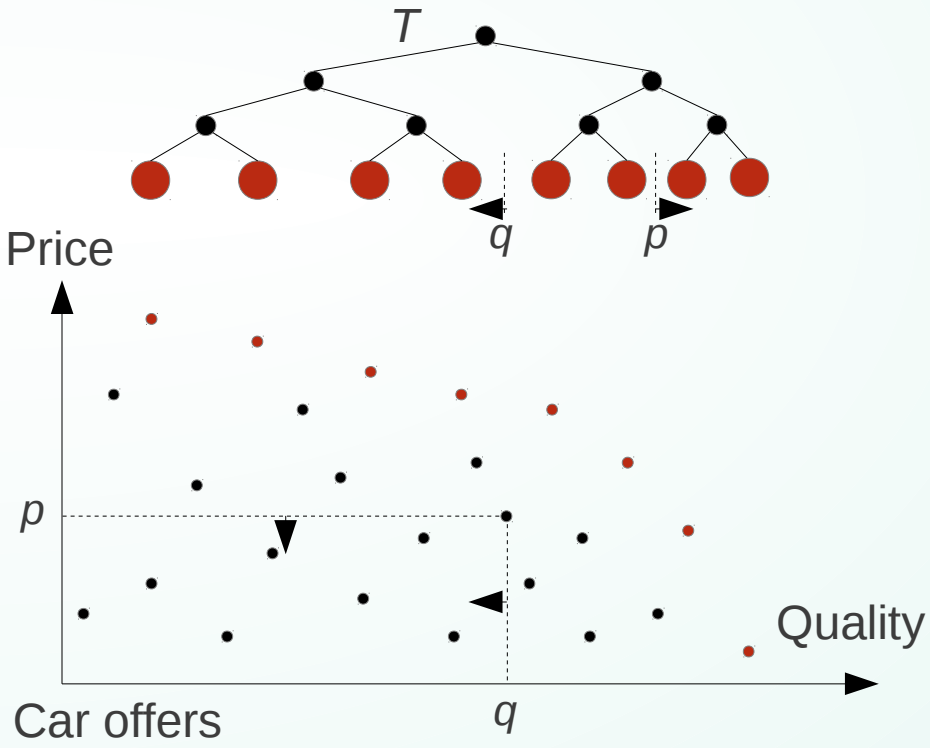

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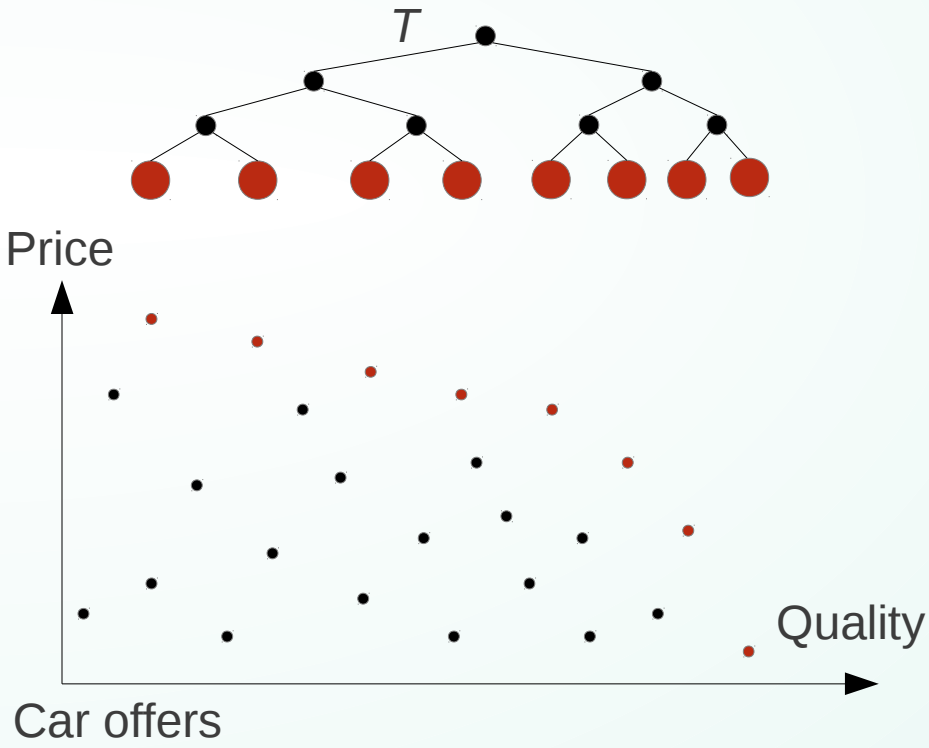

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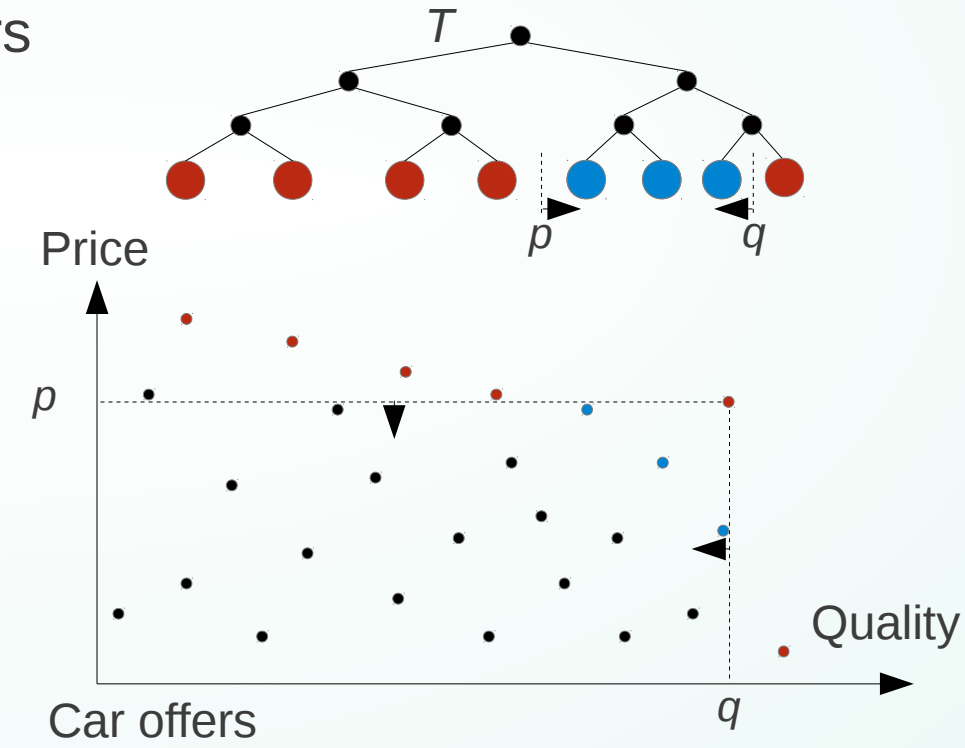

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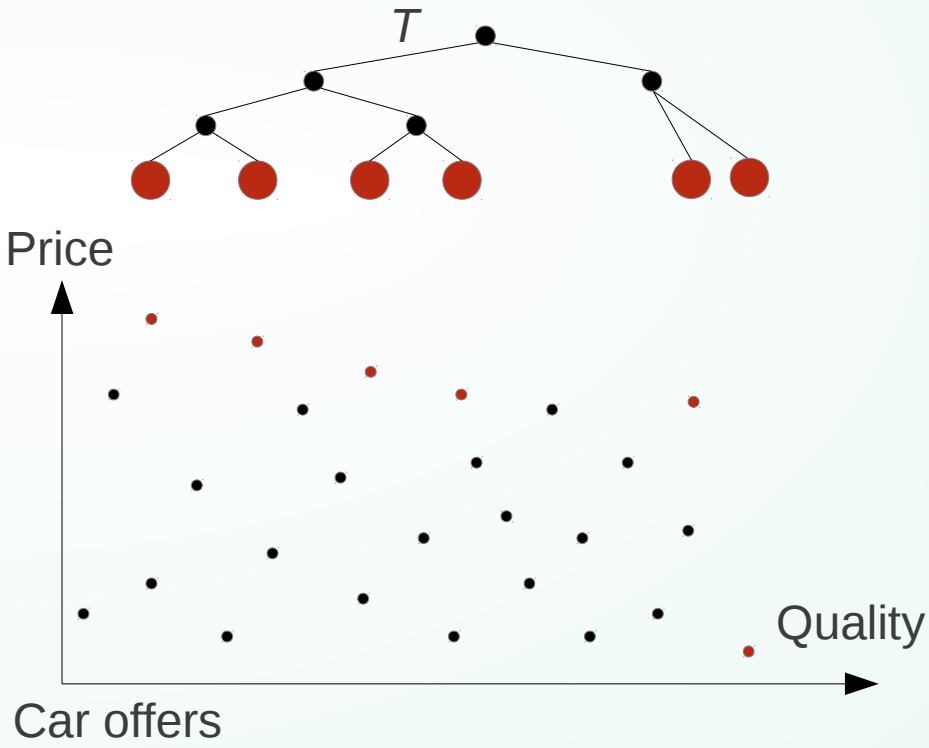

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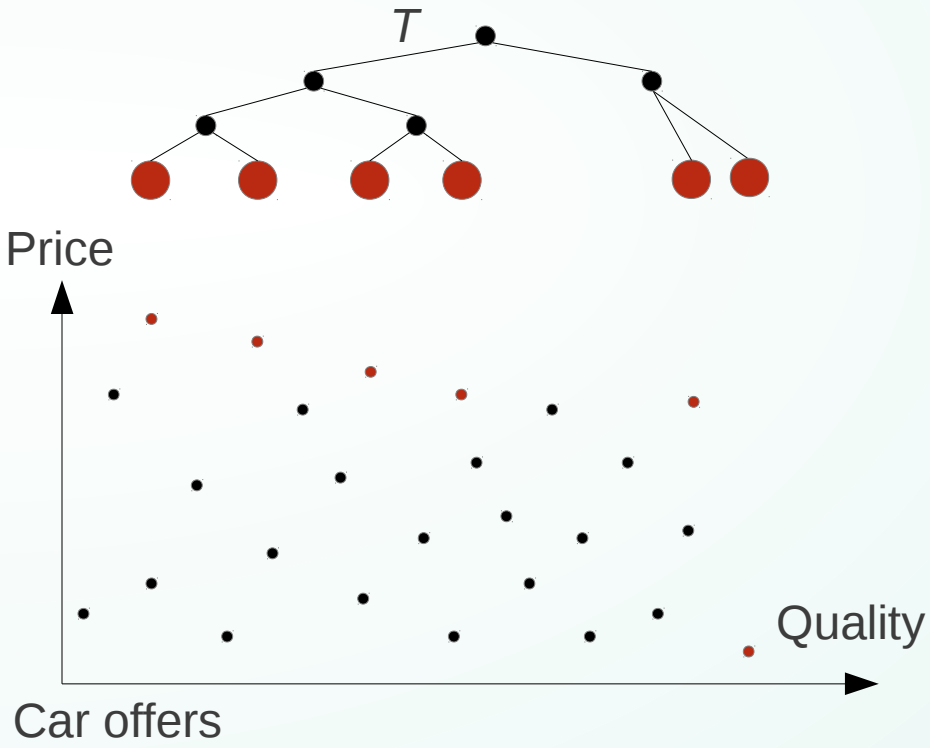

- Where *k* is the number of undominated car offers out of all offers
- Inserting a new car offer takes O(log *k*) time
- Reporting the undominated offers takes *O*(*k*) time

Designing Invariants and Dynamic Data Structures

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- Dynamic data structure and invariant design follows a cycle:
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- The process is similarly to suitcase packing:
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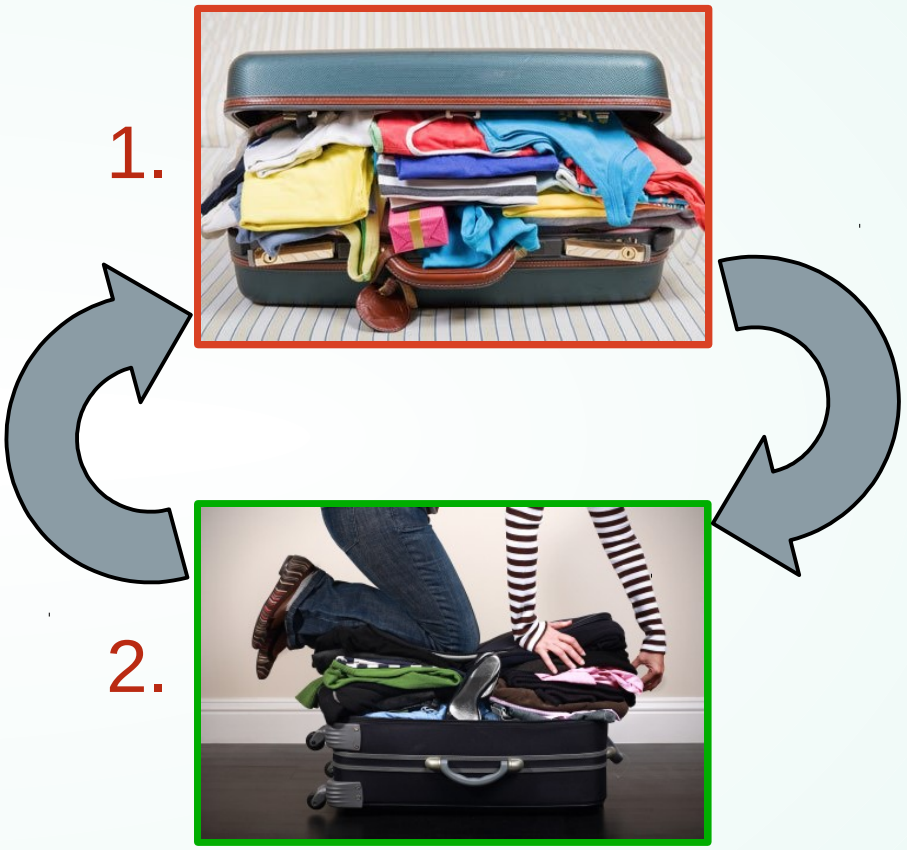

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Implicit Model

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- All operations from the RAM
- It is not allowed to *create* words, only to *move* them
- All n words have to be in contiguous positions

- Often it is assumed that all elements are distinct
- Fundamental trick: encode a bit in a pair of adjacent and distinct elements

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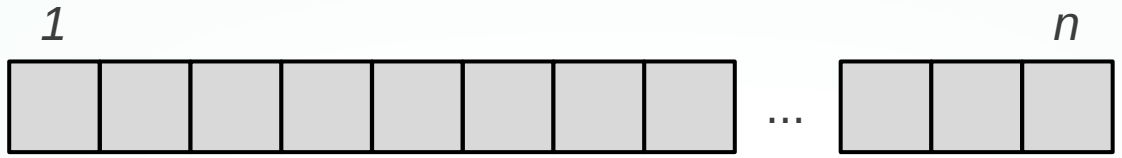

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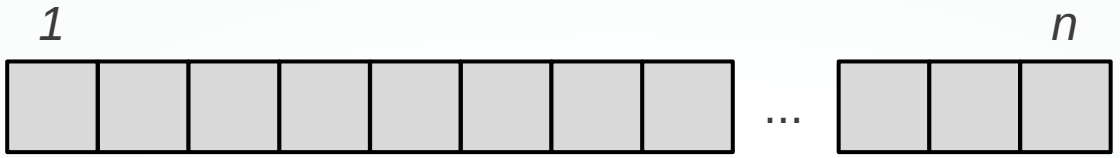

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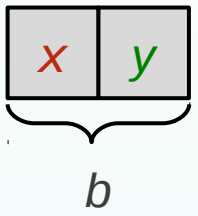

$$b = \begin{cases} 0 \text{ if } x=\min(x,y) \\ 1 \text{ if } x=\max(x,y) \end{cases}$$

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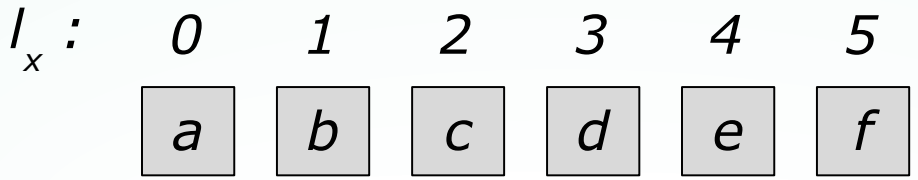

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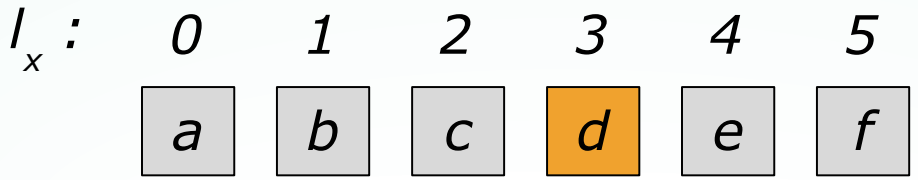
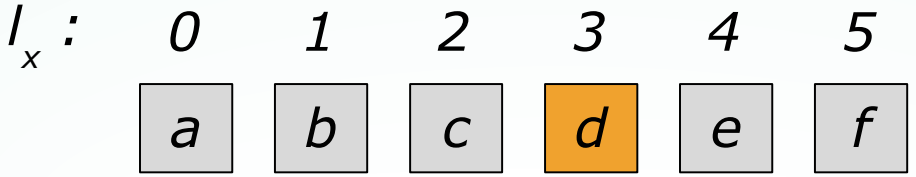


The Working-Set Property

- Element x has a *working-set number* of l_x iff:

l_x elements different from x have been searched for since we last searched for x



- *An Implicit Dictionary with the Working-Set Property:*
  - Insert(x): insert element x into the dictionary and set $l_x = 0$
  - Delete(x): delete element x from the dictionary
  - Search(x): determine if x is in the dictionary and set $l_x = 0$
  - Predecessor(x): find the address of the predecessor of x
  - Successor(x): find the address of the successor of x

Previous and Our Results

Ref.	WS prop.	Insert/Delete(e)	Search(e)	Predecessor/ Successor(e)	Additional Words
M1989	-	$O(\log^2 n)$	$O(\log^2 n)$		None
FGMP2002	-	$O(\log^2 n/\log\log n)$	$O(\log^2 n/\log\log n)$		None
FG2006	-	$O(\log n)$ amor.	$O(\log n)$	$O(\log n)$	None
FG2003	-	$O(\log n)$	$O(\log n)$	$O(\log n)$	None
I2001	+	$O(\log n)$	$O(\log l_e)$	$O(\log l_{e^*})$	$O(n)$
BHM2009	+	$O(\log n)$	$O(\log l_e)$ exp.	$O(\log n)$	$O(\log\log n)$
BHM2009	+	$O(\log n)$	$O(\log l_e)$ exp.	$O(\log l_e)$ exp.	$O(\sqrt{n})$
BKT2010	+	$O(\log n)$	$O(\log l_e)$	$O(\log n)$	None
BK2011	+	$O(\log n)$	$O(\log \min(l_{p(e)}, l_e, l_{s(e)}))$	$O(\log l_{e^*})$	None

e^* is the predecessor/successor of e

Implicit Moveable Dictionaries

Implicit Moveable Dictionaries

- A dictionary laid out in memory addresses [*i*,*j*]

- *Interface:*
  - Insert-left/right(*e*): insert element *e* into the dictionary which grows to the left/right
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- Can be constructed from O(*1*) FG dictionaries used as black boxes

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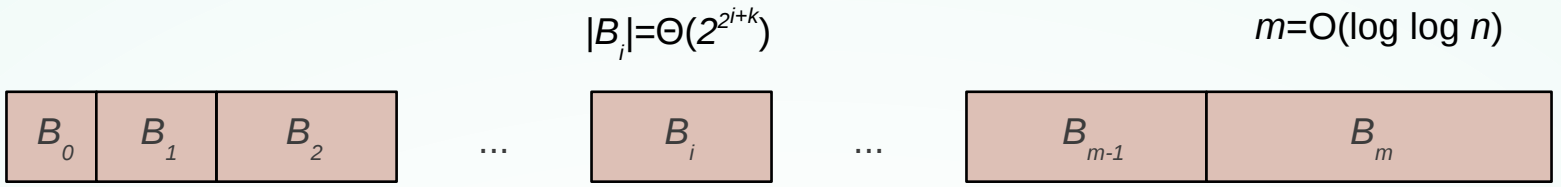
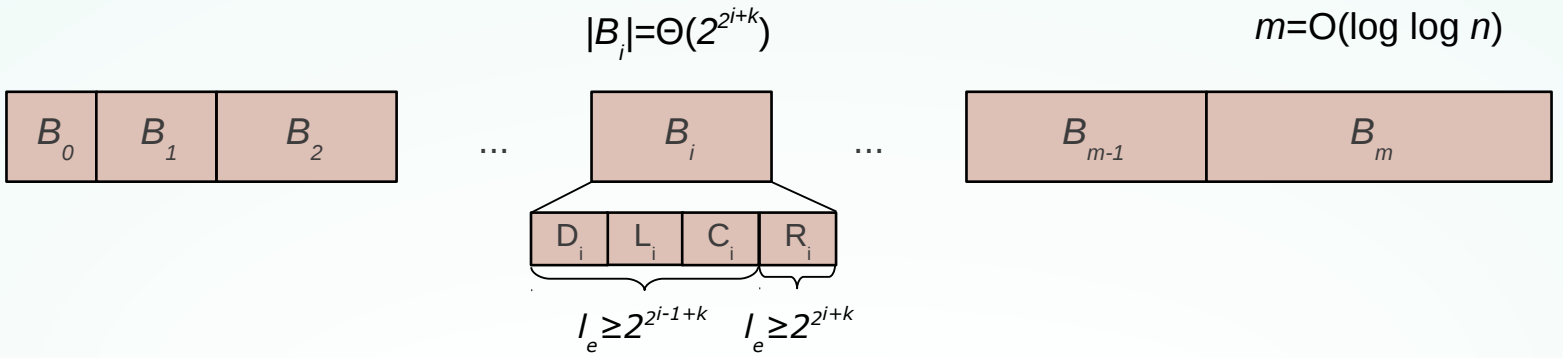
Implicit Working-Set Dictionaries



Implicit Working-Set Dictionaries

- Exponential layout



- B_i consists of O(*1*) moveable dictionaries
- All elements e in B_i have $l_e \geq 2^{2i-1+k}$ or $l_e \geq 2^{2i+k}$
- Searched and inserted elements are moved into B_0 (overflows)
- These are the ideas we used in the ISAAC 2010 paper
- Only gives O(log n) bounds for predecessor and successor searches as all B_i have to be searched: the invariants do not relate e to its prede/suc-cessor

Implicit Working-Set Dictionaries

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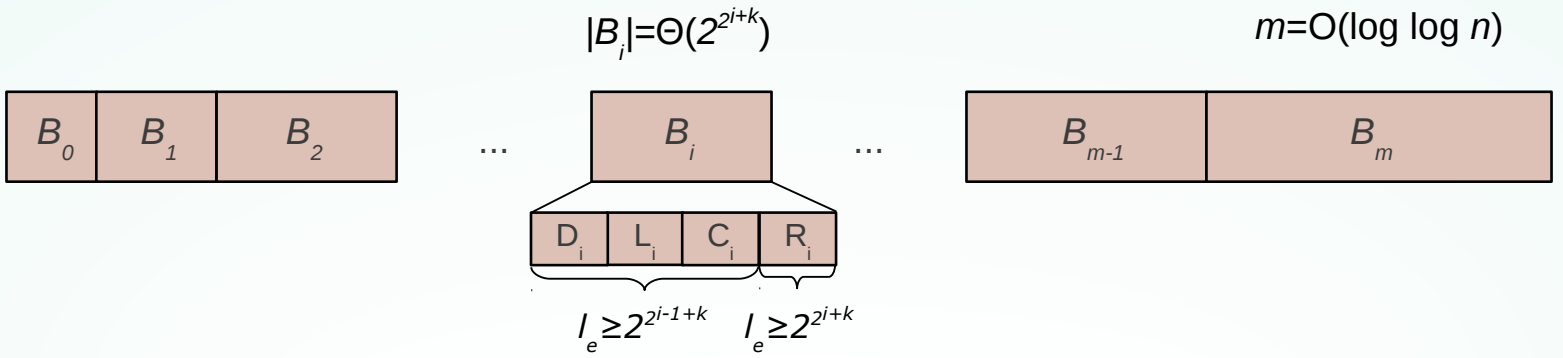

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Summary of Invariants

- Blocks of fixed size: easy word/pointer encoding
- Elements in each block are divided according to working-set number

Implicit Predecessor/Successor Working-Set Dictionaries

- Intervals to solve the predecessor and successor problems

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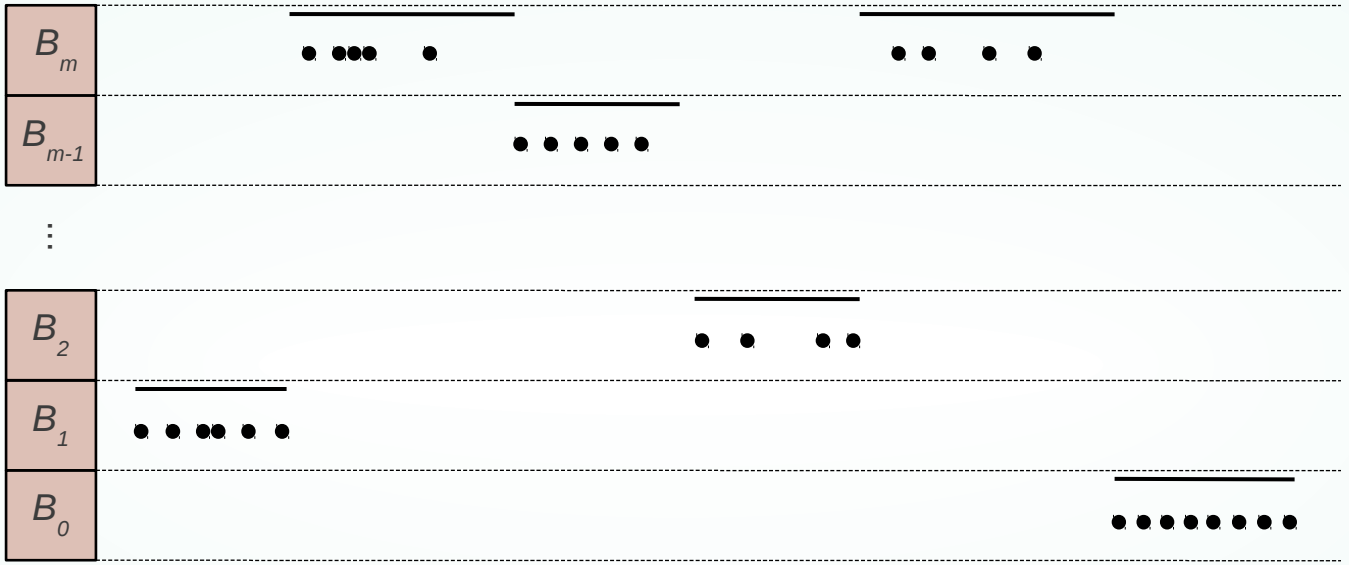

- Divide the key-space into mutually disjoint intervals aligned with the points/elements
- Invariant: any point/element, intersecting an interval at level i, lies in block B_i
- Predecessor/Successor(e) searches can terminate when an interval at level i is intersected

Implicit Predecessor/Successor Working-Set Dictionaries

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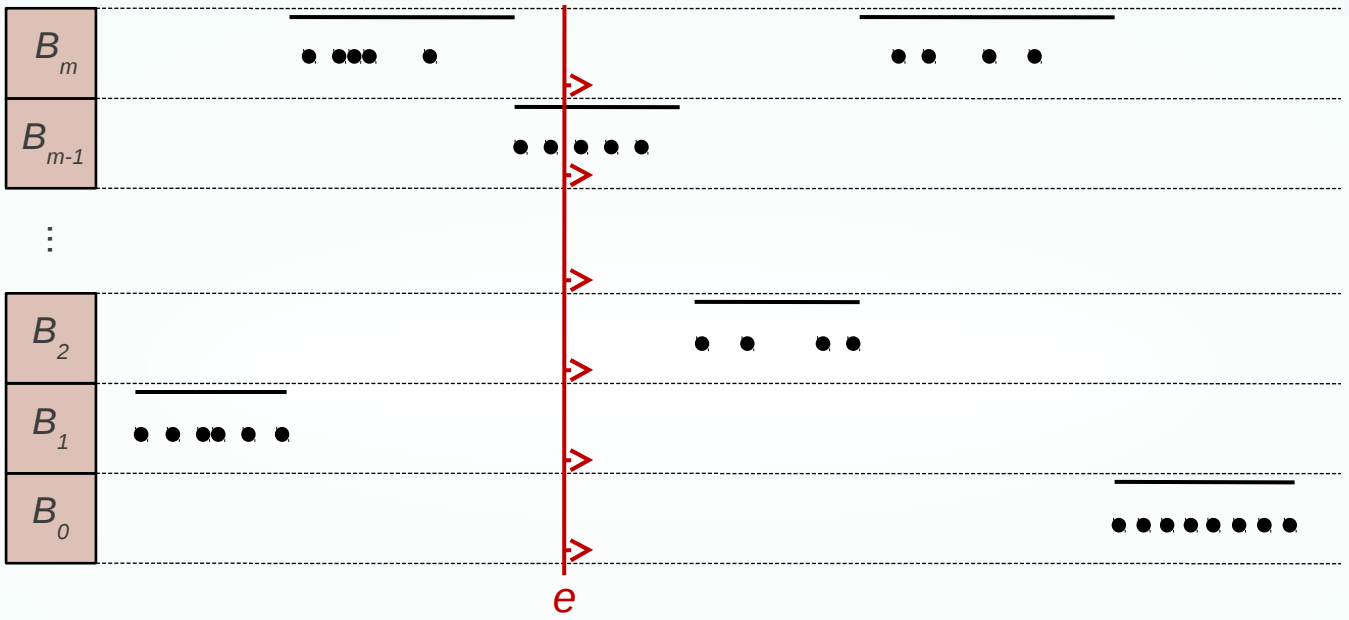

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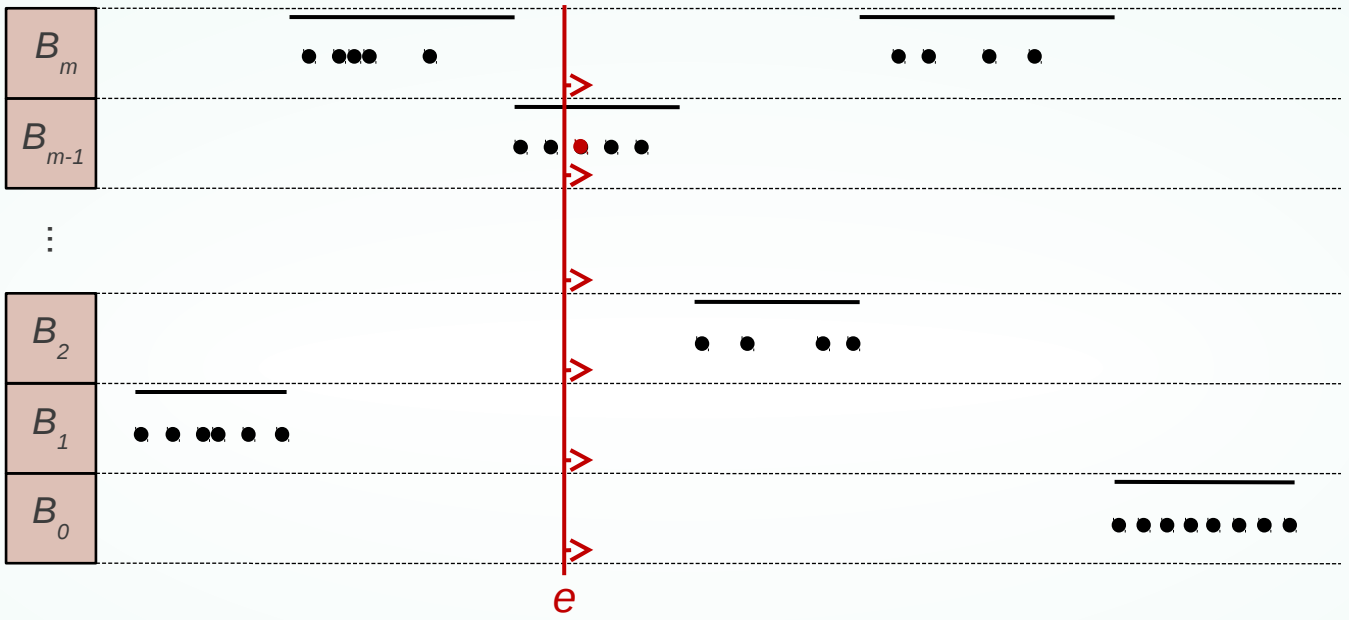

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Implicit Representation of Intervals

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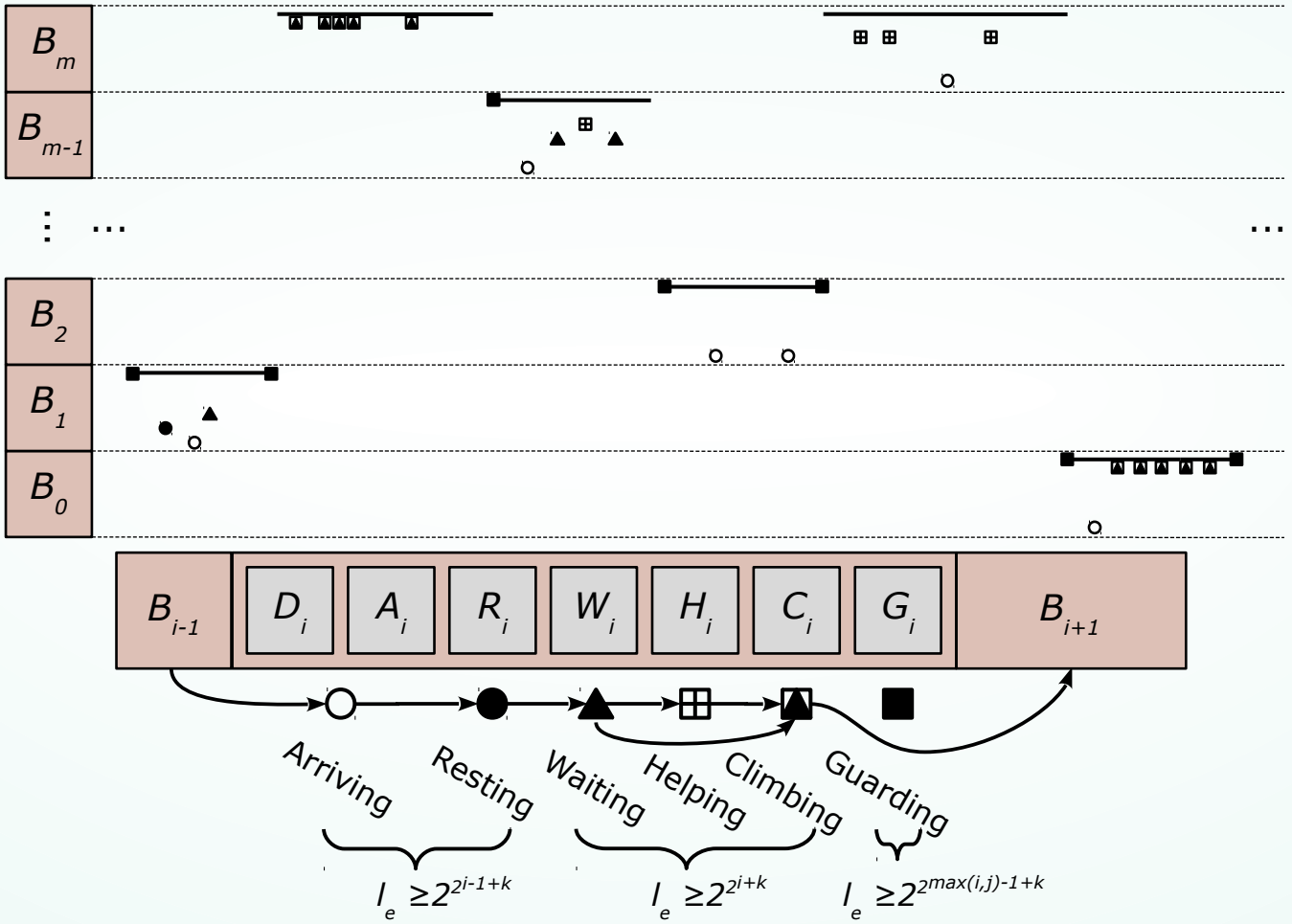

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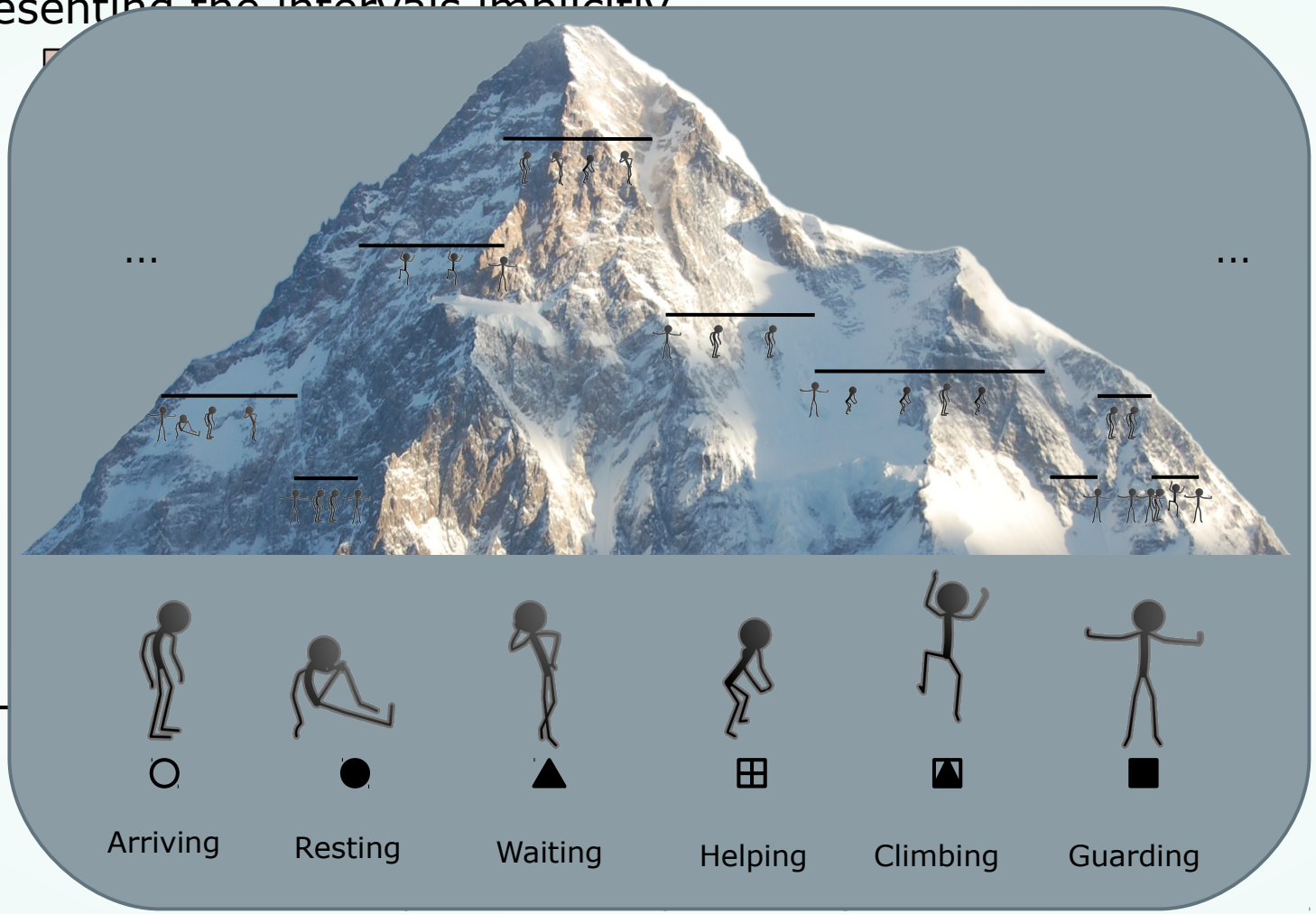

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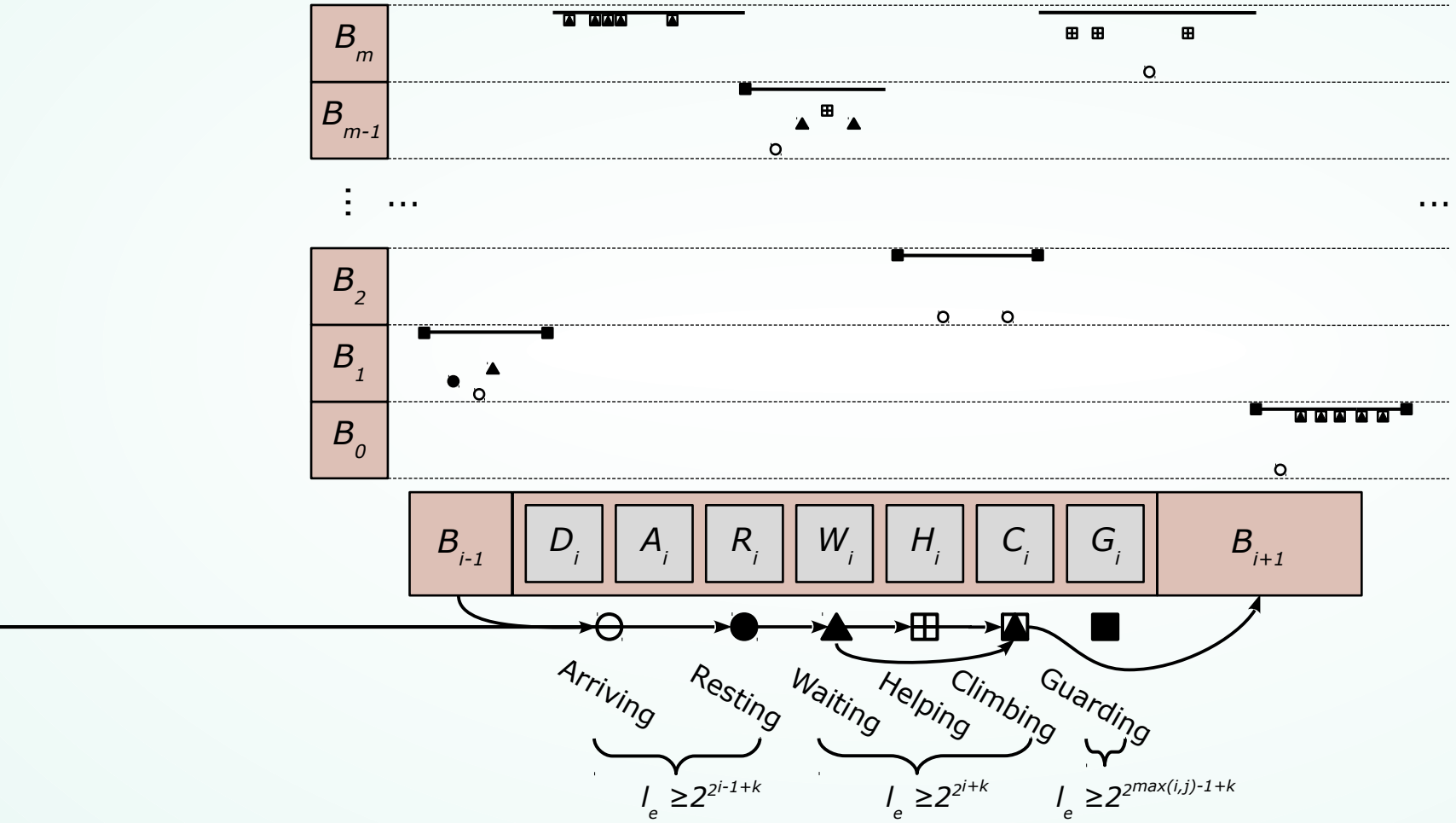

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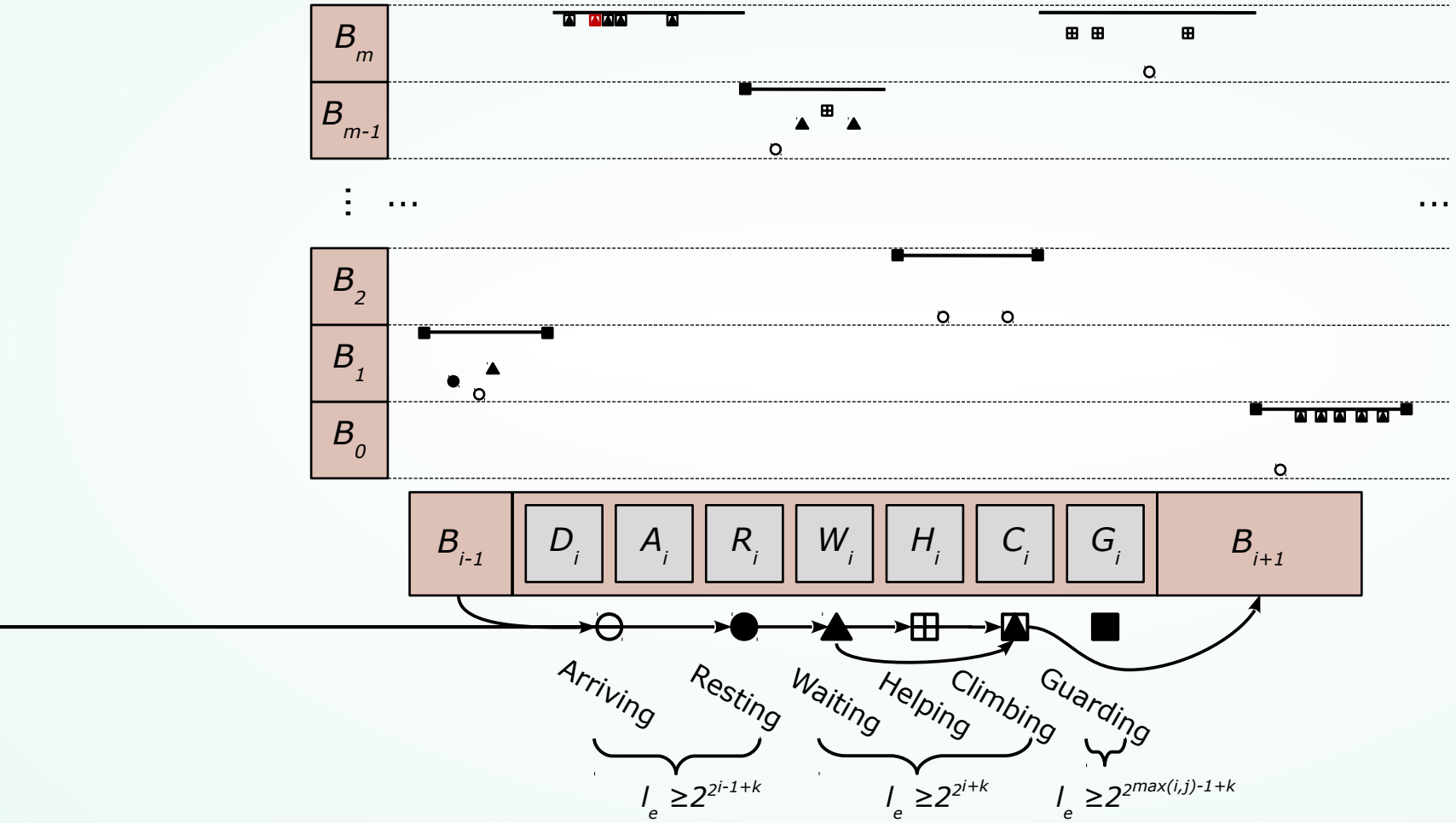

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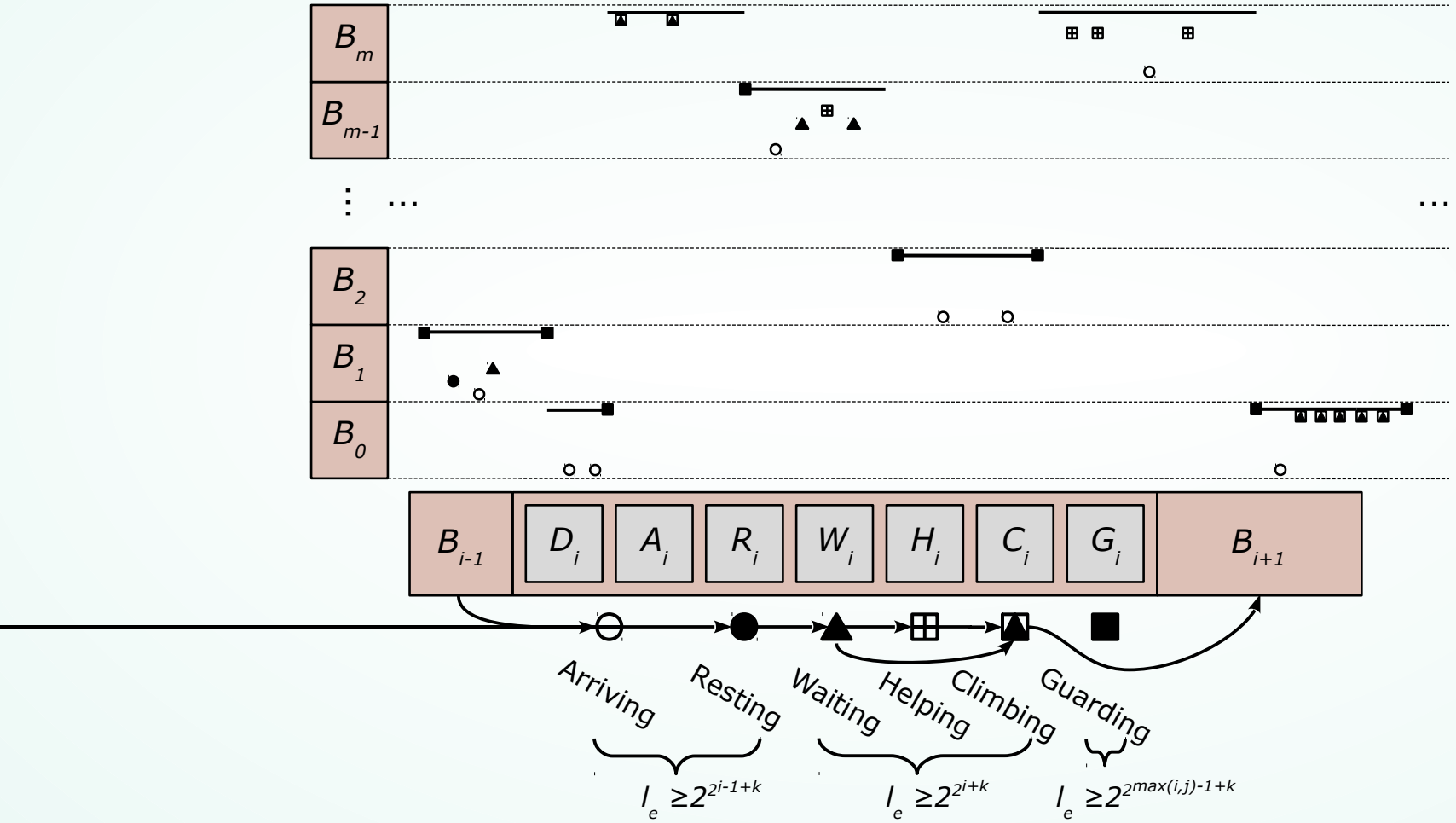

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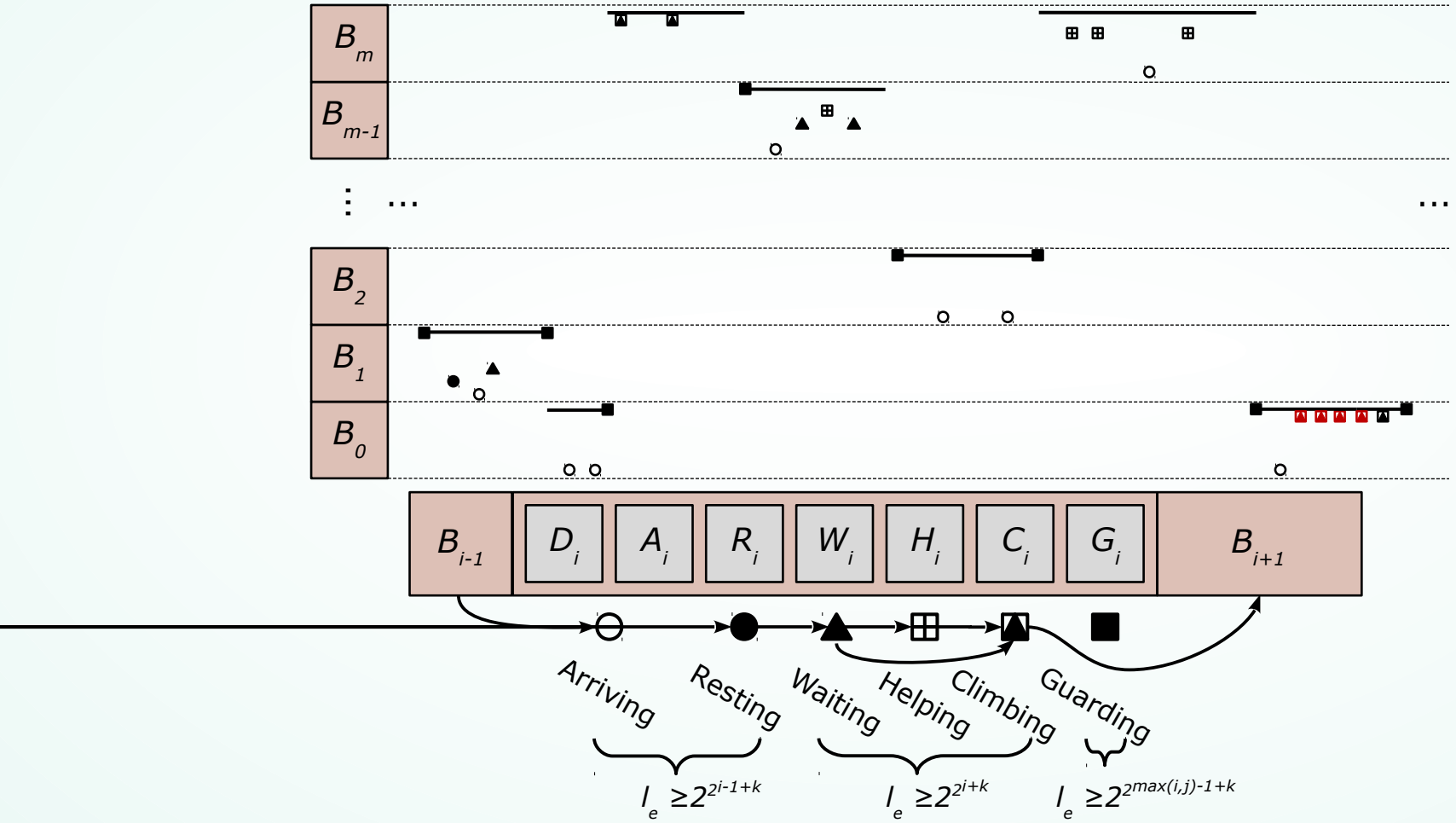

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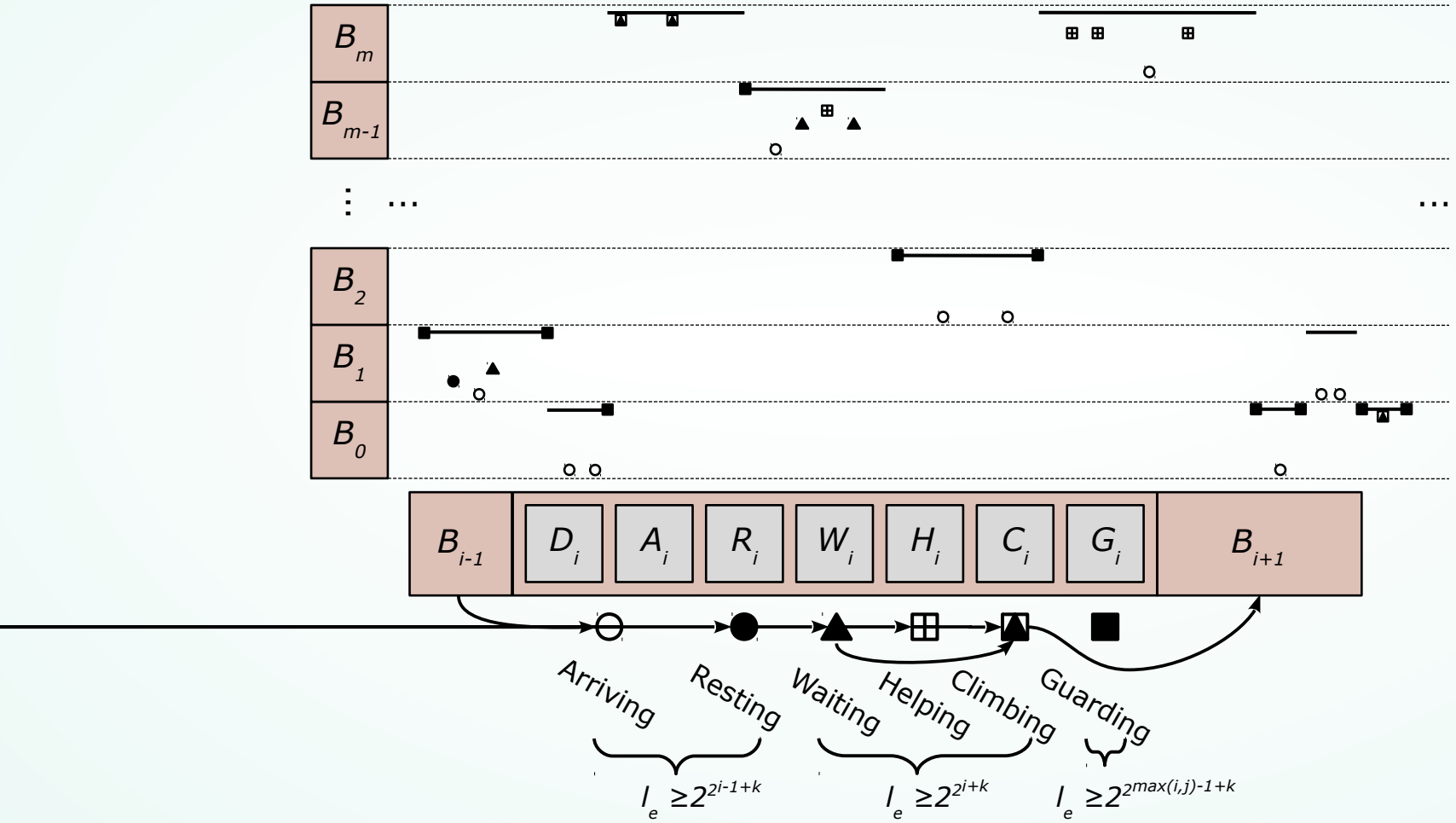

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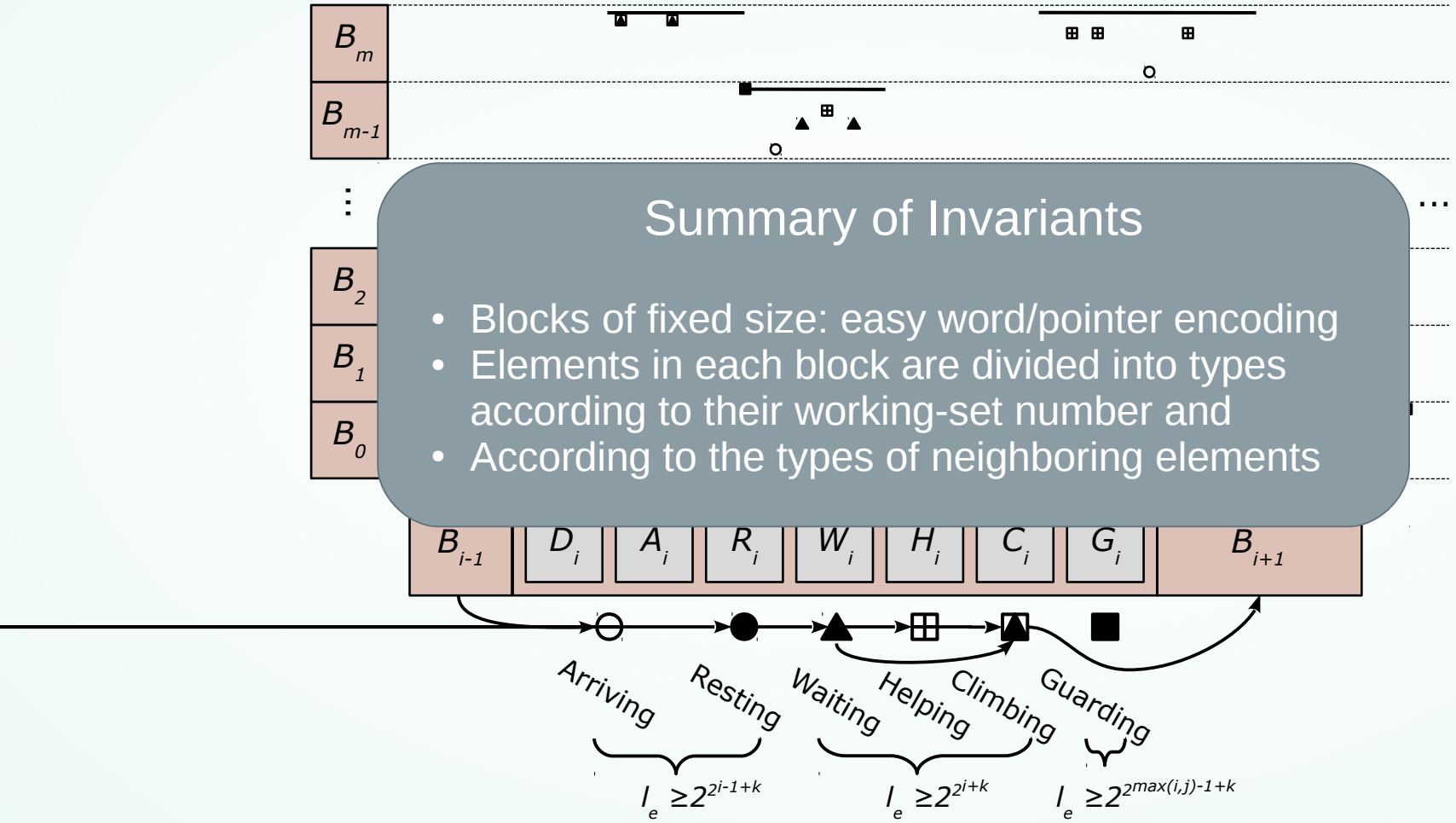

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- Blocks of fixed size: easy word/pointer encoding
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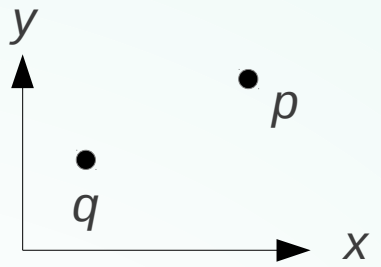
Outline

- Introduction to Algorithms and Data Structures
- Implicit Working-Set Dictionaries
- Skyline Queries
- Catenable Priority Queues with Attrition
- Conclusion

What are Skyline Queries?



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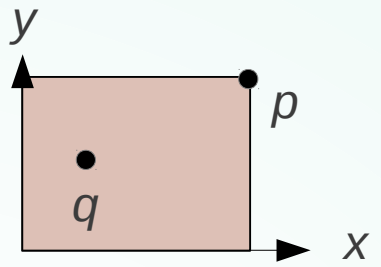
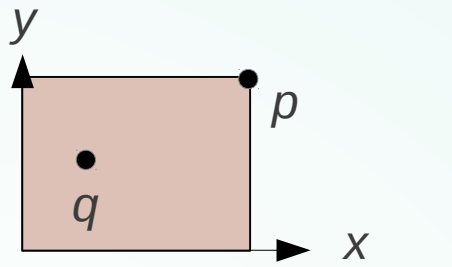
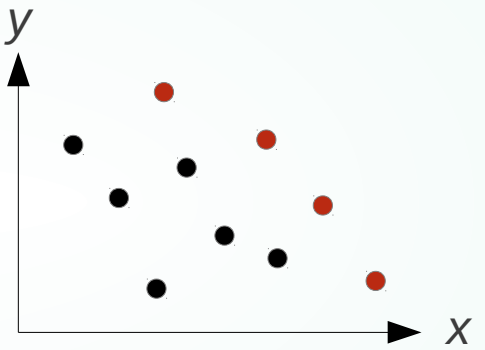


What are Skyline Queries?

- Given two points $p,q \in P \subseteq \mathbb{R}^2$ we say p *dominates* q iff $p_x \geq q_x$ and $p_y \geq q_y$



- The *maximal/skyline* points of a point set $P \subseteq \mathbb{R}^2$ are the undominated points



- Given a *dynamic* point set $P \subseteq \mathbb{R}^2$ we want to be able to find the skyline for a given query range $Q = [x_1,x_2] \times [y_1,y_2]$

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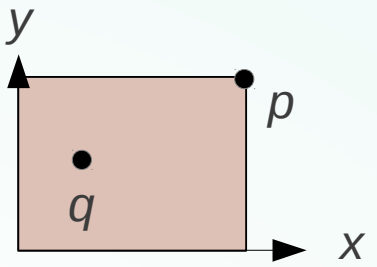

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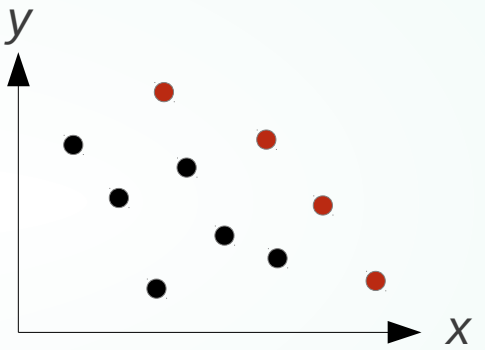

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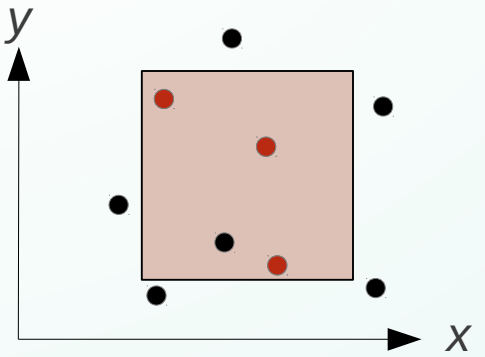
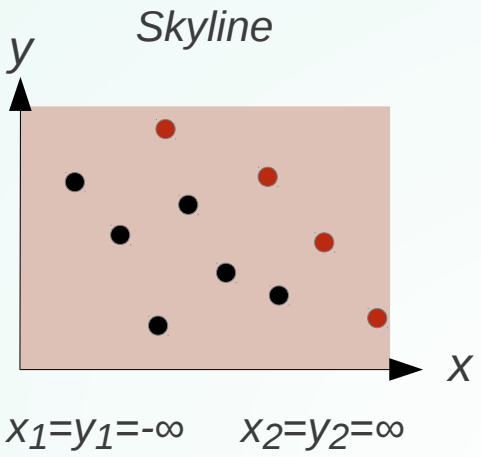


Special Cases of Skyline Queries



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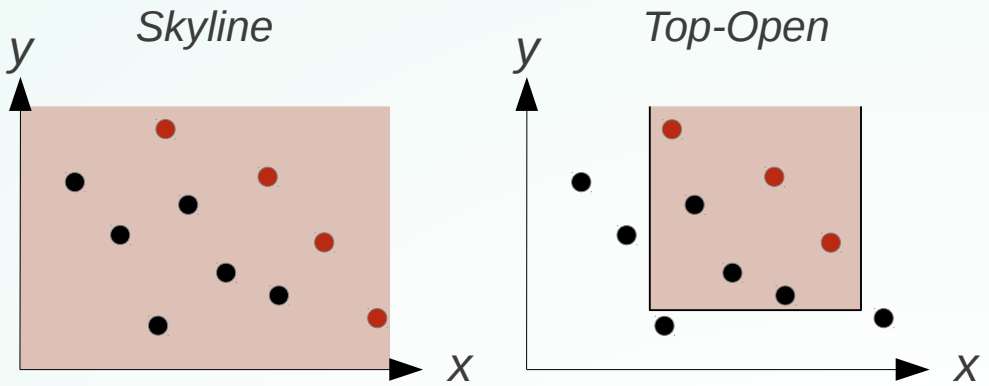

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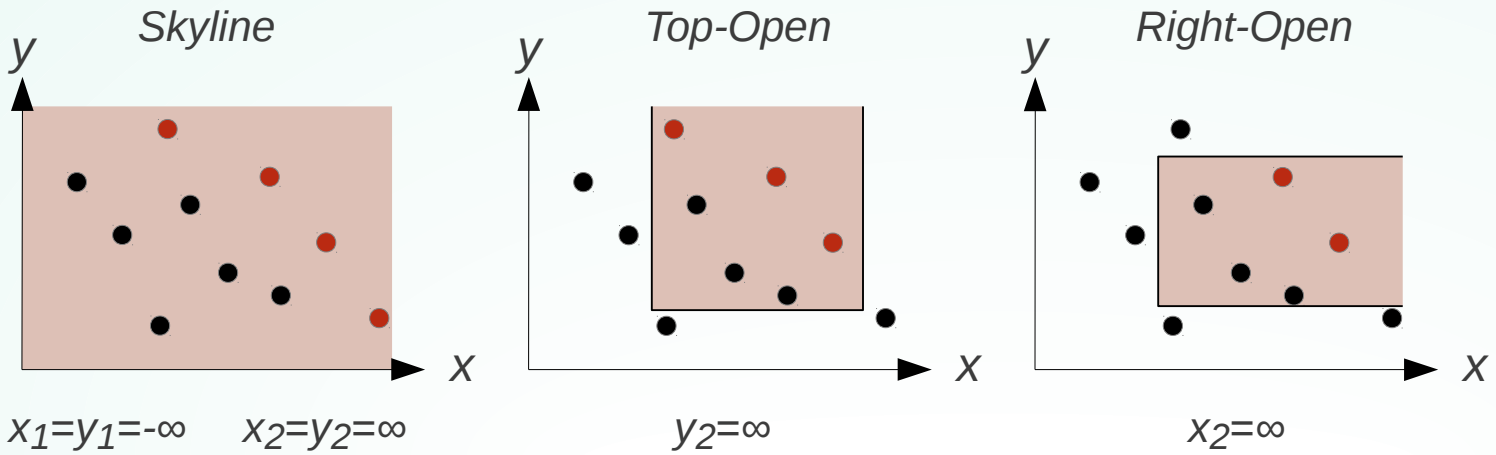

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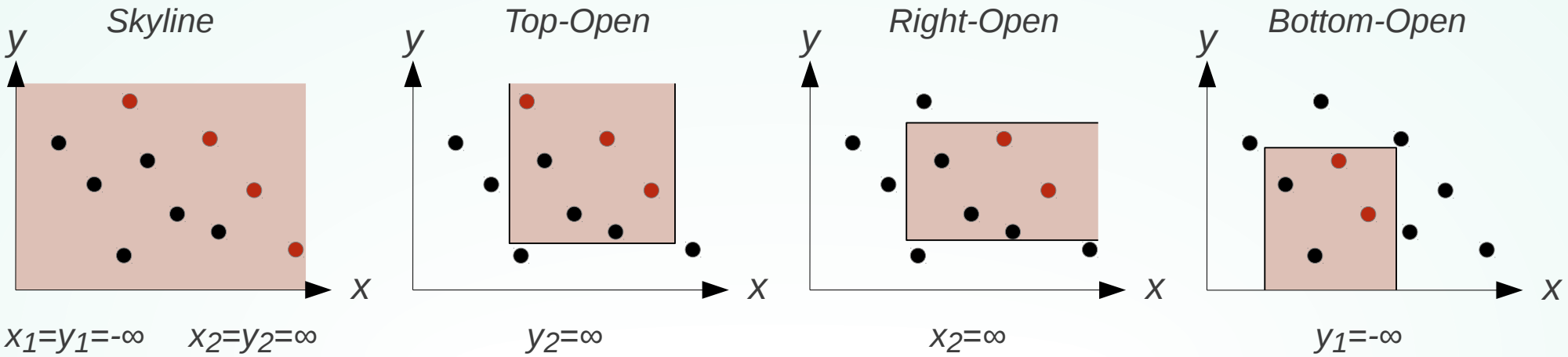

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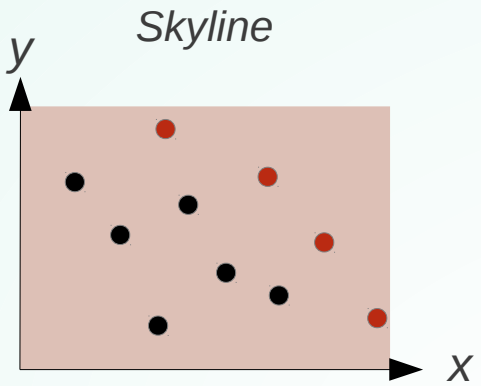

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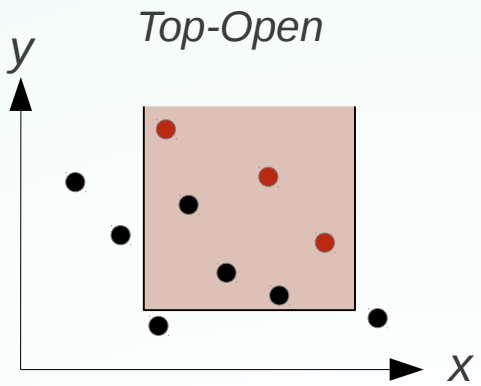

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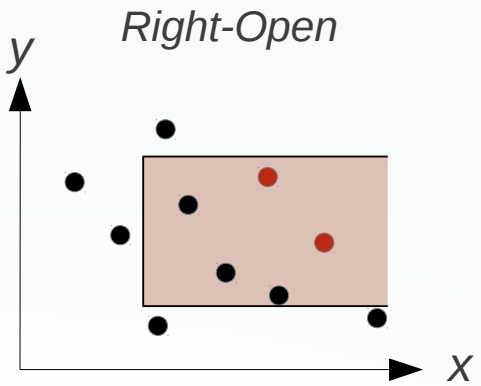

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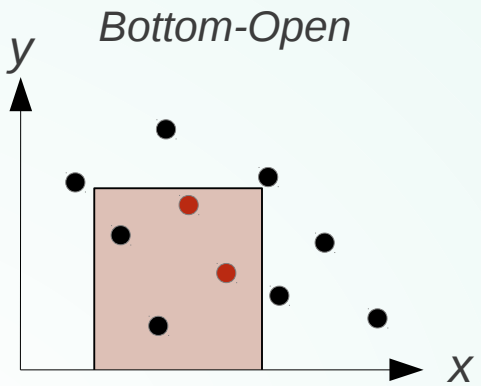

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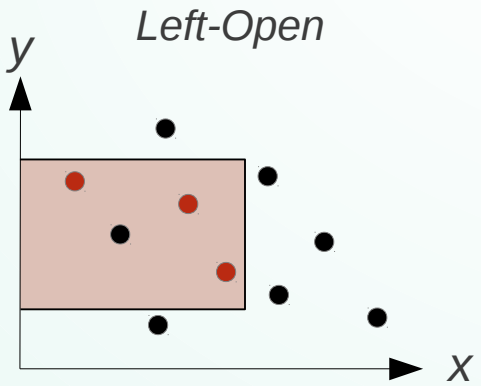

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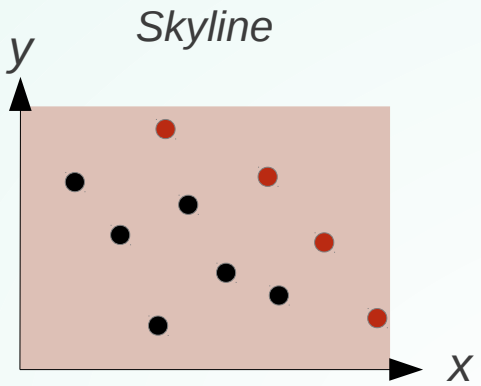

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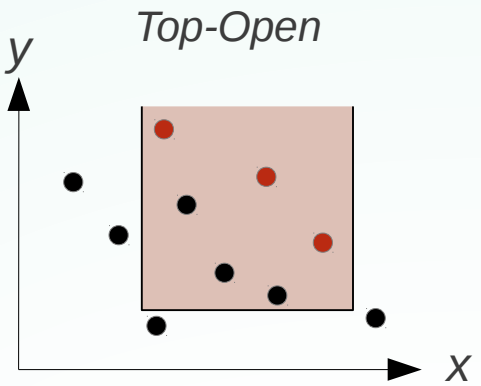

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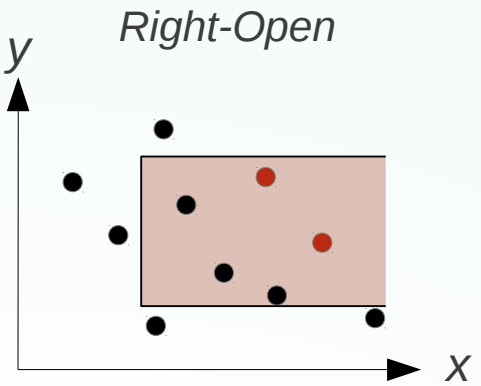

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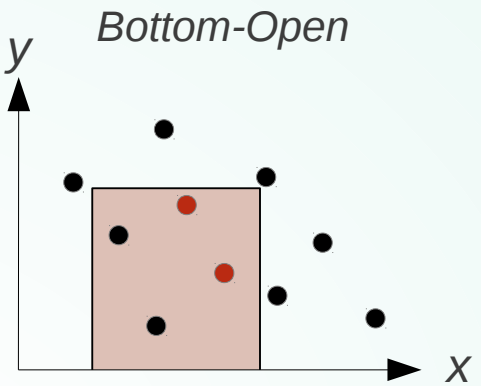

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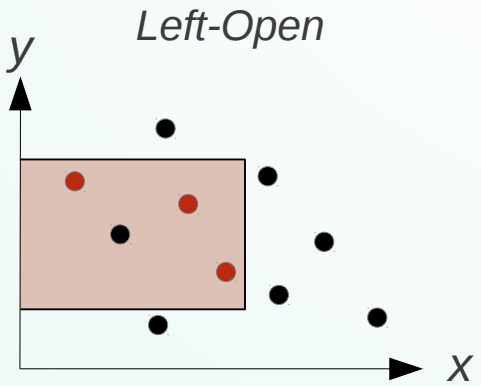

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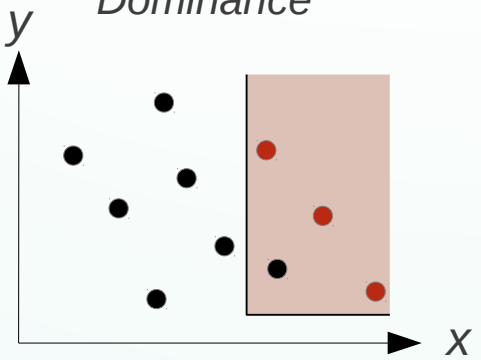

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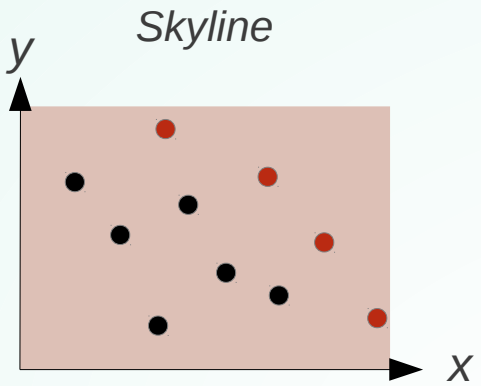

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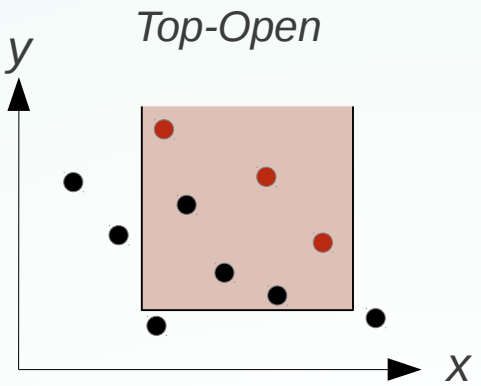

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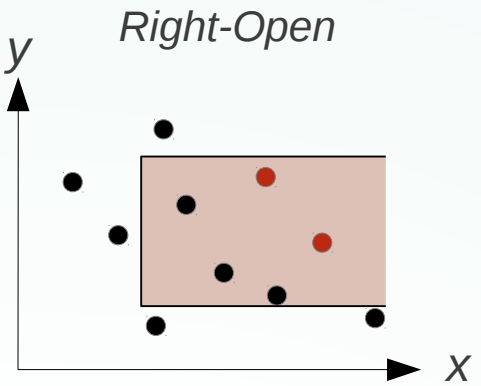

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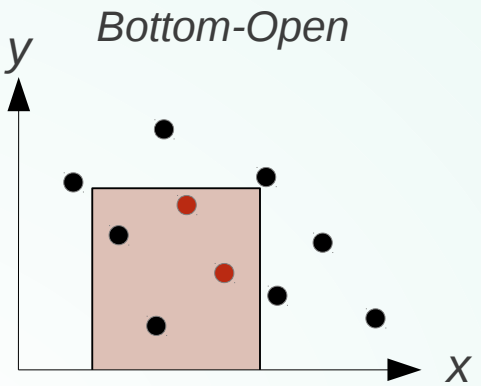

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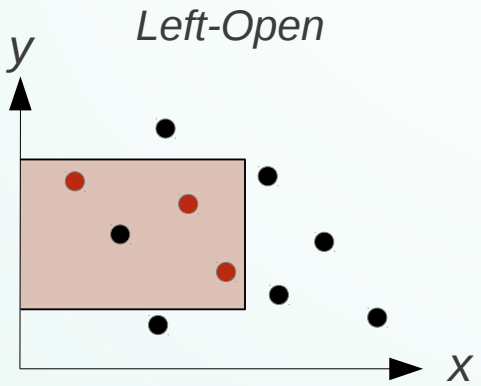

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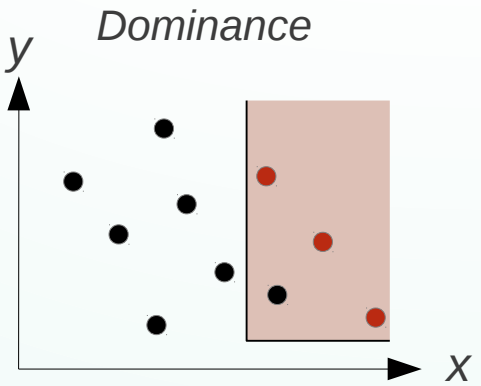

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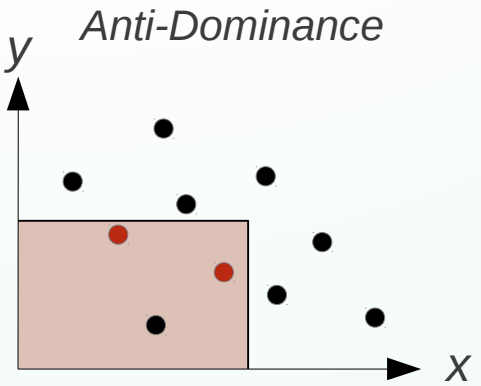
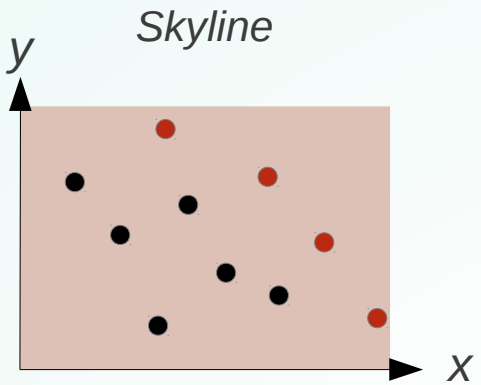
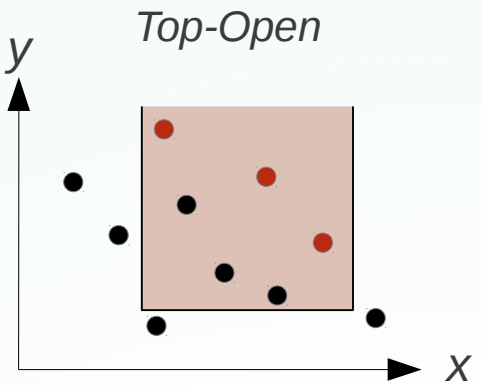


$x_1=y_1=-\infty$

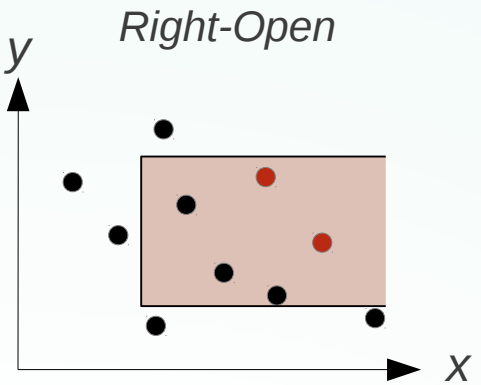
Special Cases of Skyline Queries



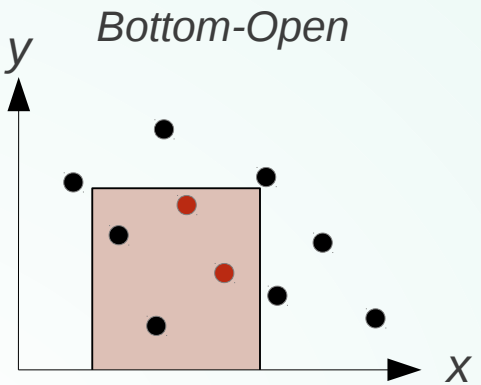
$x_1=y_1=-\infty$ $x_2=y_2=\infty$



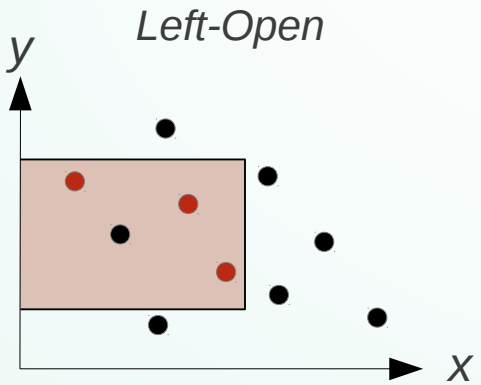
$y_2=\infty$



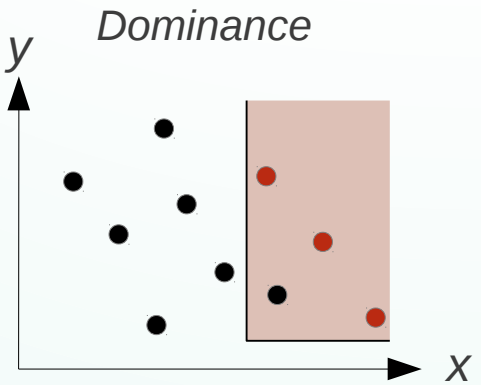
$x_2=\infty$



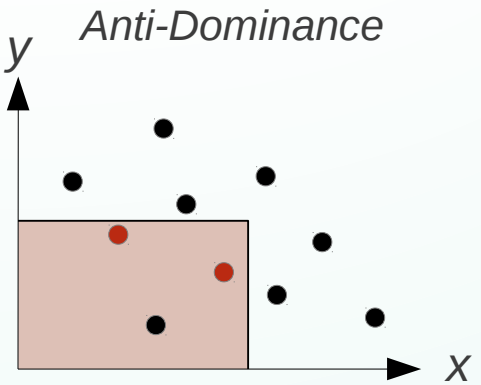
$y_1=-\infty$



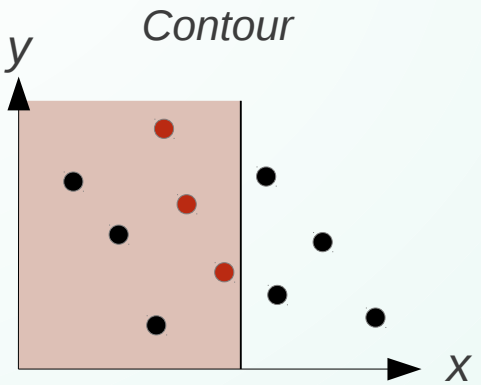
$x_1=-\infty$



$x_2=y_2=\infty$

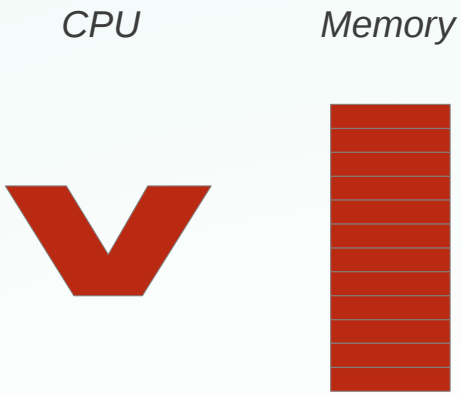


$x_1=y_1=-\infty$

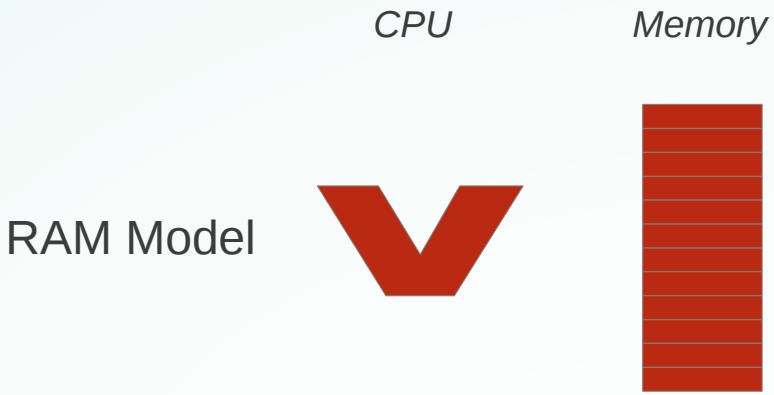


$x_1=y_1=-\infty$ $y_2=\infty$

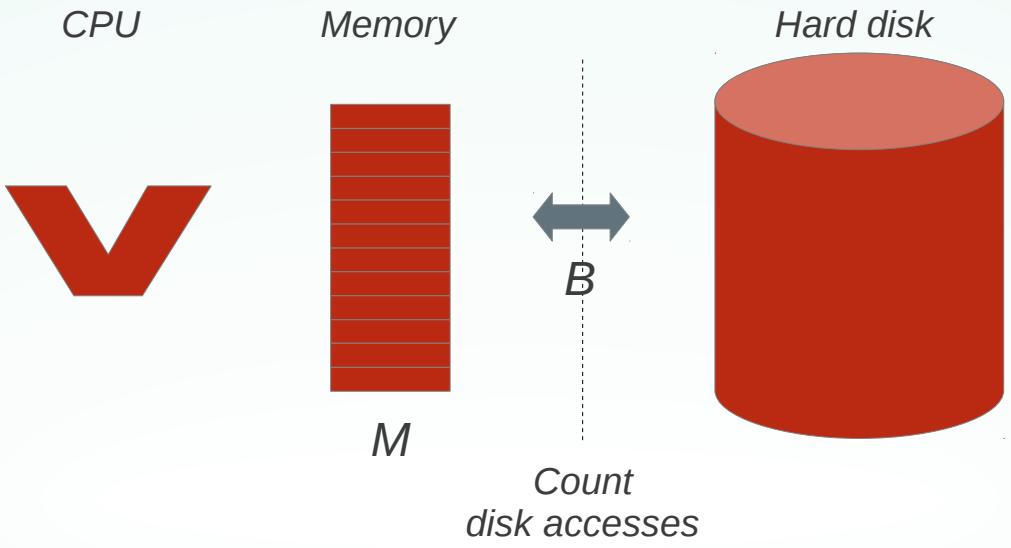
EM Model



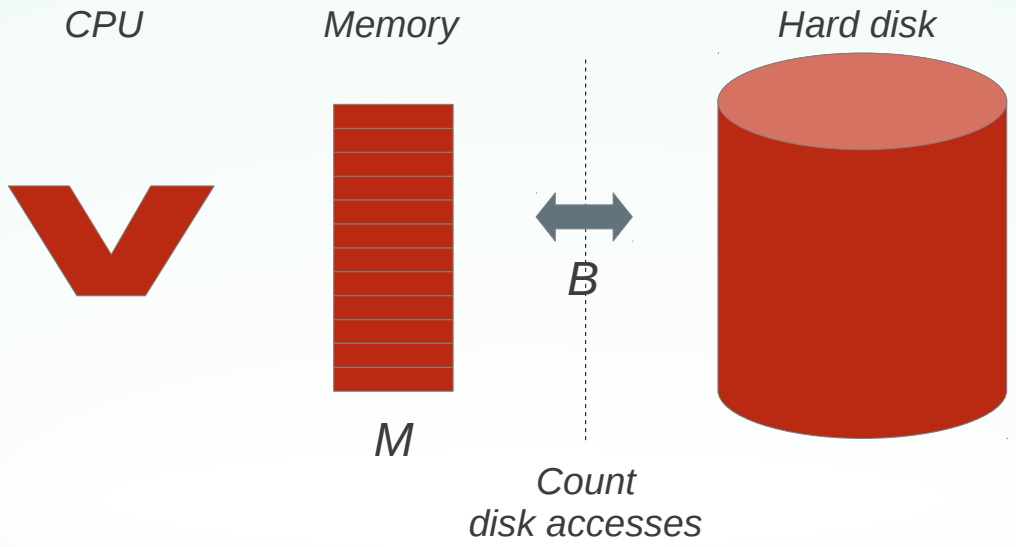
EM Model



EM Model

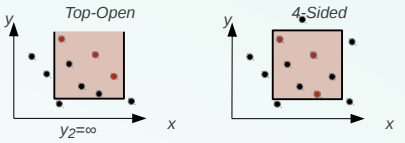


EM Model



- We count the number of disk accesses, not CPU instructions
- When reading/writing from/to disk, we can access B consecutive elements in one I/O
- Our algorithms should spent $O(1/B)$ I/Os to access one element
- Scanning uses $O(n/B)$ I/Os and search trees uses $O(\log_B n)$ I/Os

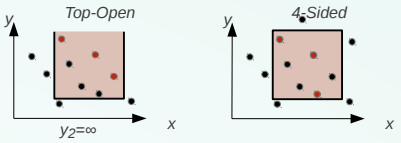
Previous and Our Results



Problem	Space	Pre-proces	Query	Query	Update	Domain	Source
Top-Open	$O(n)$	$O(n \log n)$	$O(1+k)$	-	-	$\mathbb{R}$	YA10
Top-Open	$O(n)$	$O(n \log n)$	$O(\log n+k)$	-	$O(\log n)$	$\mathbb{R}$	BT11
Top-Open	$O(n)$	$O(n \log n/\log\log n)$	$O(\log n/\log\log n+k)$	-	$O(\log n/\log\log n)$	$\mathbb{R}$	BT11
4-sided	$O(n \log n)$	$O(n \log n)$	$O(\log n+k)$	-	-	$\mathbb{R}$	KDKS11
4-sided	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n+k)$	-	-	Rank	GKASK97
4-sided	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n+k)$	-	$O(\log^2 n)$	$\mathbb{R}$	BT11
Top-Open	$O(n/B)$	$O(n/B \log_{M/B} n/B)$	$O(n/B)$	-	-	$\mathbb{R}$	PTFS05 ST11
Top-Open	Heuristics, various update types: HKIT06,PTFS05,TO06,WAEA07					$\mathbb{R}$	
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_B n+k/B)$	-	-	$\mathbb{R}$	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log\log_B U+k/B)$	-	-	U	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(1+k/B)$	-	-	Rank	KTTTY13
4-sided	$O(n/B)$	$O(n/B^*)$	$O((n/B)^\varepsilon+k/B)$	$\Omega((n/B)^\varepsilon+k/B)$	-	$\mathbb{R}$	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_{2B^\varepsilon} n/B+k/B^{1-\varepsilon})$	-	$O(\log_{2B^\varepsilon} n/B)$	$\mathbb{R}$	KTTTY13
4-sided	$O(n/B)$	$O(n/B^*)$	$O((n/B)^\varepsilon+k/B)$	$\Omega((n/B)^\varepsilon+k/B)$	$O(\log n/B)$	$\mathbb{R}$	KTTTY13

*** Assumes pre-sorting**

Previous and Our Results



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Top-Open	$O(n)$	$O(n \log n/\log\log n)$	$O(\log n/\log\log n+k)$	-	$O(\log n/\log\log n)$	$\mathbb{R}$	BT11
4-sided	$O(n \log n)$	$O(n \log n)$	$O(\log n+k)$	-	-	$\mathbb{R}$	KDKS11
4-sided	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n+k)$	-	-	Rank	GKASK97
4-sided	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n+k)$	-	$O(\log^2 n)$	$\mathbb{R}$	BT11
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Top-Open	Heuristics, various update types: HKIT06,PTFS05,TO06,WAEA07					$\mathbb{R}$	
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_B n+k/B)$	-	-	$\mathbb{R}$	KTTTY13
Top-Open	$O(n/B)$			-	-	U	KTTTY13
Top-Open	$O(n/B)$			-	-	Rank	KTTTY13
4-sided	$O(n/B)$	$O(n/B^*)$	$O((n/B)^\varepsilon+k/B)$	$\Omega((n/B)^\varepsilon+k/B)$	-	$\mathbb{R}$	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_{2B^\varepsilon} n/B+k/B^{1-\varepsilon})$	-	$O(\log_{2B^\varepsilon} n/B)$	$\mathbb{R}$	KTTTY13
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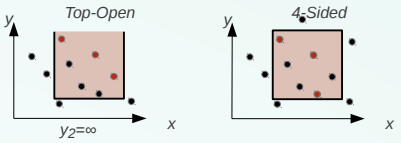
Indexability Model
Indivisibility Assumption

*** Assumes pre-sorting**

Previous and Our Results



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4-sided	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n)$	$O(n \log n/\log\log n+k)$	-	-	Rank	GKASK97
4-sided	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n+k)$	-	$O(\log^2 n)$	$\mathbb{R}$	BT11
Top-Open	$O(n/B)$	$O(n/B \log_{M/B} n/B)$	$O(n/B)$	-	-	$\mathbb{R}$	PTFS05 ST11
Top-Open	Heuristics, various update types: HKIT06,PTFS05,TO06,WAEA07					$\mathbb{R}$	
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_B n+k/B)$	-	-	$\mathbb{R}$	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log\log_B U+k/B)$	-	-	U	KTTTY13
Top-Open	$O(n/B)$	$O(n/B^*)$	$O(1+k/B)$	-	-	Rank	KTTTY13
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Top-Open	$O(n/B)$	$O(n/B^*)$	$O(\log_{2B^\varepsilon} n/B+k/B^{1-\varepsilon})$	-	$O(\log_{2B^\varepsilon} n/B)$	$\mathbb{R}$	KTTTY13
4-sided	$O(n/B)$	$O(n/B^*)$	$O((n/B)^\varepsilon+k/B)$	$\Omega((n/B)^\varepsilon+k/B)$	$O(\log n/B)$	$\mathbb{R}$	KTTTY13

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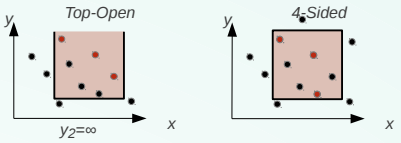
Previous and Our Results



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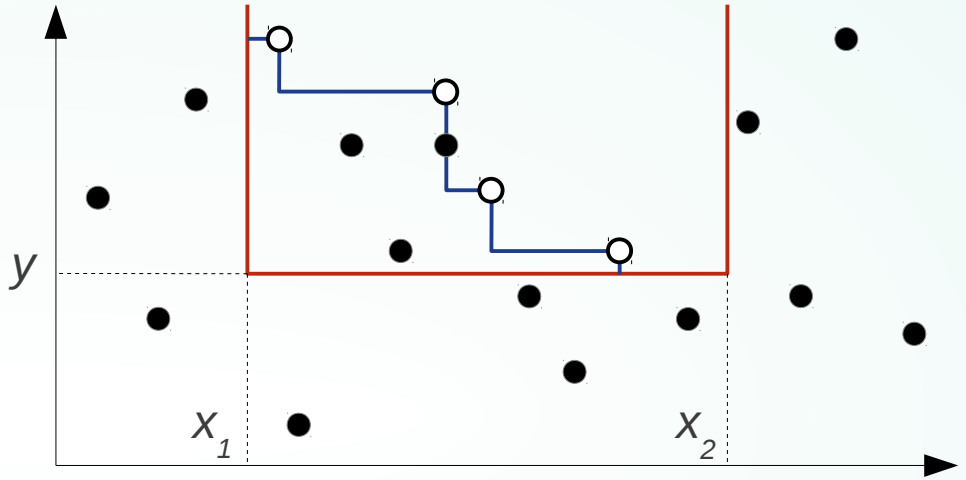
For Today!

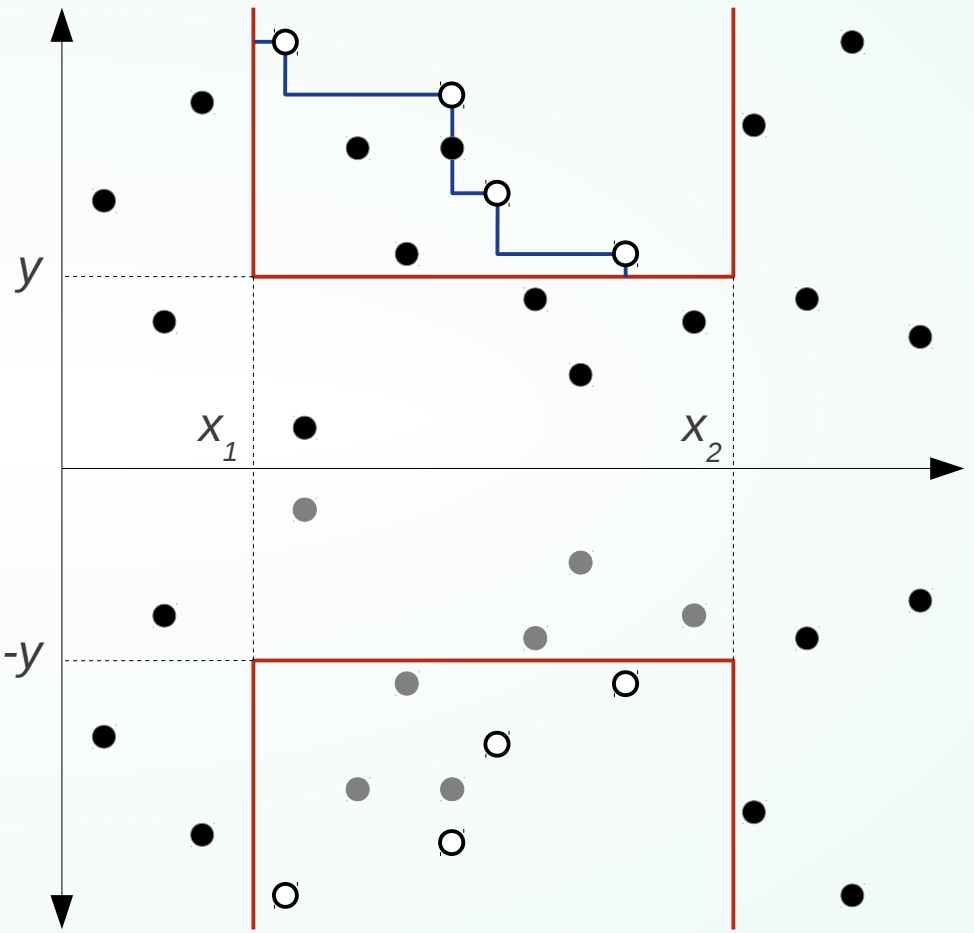
* Assumes pre-sorting

Observation for Top-Open Queries



Observation for Top-Open Queries

- Consider mirroing the point set in the *y*-axis
- Let *x* represent the insertion time and *y* the key space
- When inserting element *e* it deletes/*attrites* all elements inserted before it with a larger key
- Non-attrited points and undominated points are equivalent



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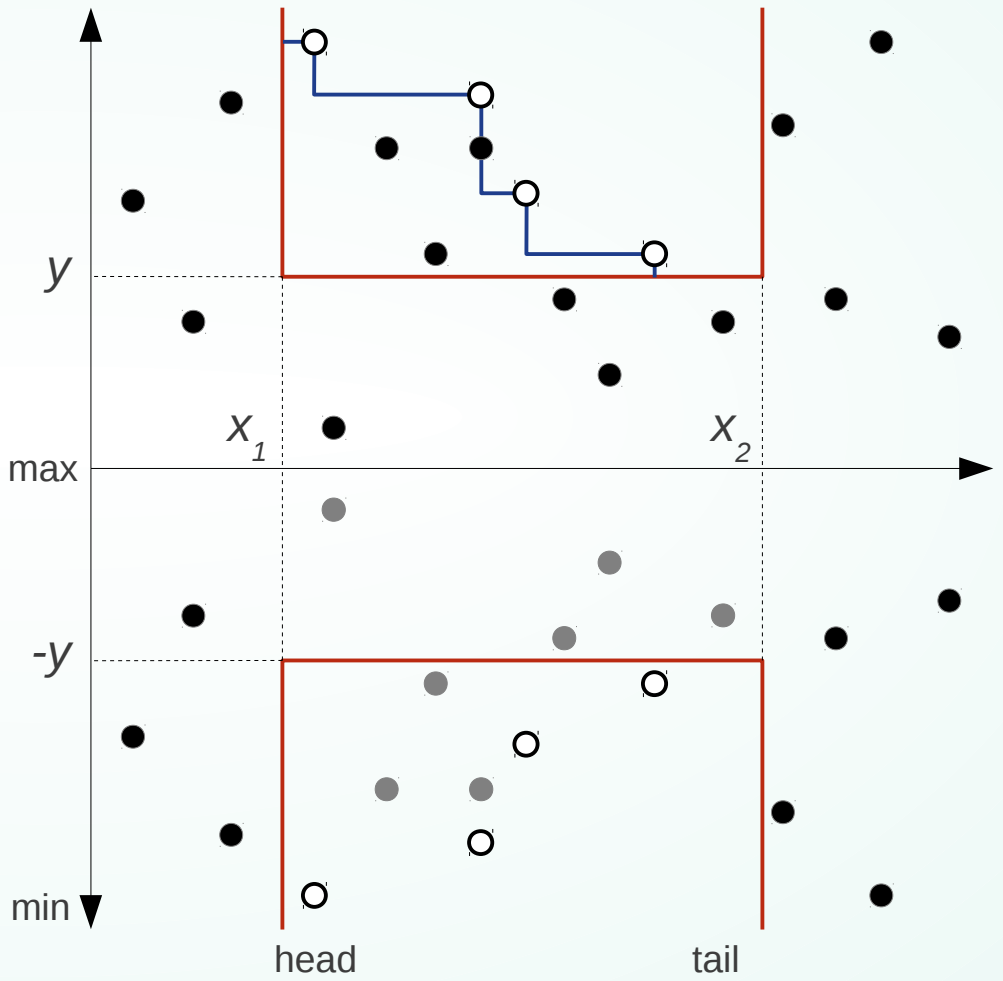

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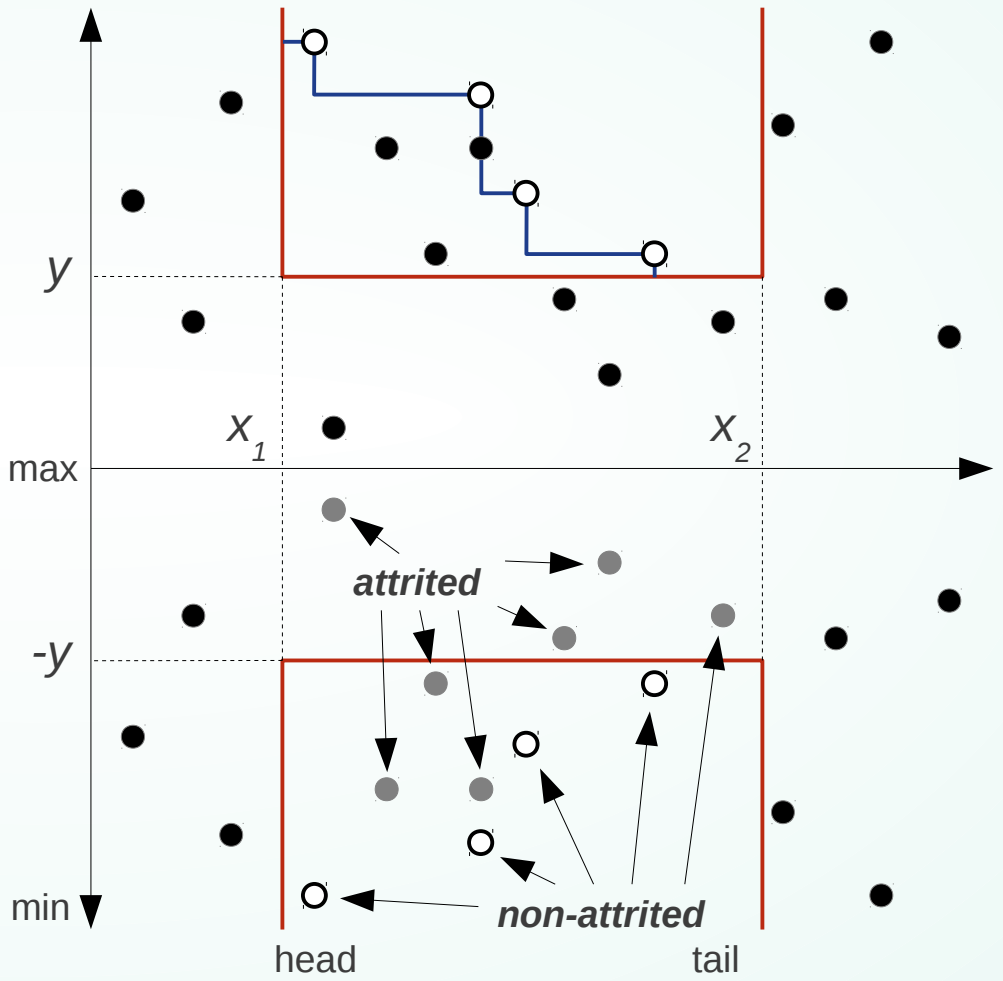
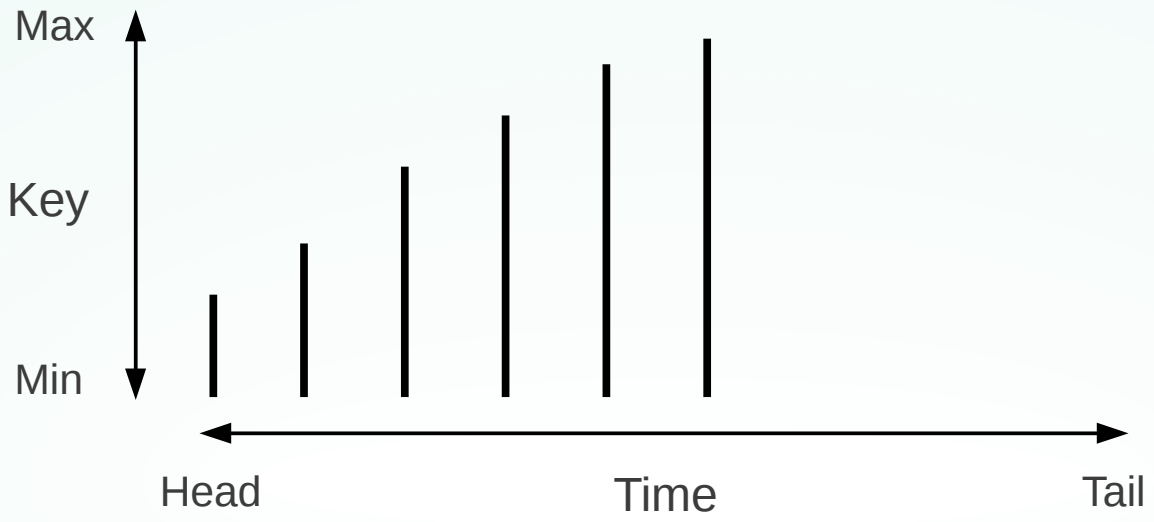
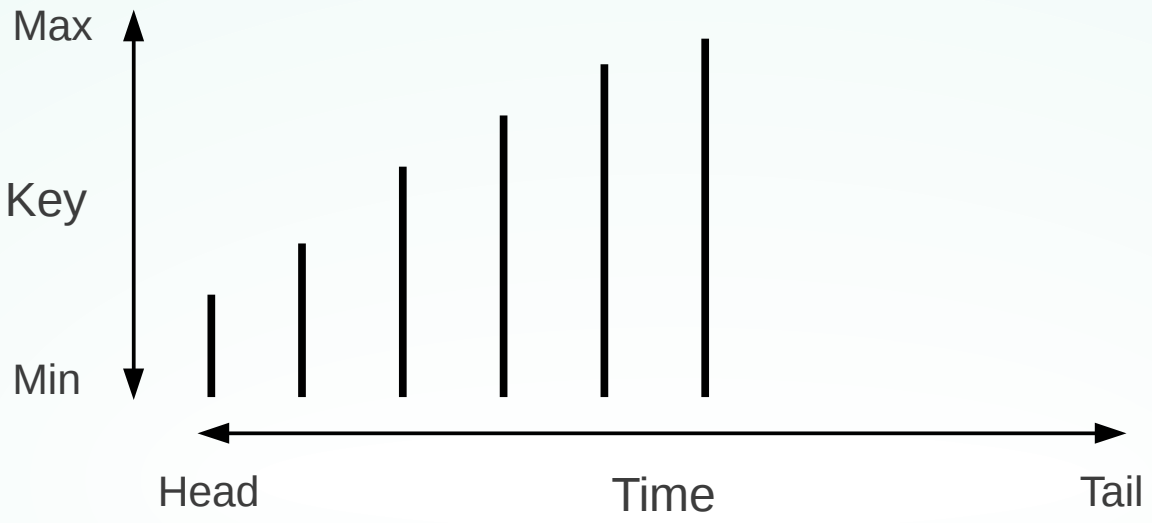


Priority Queues with Attrition

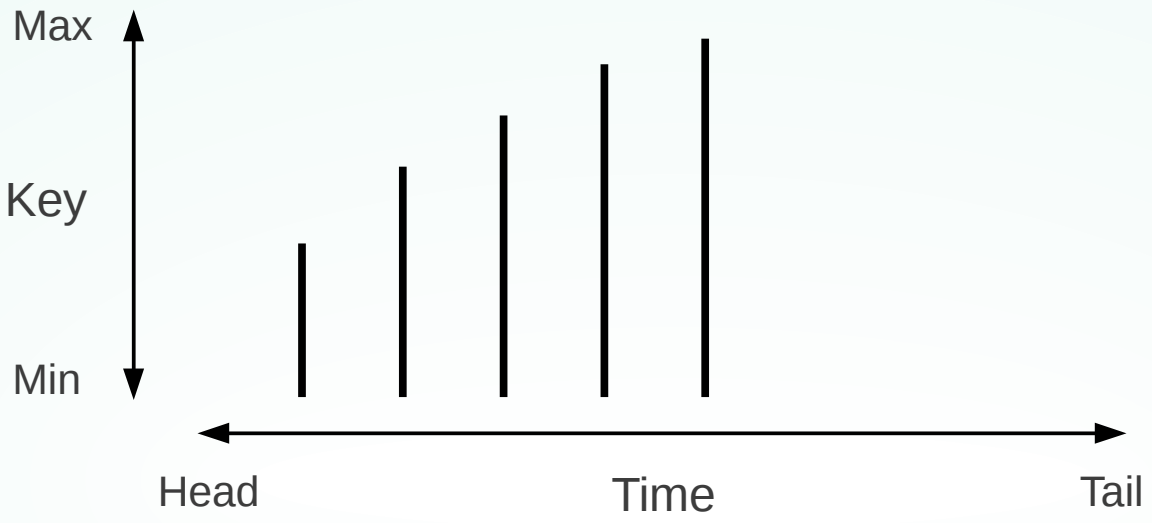


Priority Queues with Attrition



- DeleteMin(): Deletes the head/minimum element of the queue
- InsertAndAttrite(e): Inserts e and deletes/attrites all elements before e with a key larger or equal to e
- ConcatenateAndAttrite(Q_1,Q_2): Deletes/attrites all elements in Q_1 with a key larger or equal to min(Q_2) and appends Q_2

Priority Queues with Attrition



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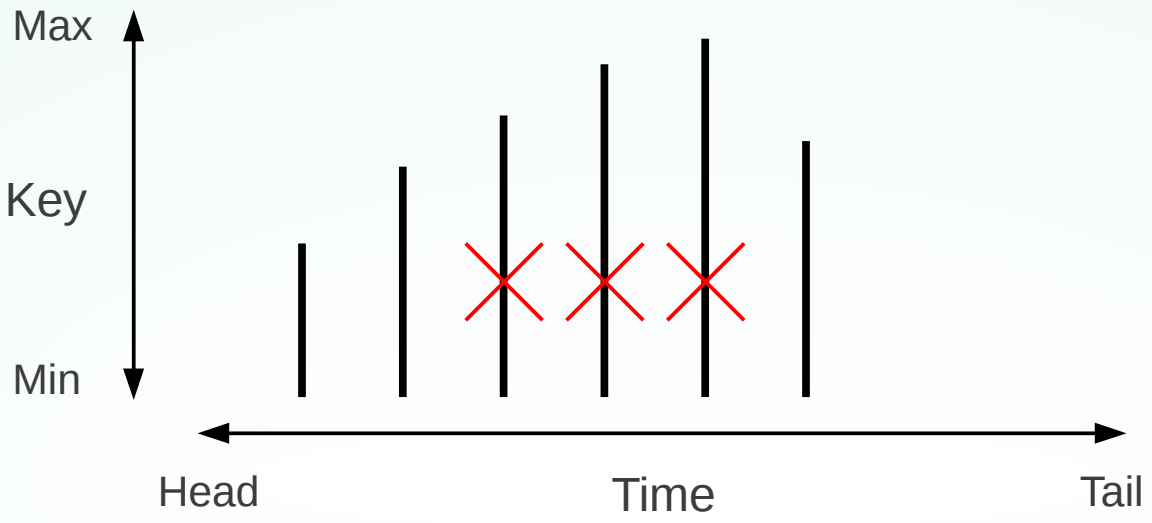

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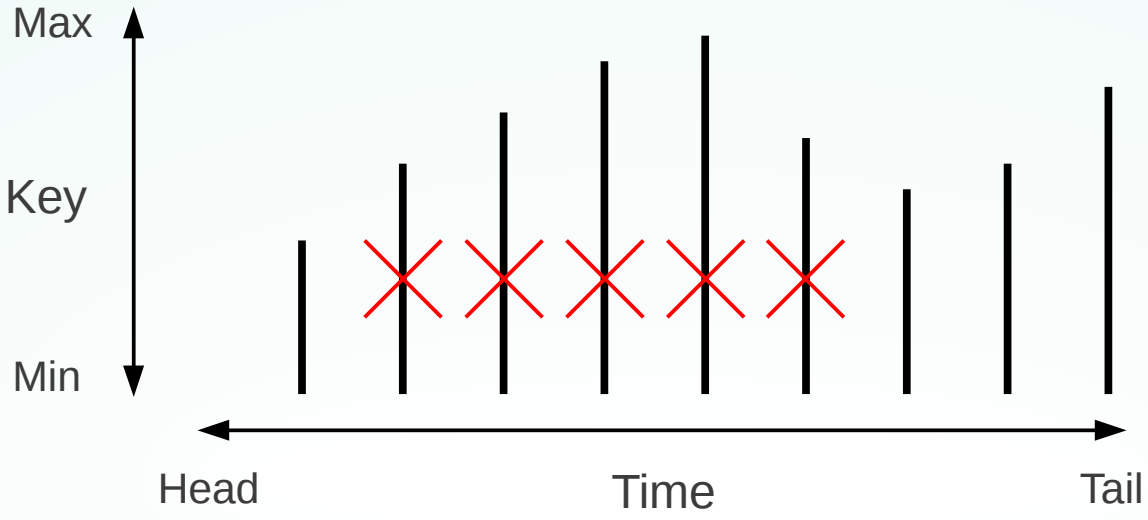

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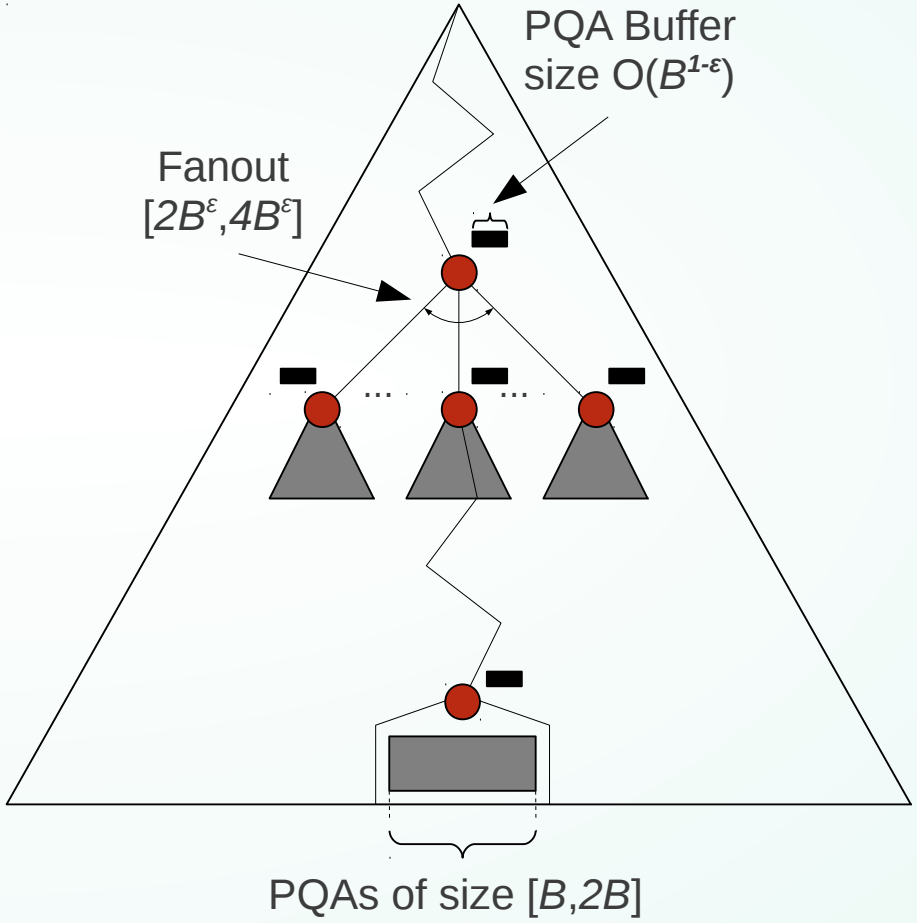
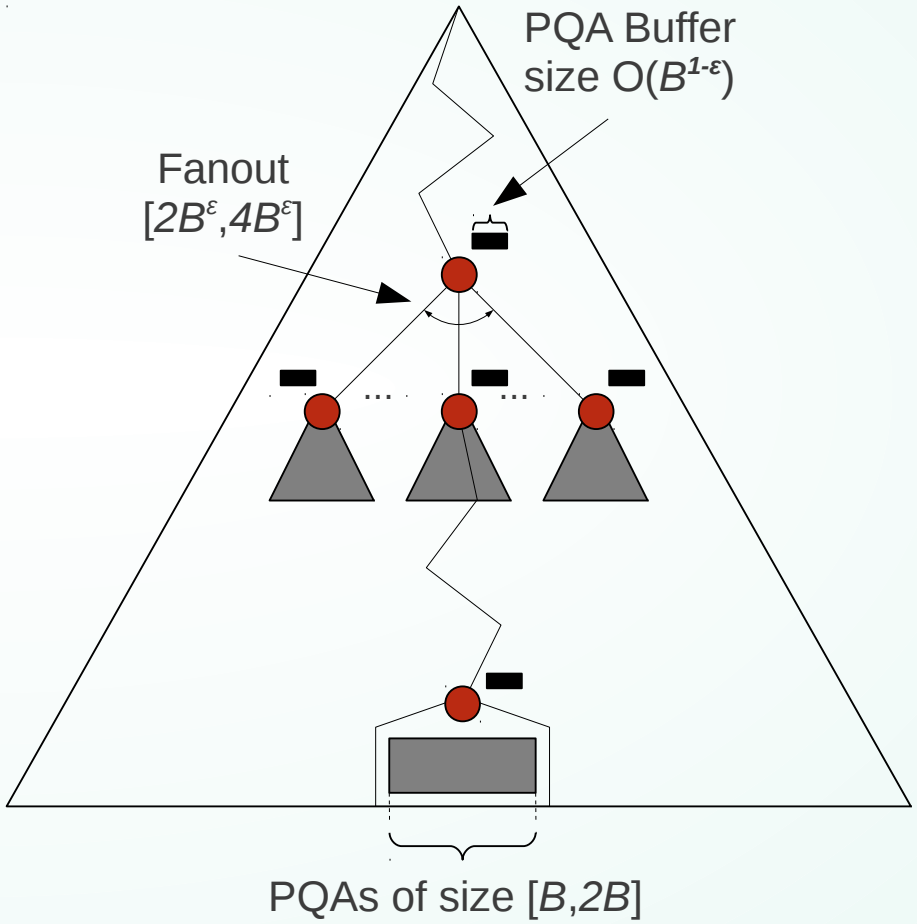
Data Structure and Invariants



Data Structure and Invariants

- A $(2B^{\varepsilon},4B^{\varepsilon})$-tree augmented with PQAs
- Internal node have between $2B^{\varepsilon}$ and $4B^{\varepsilon}$ children and stores a PQA which is the concatenation of its childrens PQAs
- Leafs stores a PQA over the B to $2B$ elements it contains
- Updating element e discards the PQAs on the path to e and rebuilds them again



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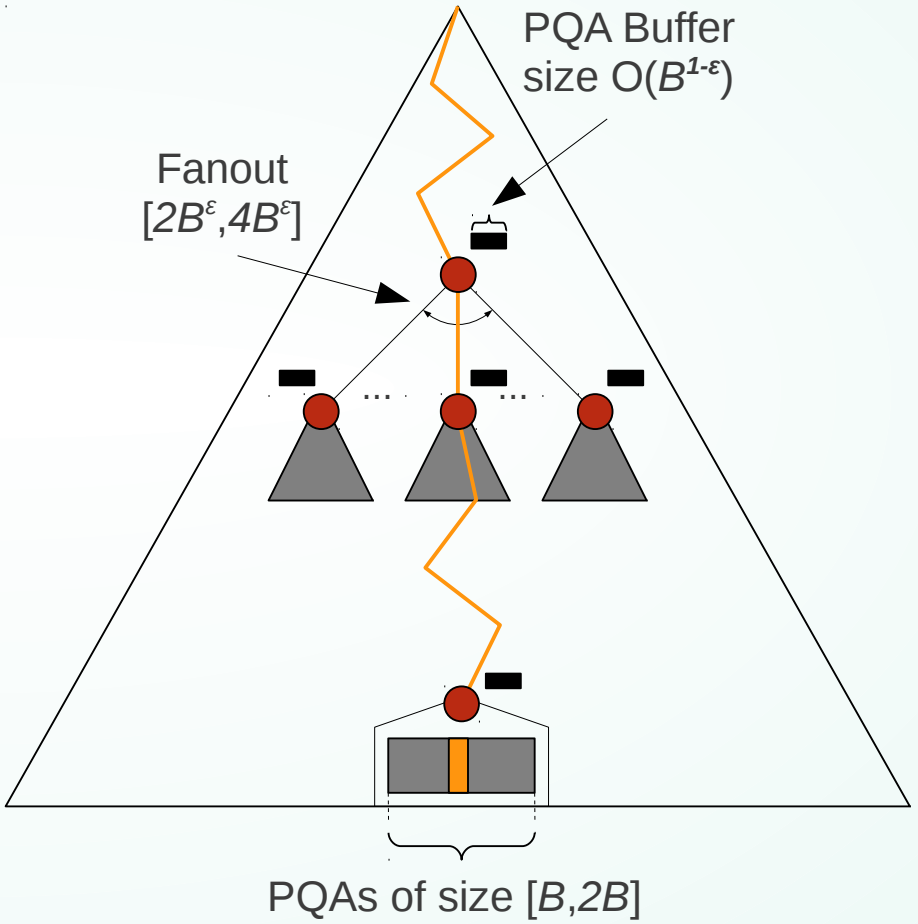

Data Structure and Invariants

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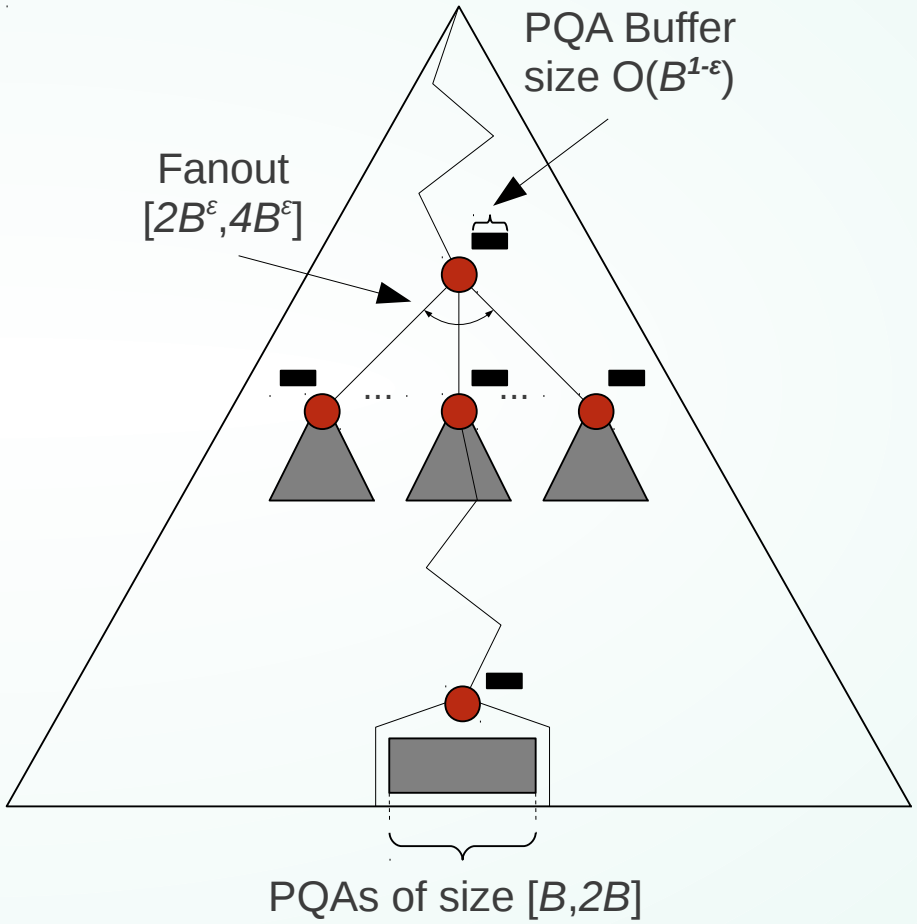
- Internal node have between $2B^{\varepsilon}$ and $4B^{\varepsilon}$ children and stores a PQA which is the concatenation of its childrens PQAs
- Leafs stores a PQA over the B to $2B$ elements it contains
- Updating element e discards the PQAs on the path to e and rebuilds them again



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  conc
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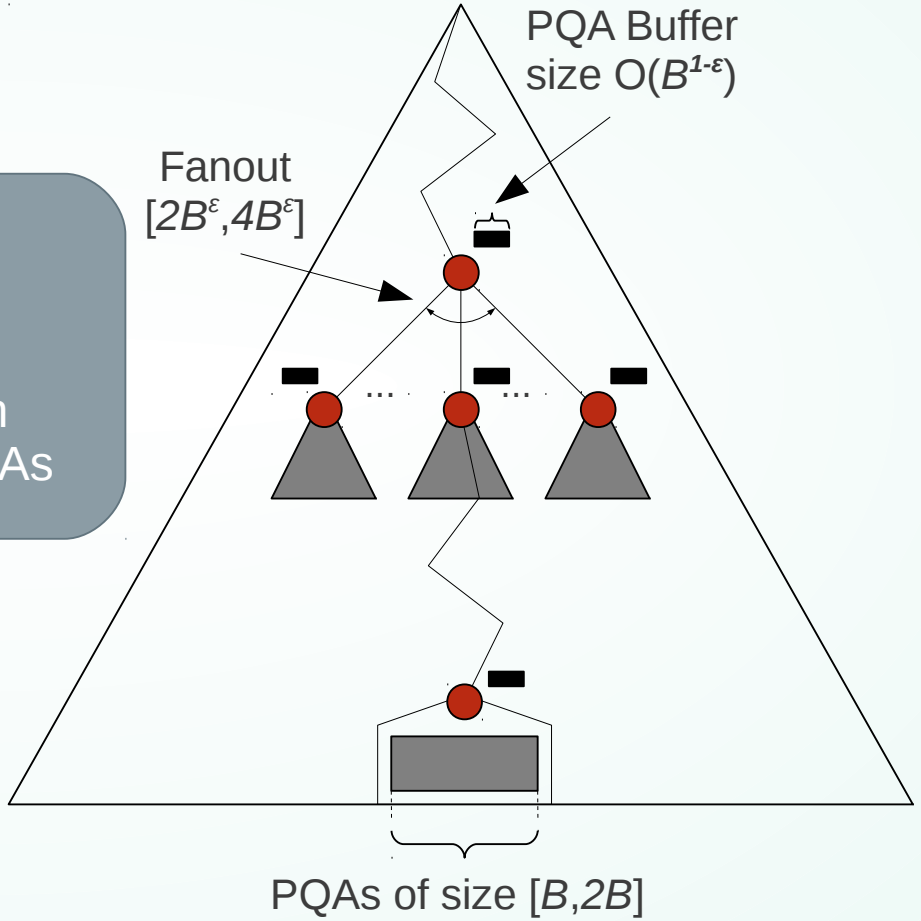
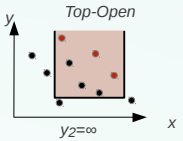
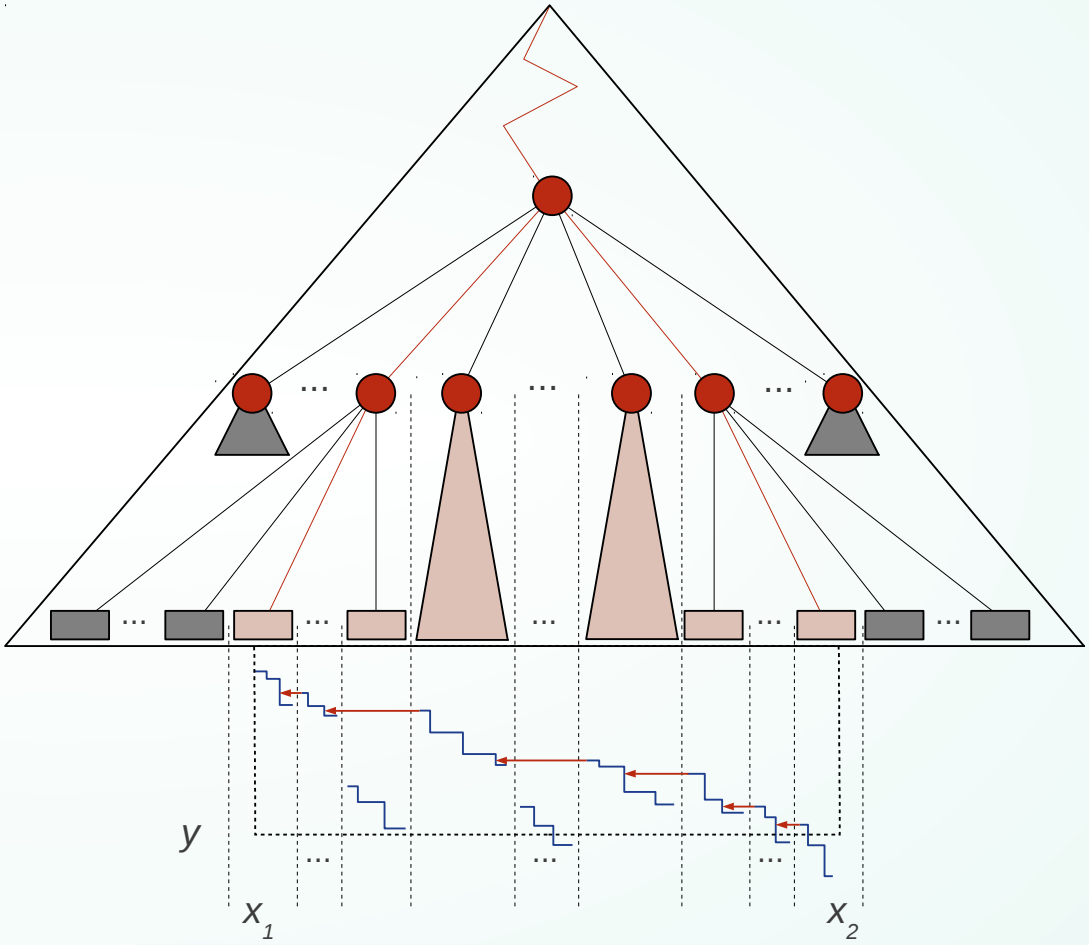
Summary of Invariants

- Nodes have a bounded degree
- All leaves have the same depth
- Nodes are augmented with PQAs



Top-Open Skyline Queries





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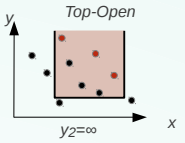
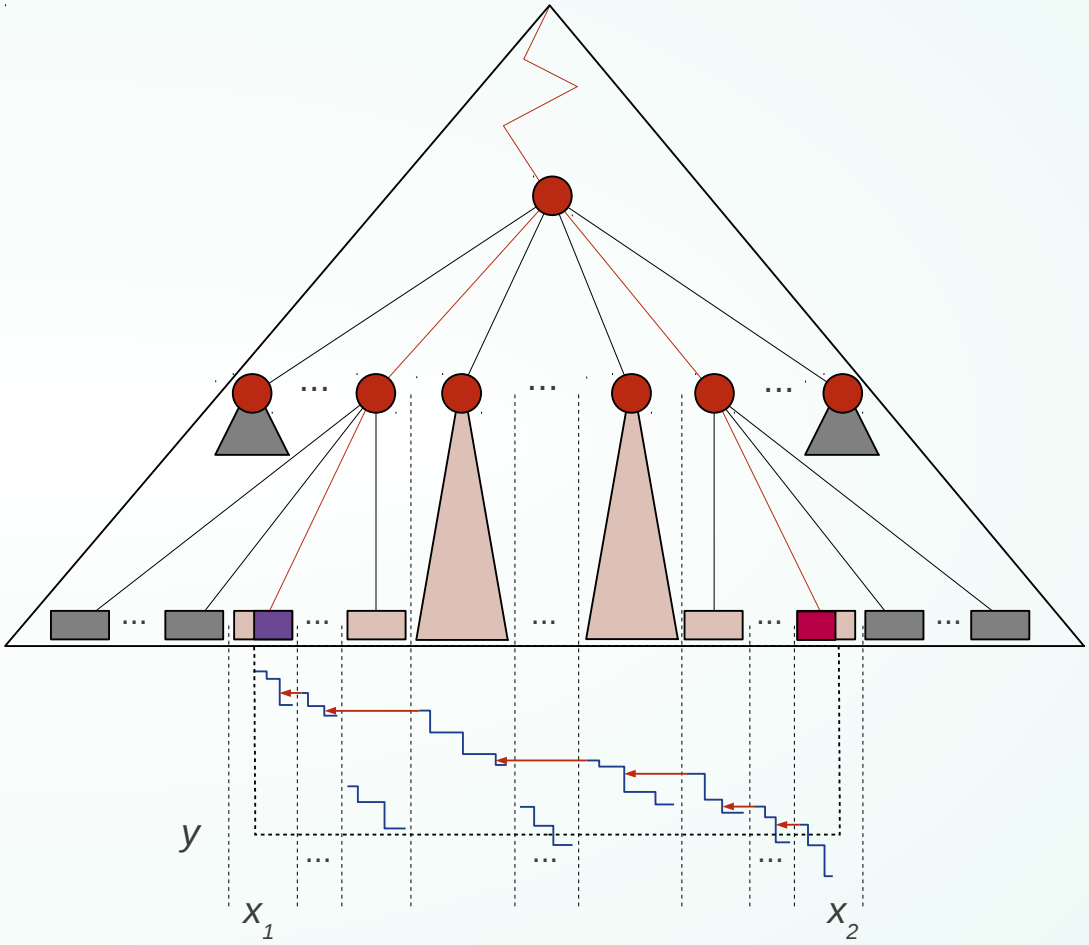


- We find the leafs of x_1 and x_2 and make two PQAs of the elements within $[x_1,x_2]$ called Q_1 and Q_2
- We concatenate Q_1, all PQAs of subtrees inside $[x_1,x_2]$ and Q_2 into one PQA Q (Divide and conquer)
- We call DeleteMin on Q and report the returned element e unless e has y-value larger than $-y$



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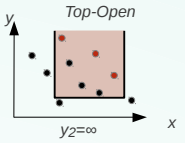

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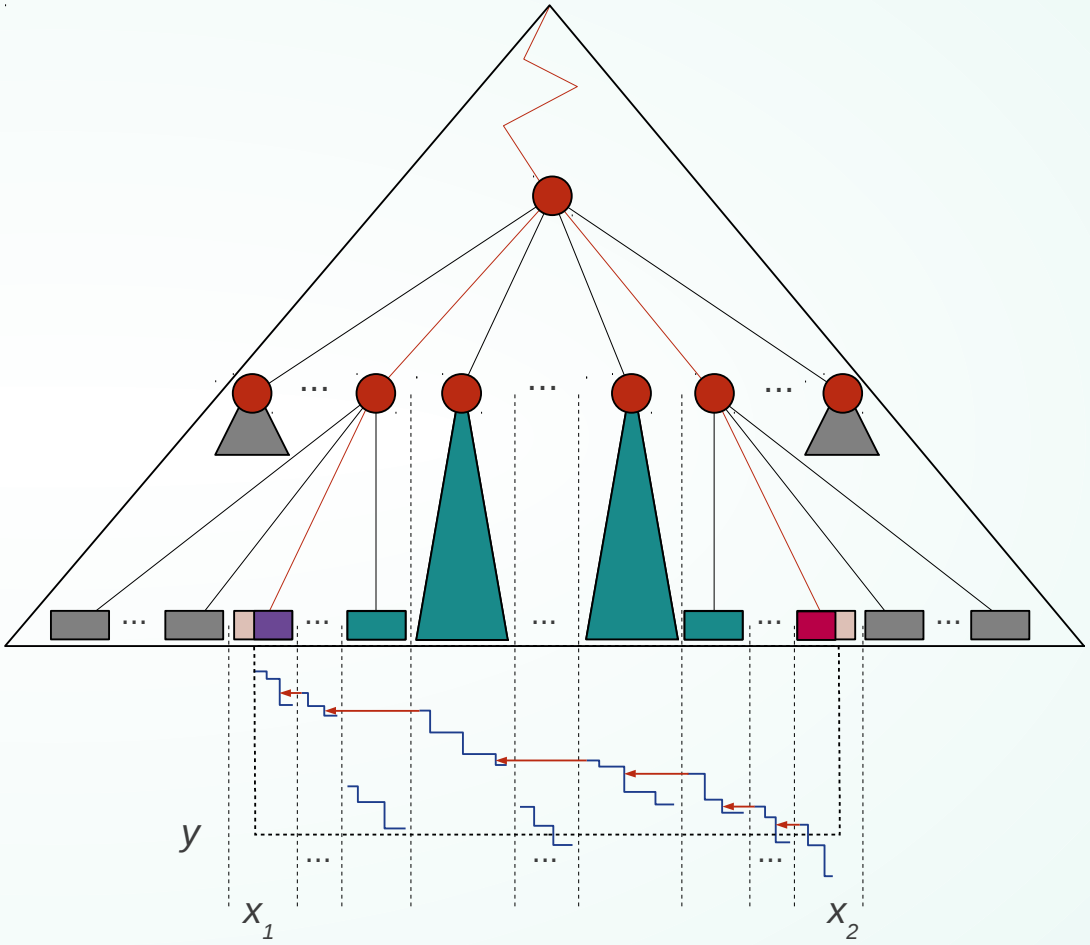

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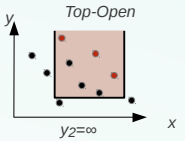

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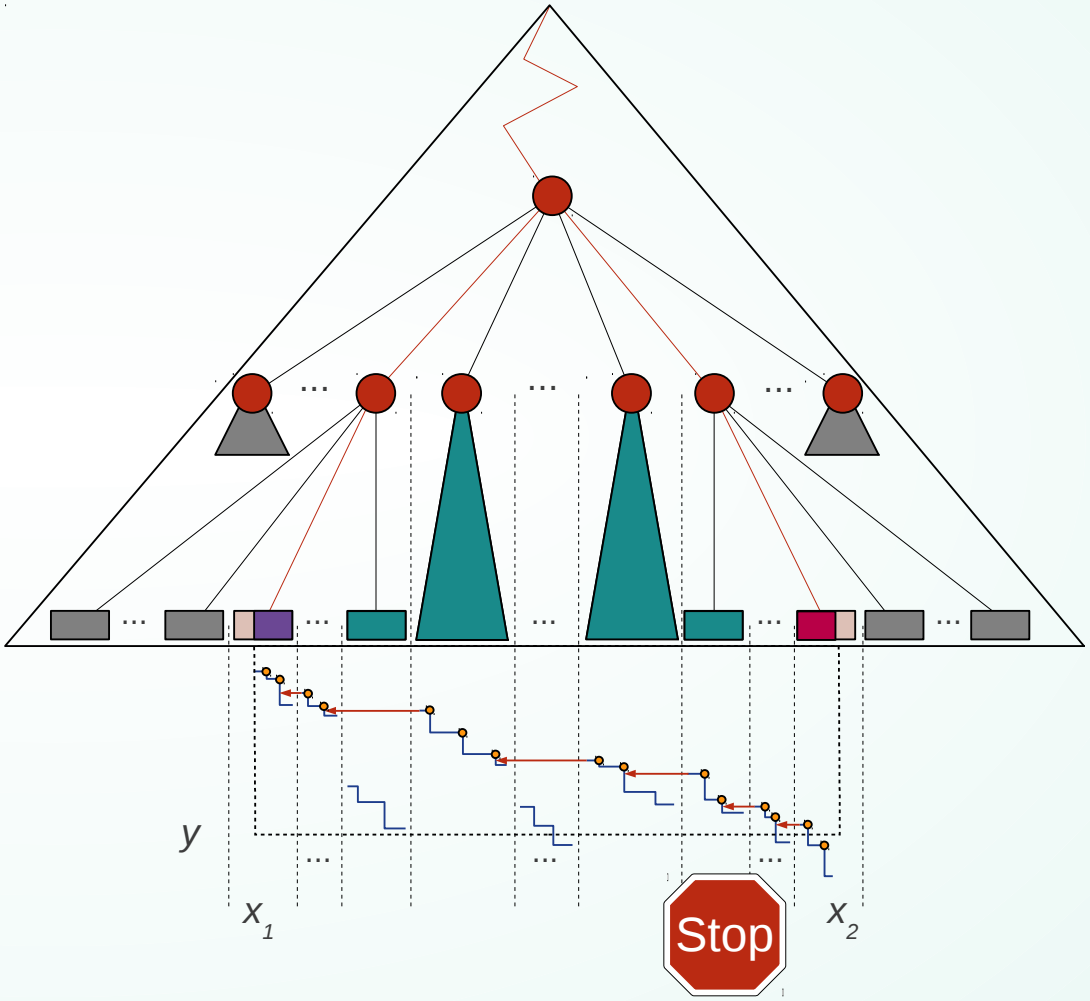

Outline

- Introduction to Algorithms and Data Structures
- Implicit Working-Set Dictionaries
- Skyline Queries
- Catenable Priority Queues with Attrition
- Conclusion

Previous and Our Results

Authors	Find-Min	Delete-Min	Insert-And-Attrite	Catenate-And-Attrite	Model
Sun89	O(*1*)	O(*1*)	O(*1*)		PM
Sun89	O(*1*)	O(*1*)	O(*1*)		EM
KTTTY13	O(*1*)	O(*1*)	O(*1*)	O(*1*)	PM
KTTTY13	O(*1/B*)	O(*1/B*)	O(*1/B*)	O(*1/B*)	EM

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- In External Memory our O(*1/B*) I/O bounds assume that for *k* PQAs we keep O(*1*) blocks in memory for each PQA
- Hence we require that $M = \Omega(kB)$ when maintaining *k* PQAs
- We can concatenate an arbitrary number of PQAs into one in O(*1*) I/Os if we maintain an extra invariant

Observations and Invariants

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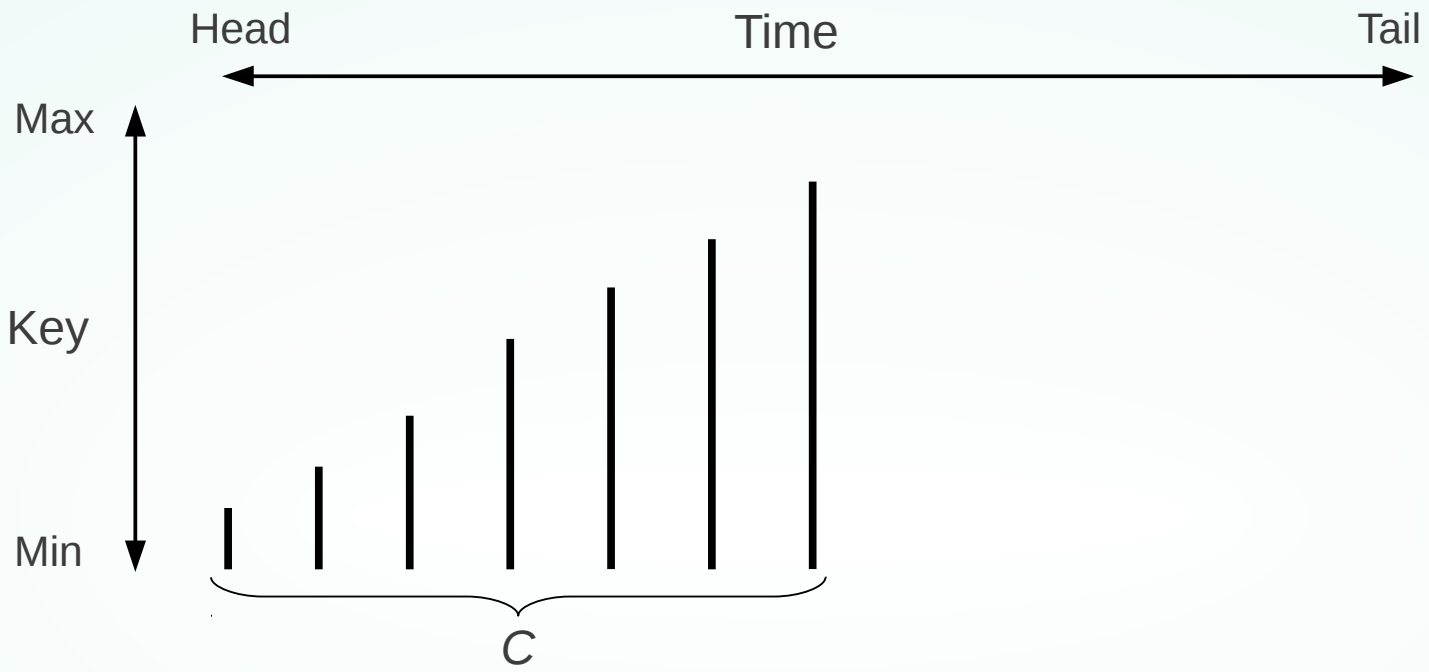

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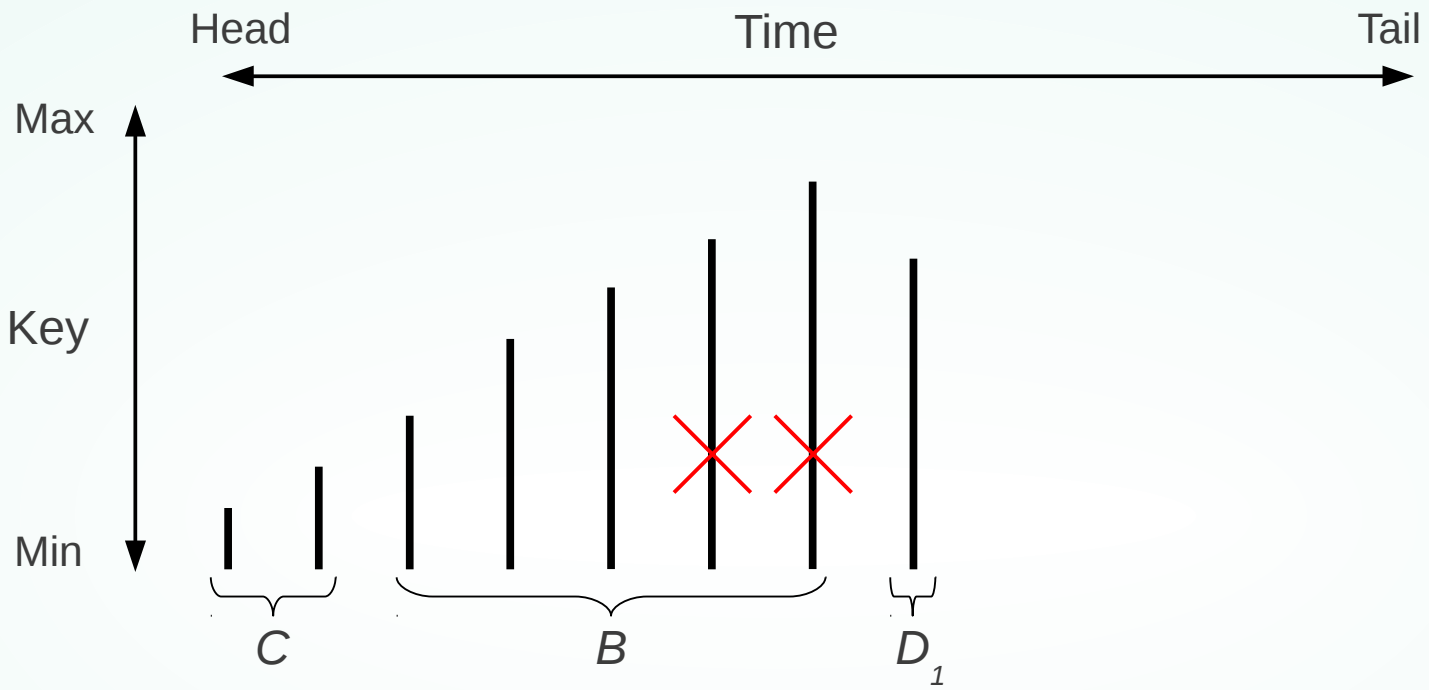

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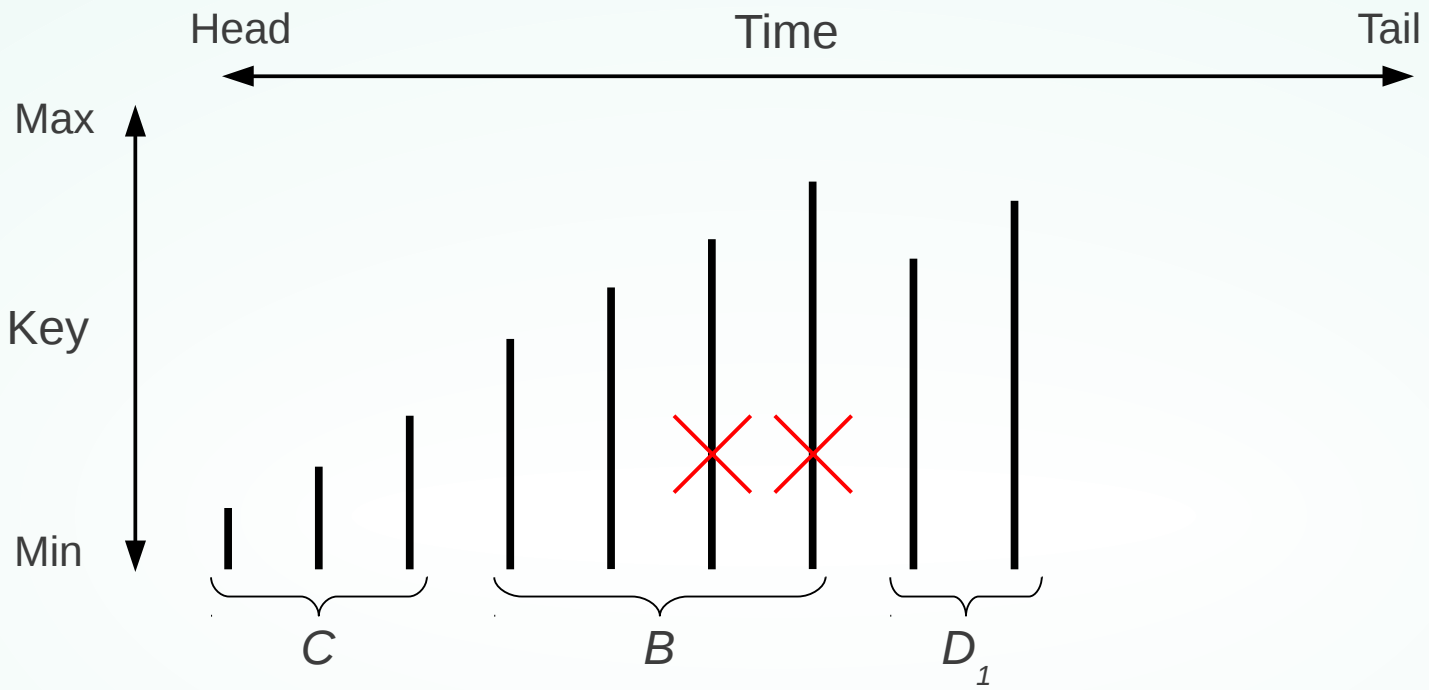

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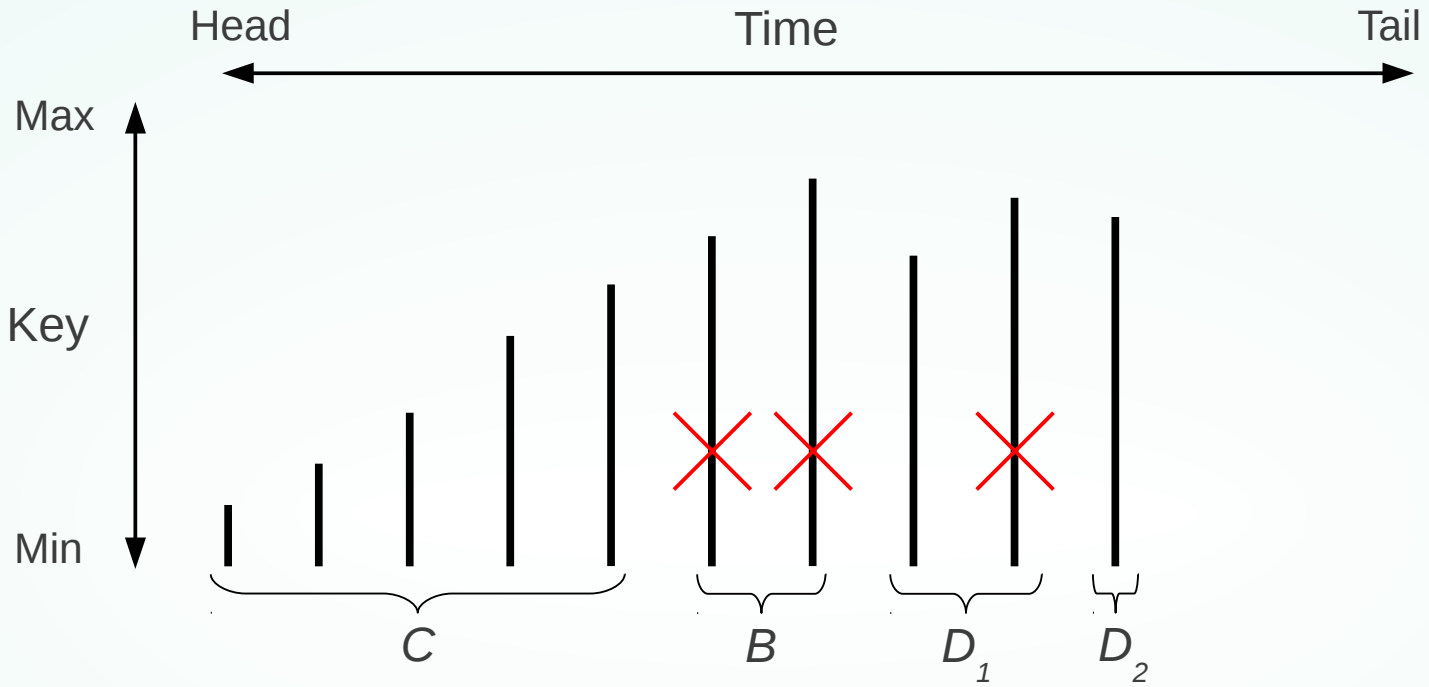

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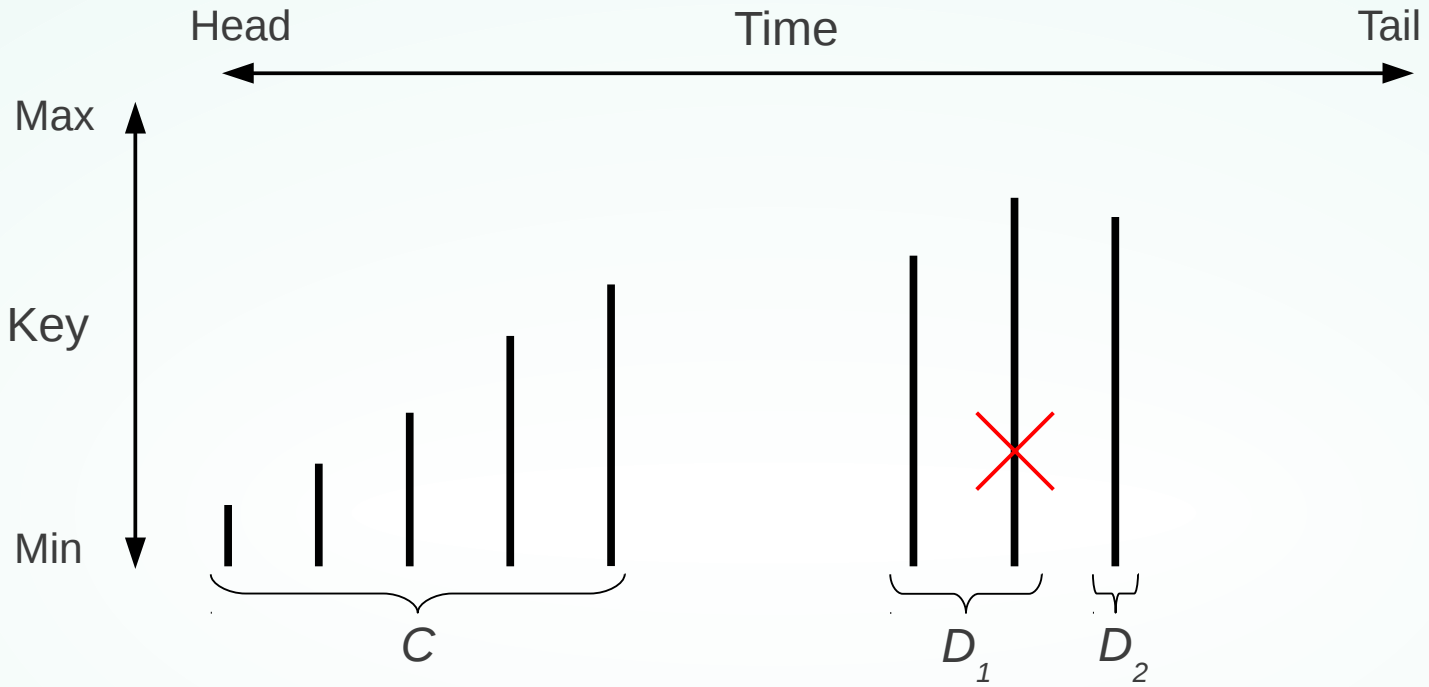

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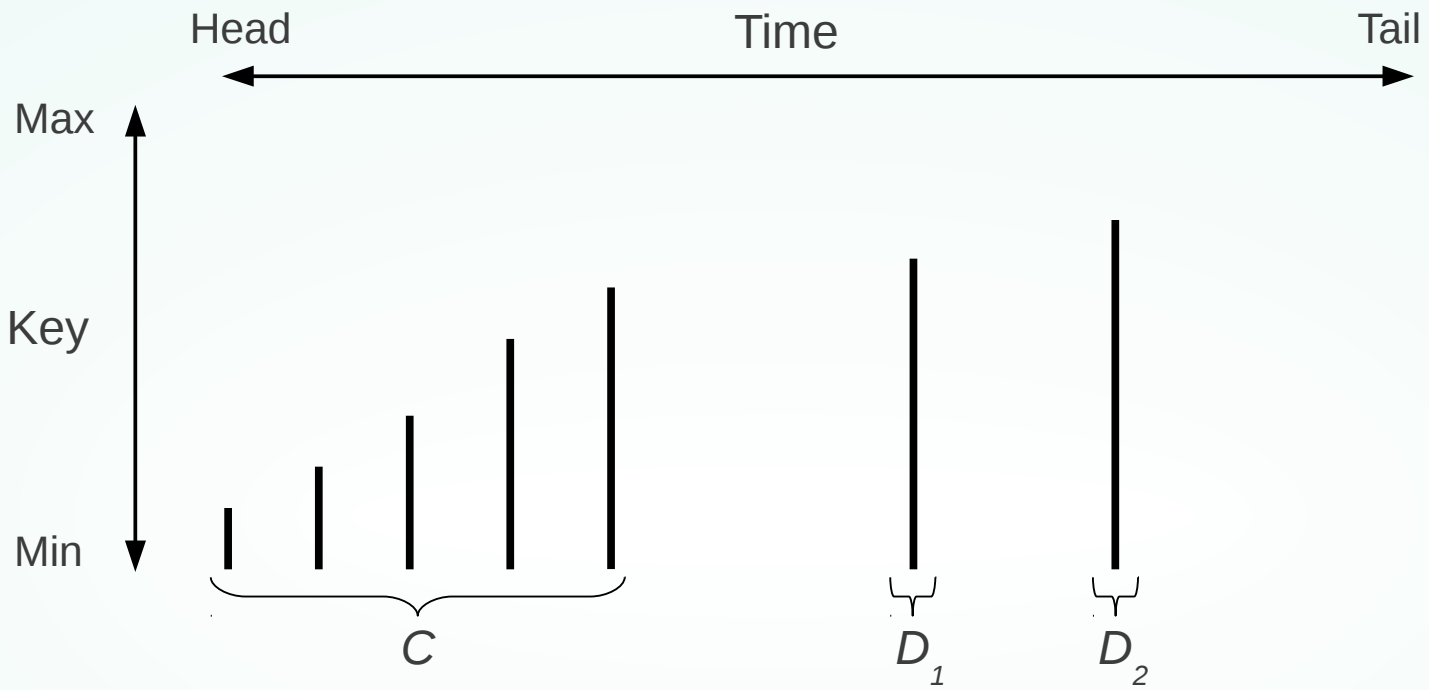

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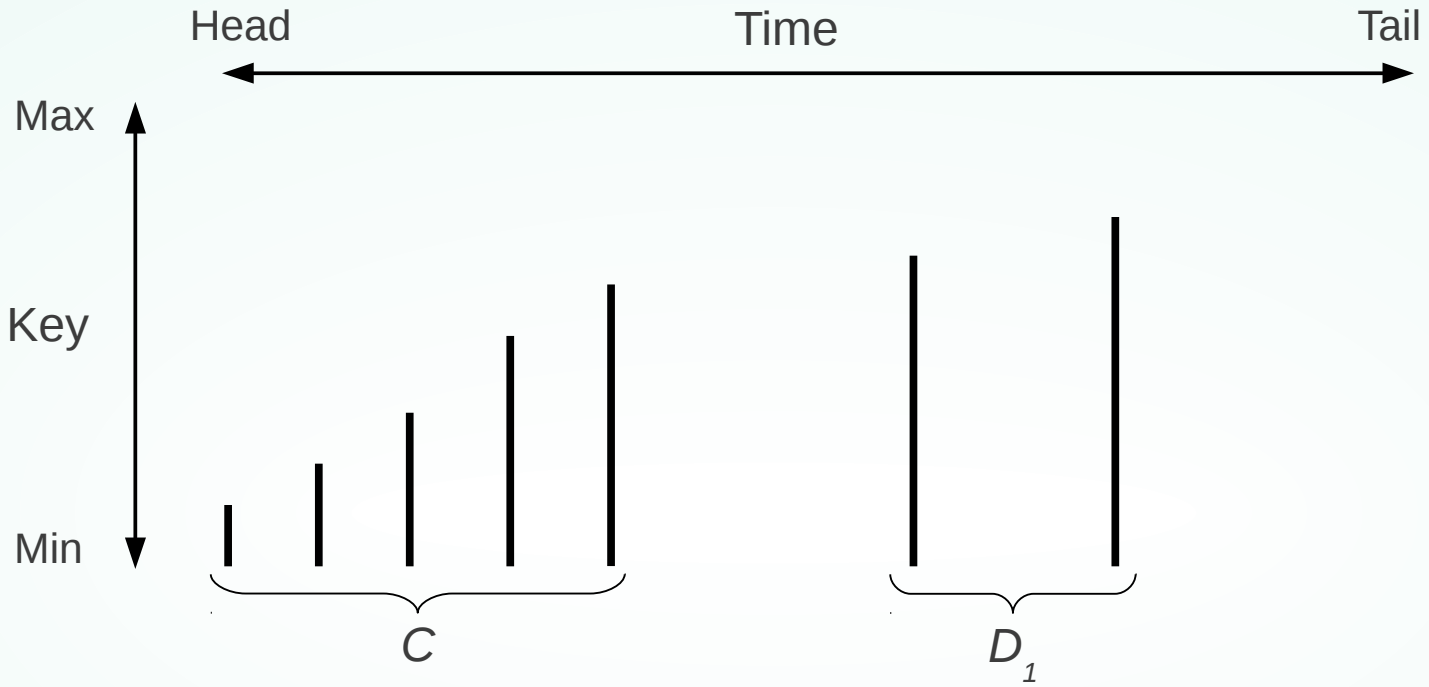

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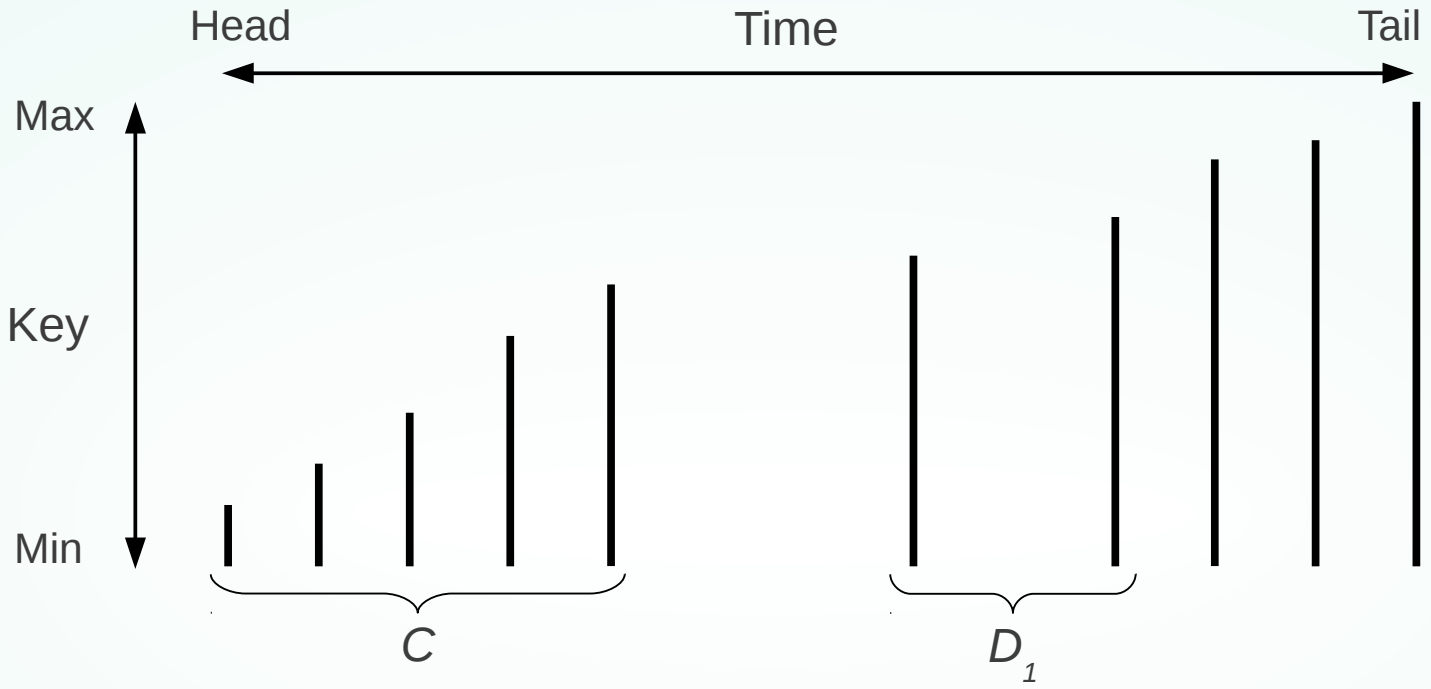

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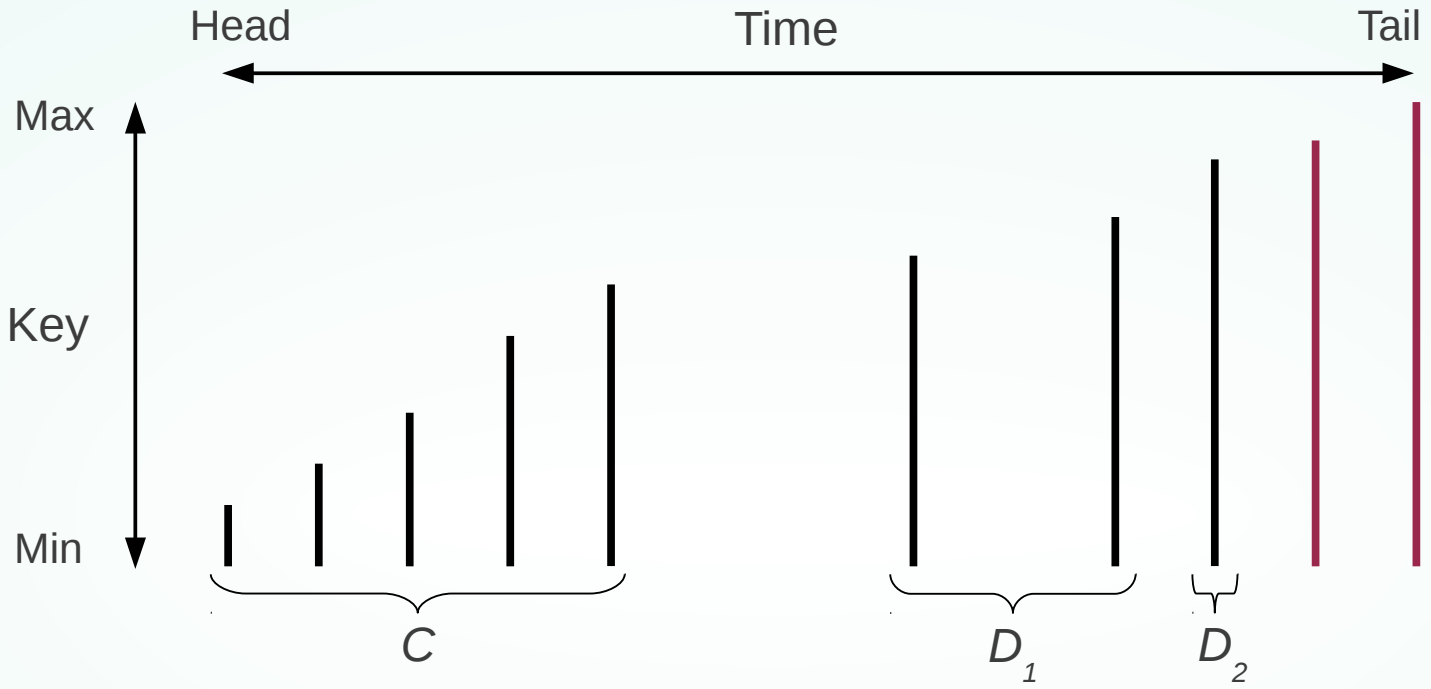

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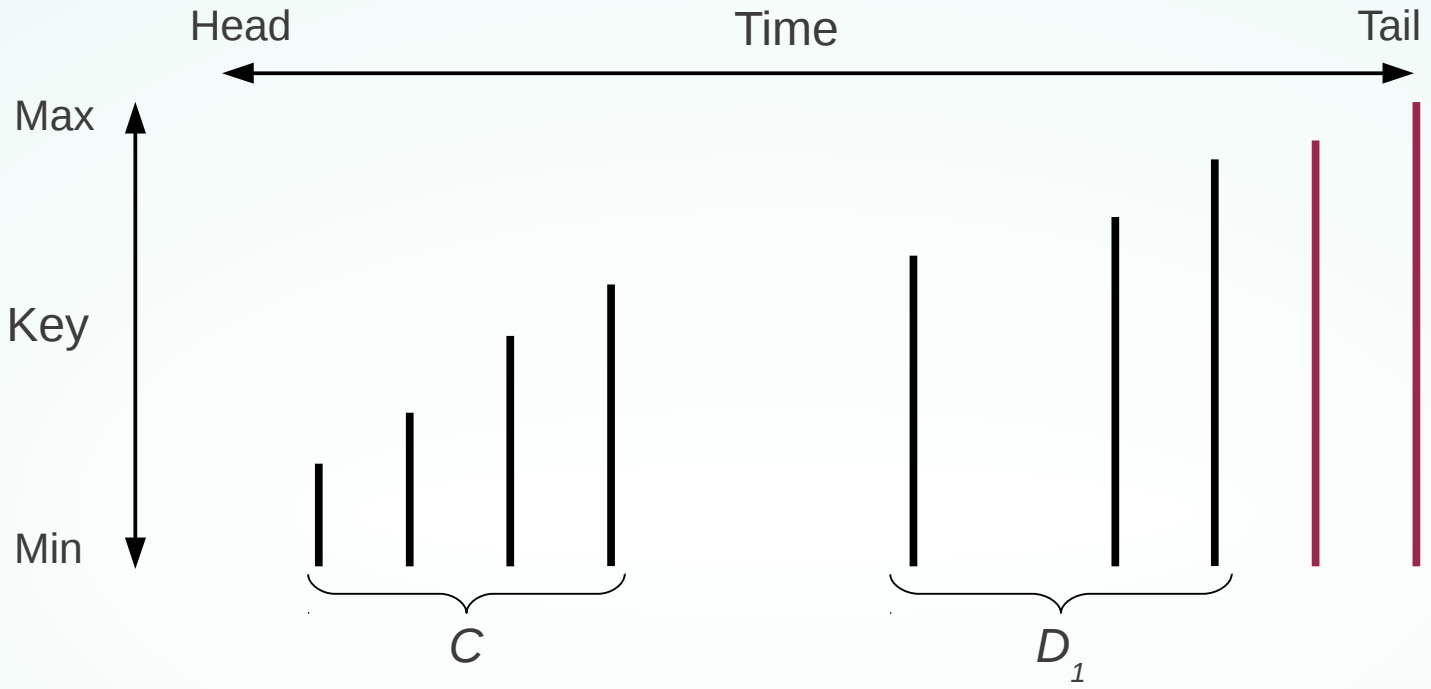

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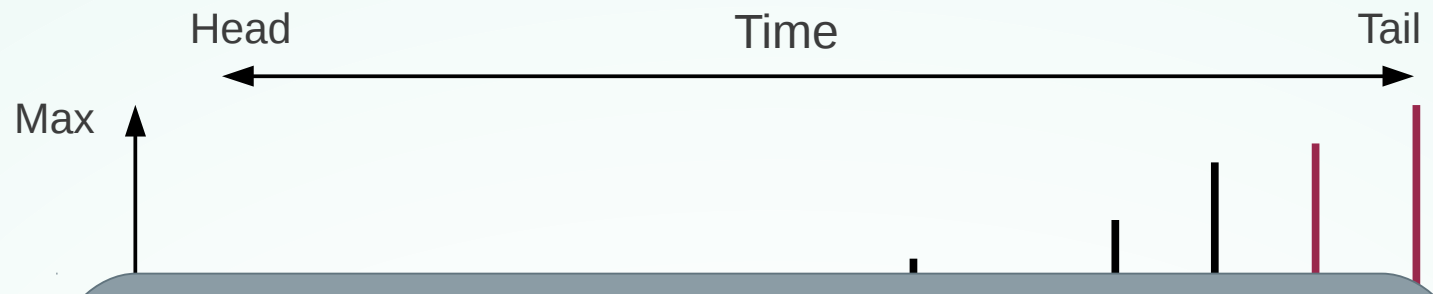

Summary of Invariants

- The clean queue holds non-attrited elements and is larger than the combined size of all dirty queues + #dirty queues
- The buffer queue holds both (non)-attrited elements
- The dirty queues might attrite into each other
- Each element in a dirty queue might contain another PQA

- Insert… before *e* with a key larger or equal to *e*
- ConcatenateAndAttrite(Q_1,Q_2): Deletes/attrites all elements in Q_1 with a key larger or equal to min(Q_2) and appends Q_2
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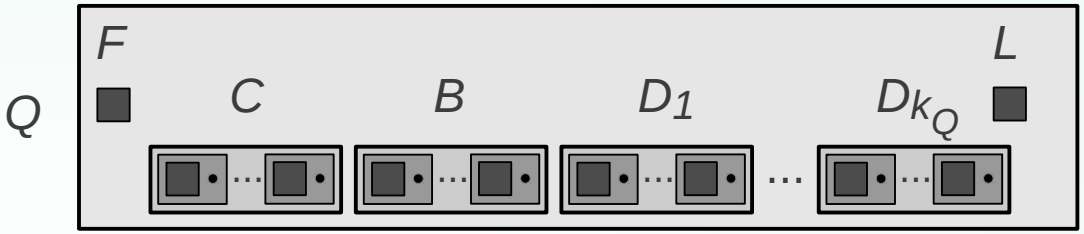
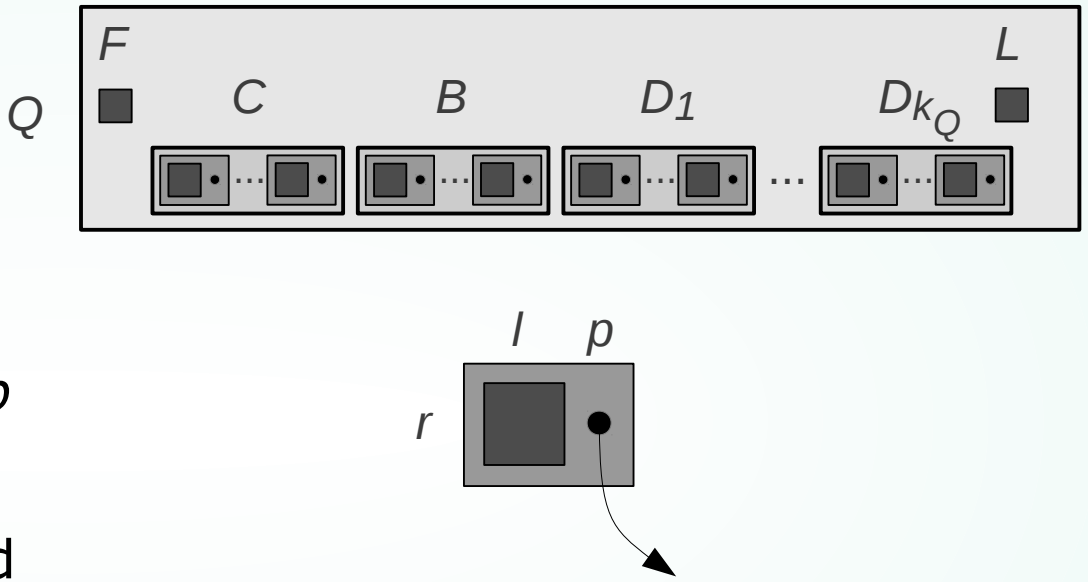
Concatenable PQA: Data Structure



Concatenable PQA: Data Structure

- A *PQA* consists of $2+k_Q$ deques C, B, D_1, …, D_{k_Q} of records and buffers F and L
- A record $r=(l,p)$ contains a buffer l of $[b,4b]$ elements and a pointer p to a PQA, if p is nil then r is simple
- A *PQA* is a tree of unbounded degree with *PQA*s as internal and leaf nodes



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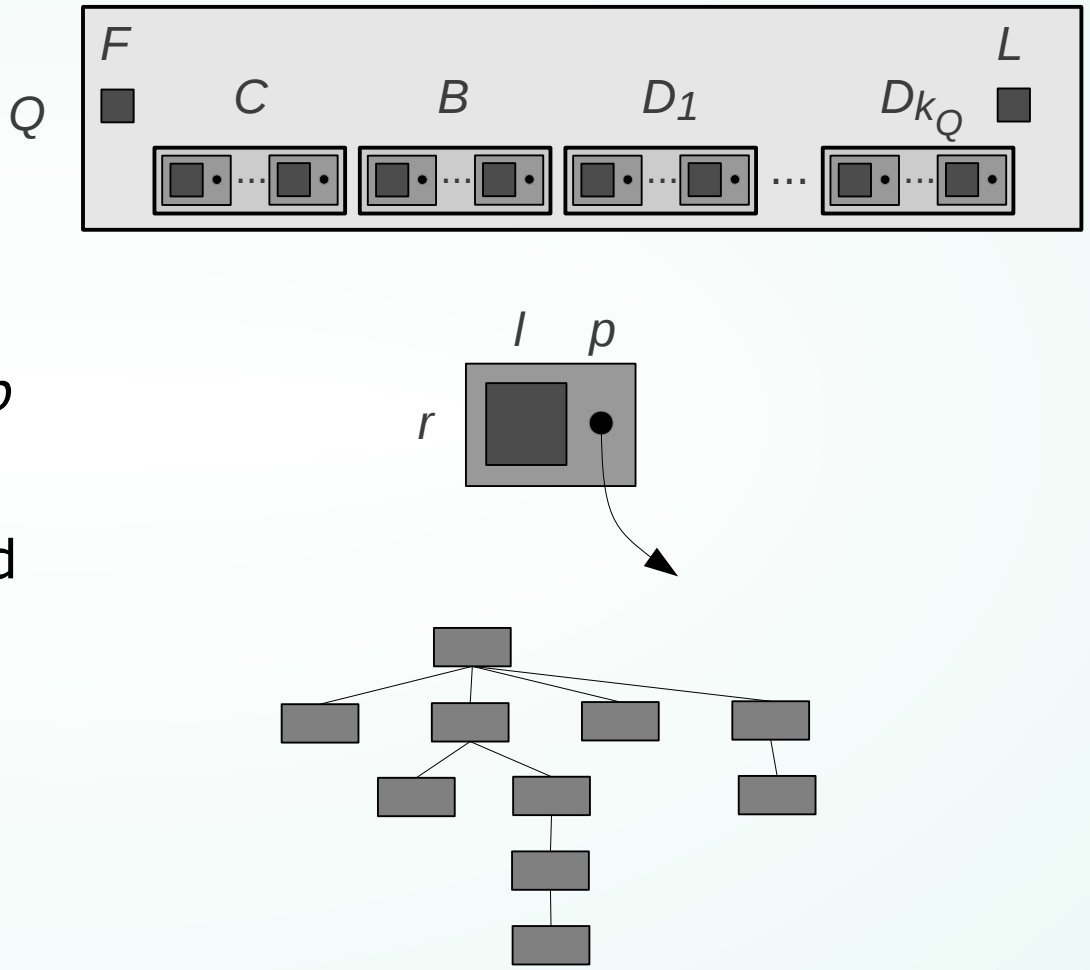


Invariants and Operations

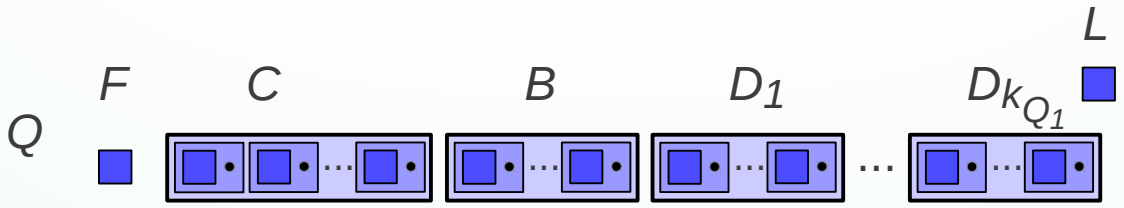
$$max(C(Q)) < min(B(Q)) < min(D_1(Q)) < min(D_i(Q)), \ for\ i > 1$$

$$C(Q) \text{ and } B(Q) \text{ are simple}$$

$$max(F(Q)) < min(C(Q)) \quad min(D_1(Q)) < min(L(Q))$$

$$|C(Q)| \geqslant \sum_{i=1}^{k_Q} |D_i(Q)| + k_Q$$

DeleteMin



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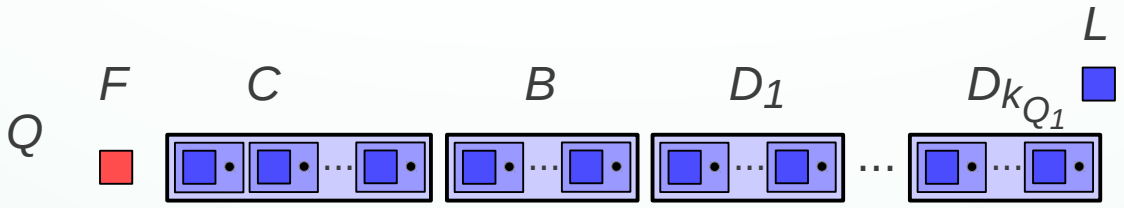

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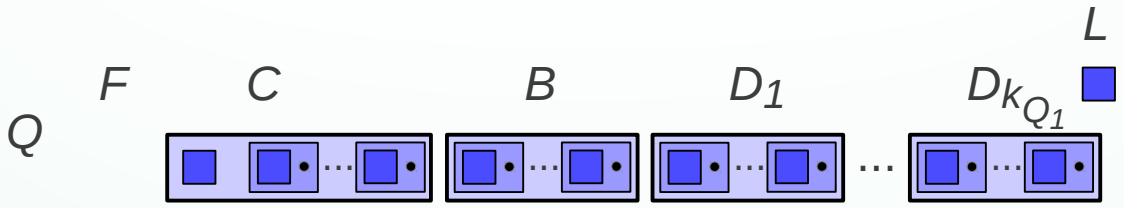

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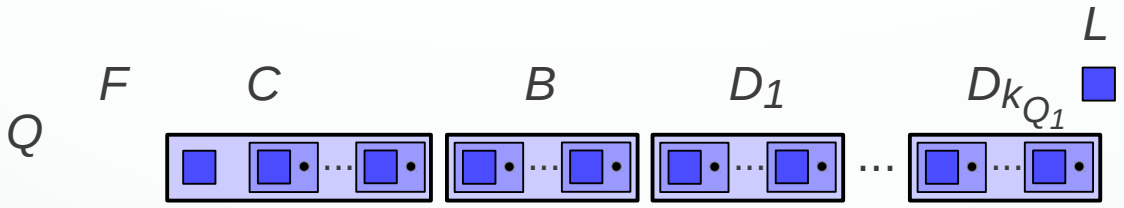

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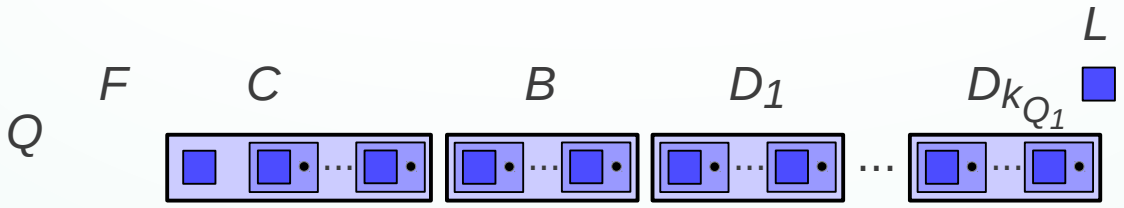
Bias to the rescue!



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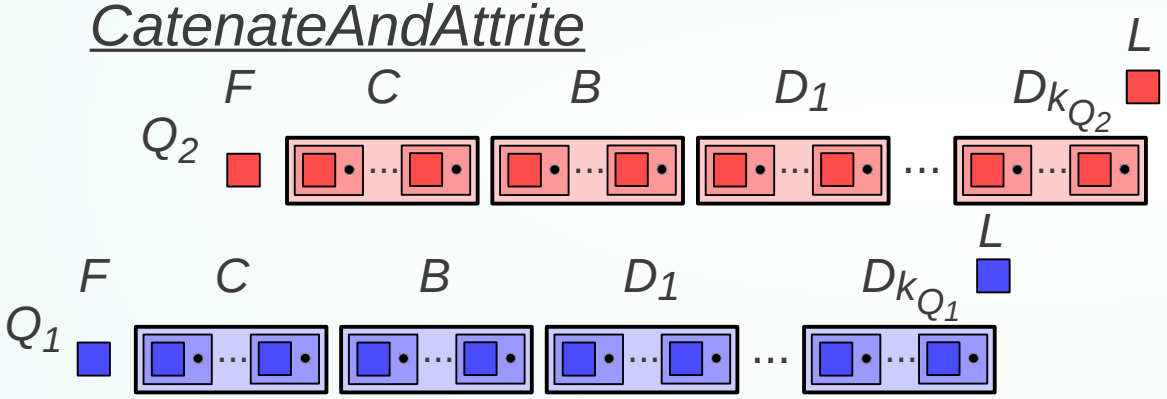
<u>CatenateAndAttrite</u>



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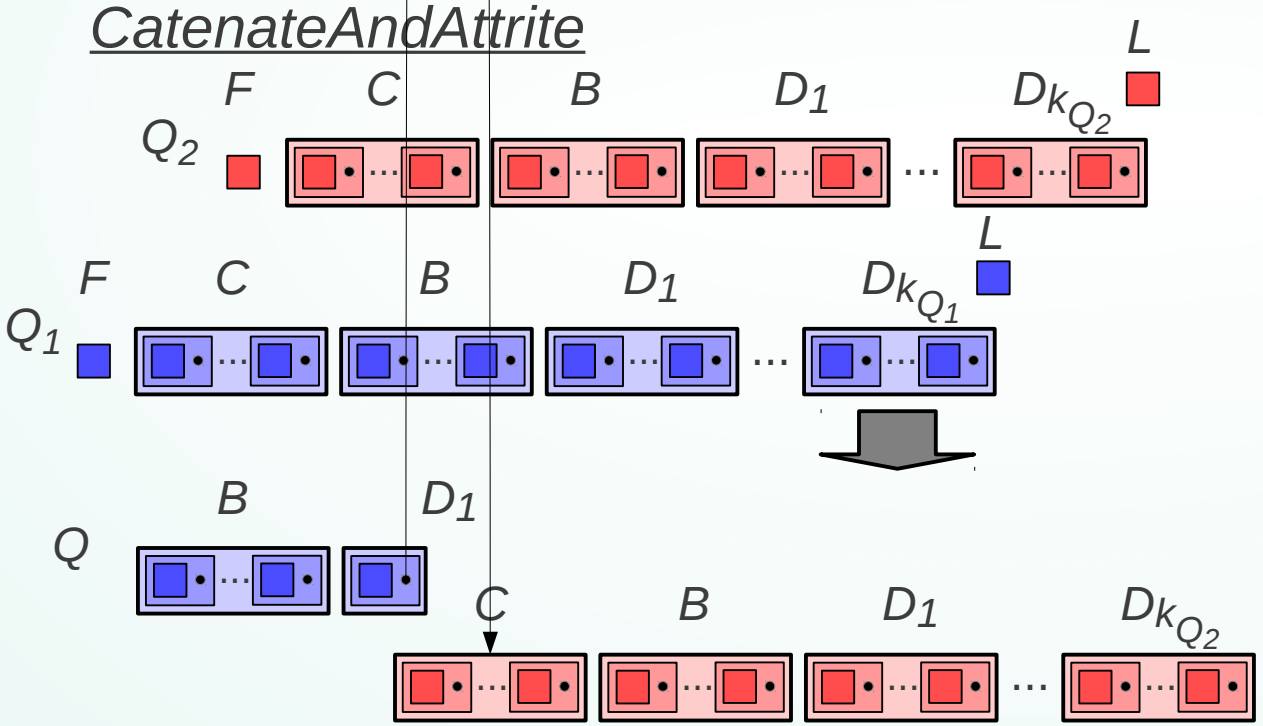

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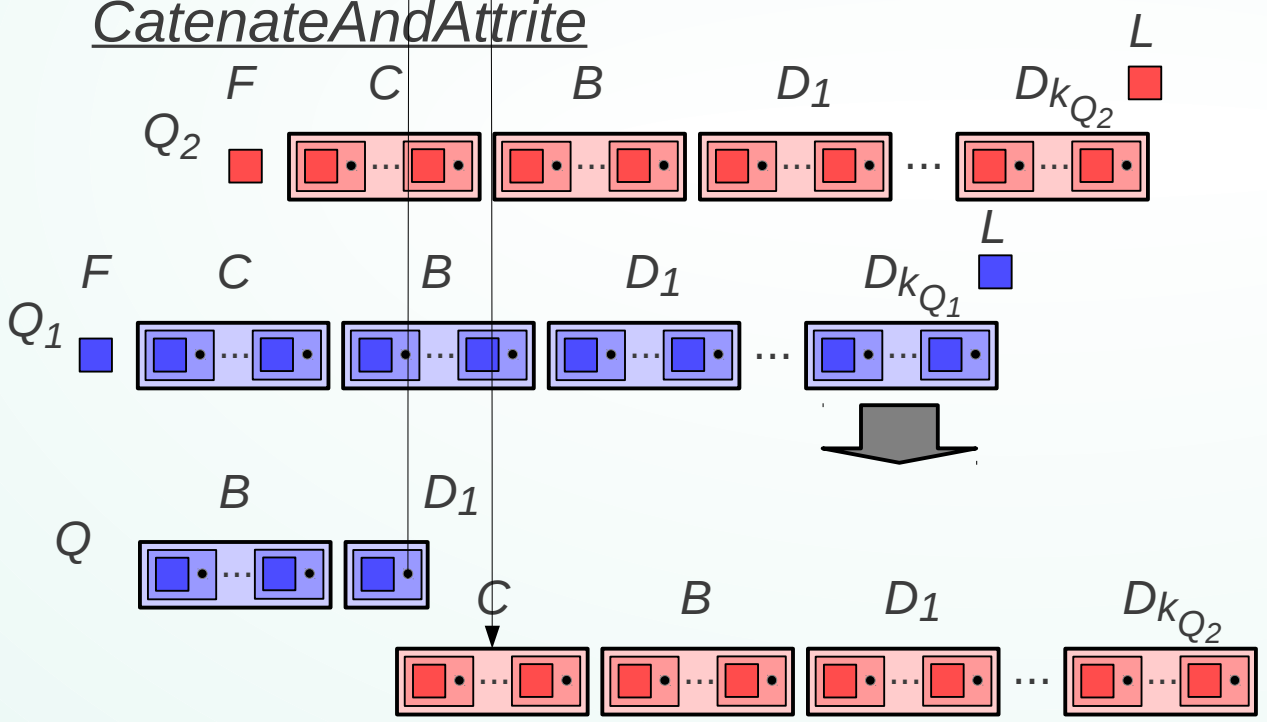

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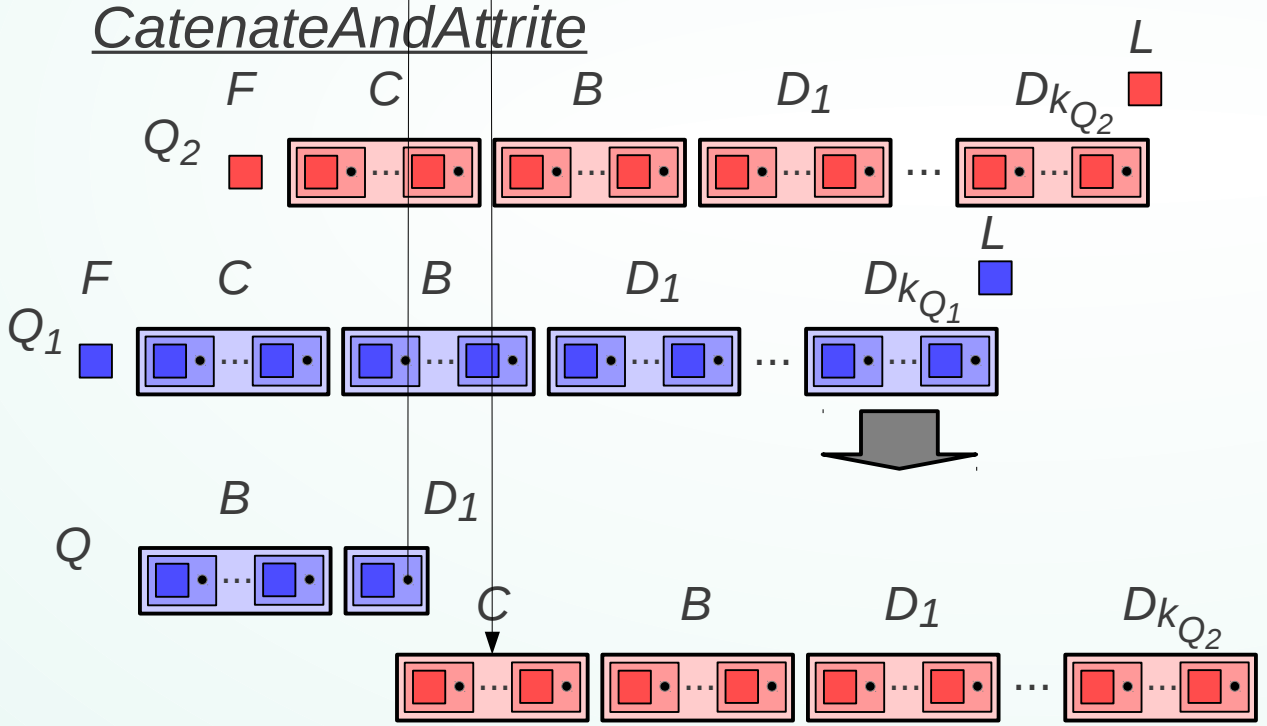

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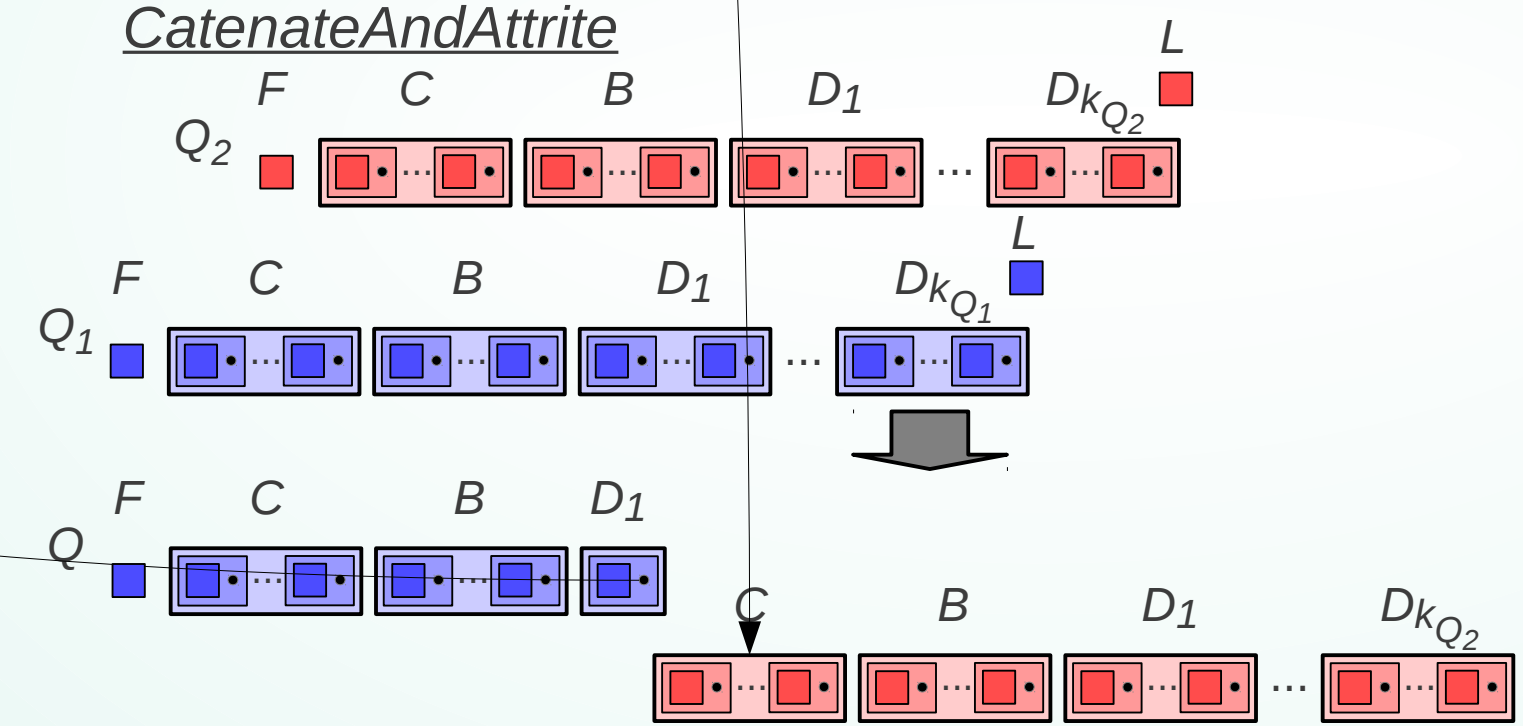

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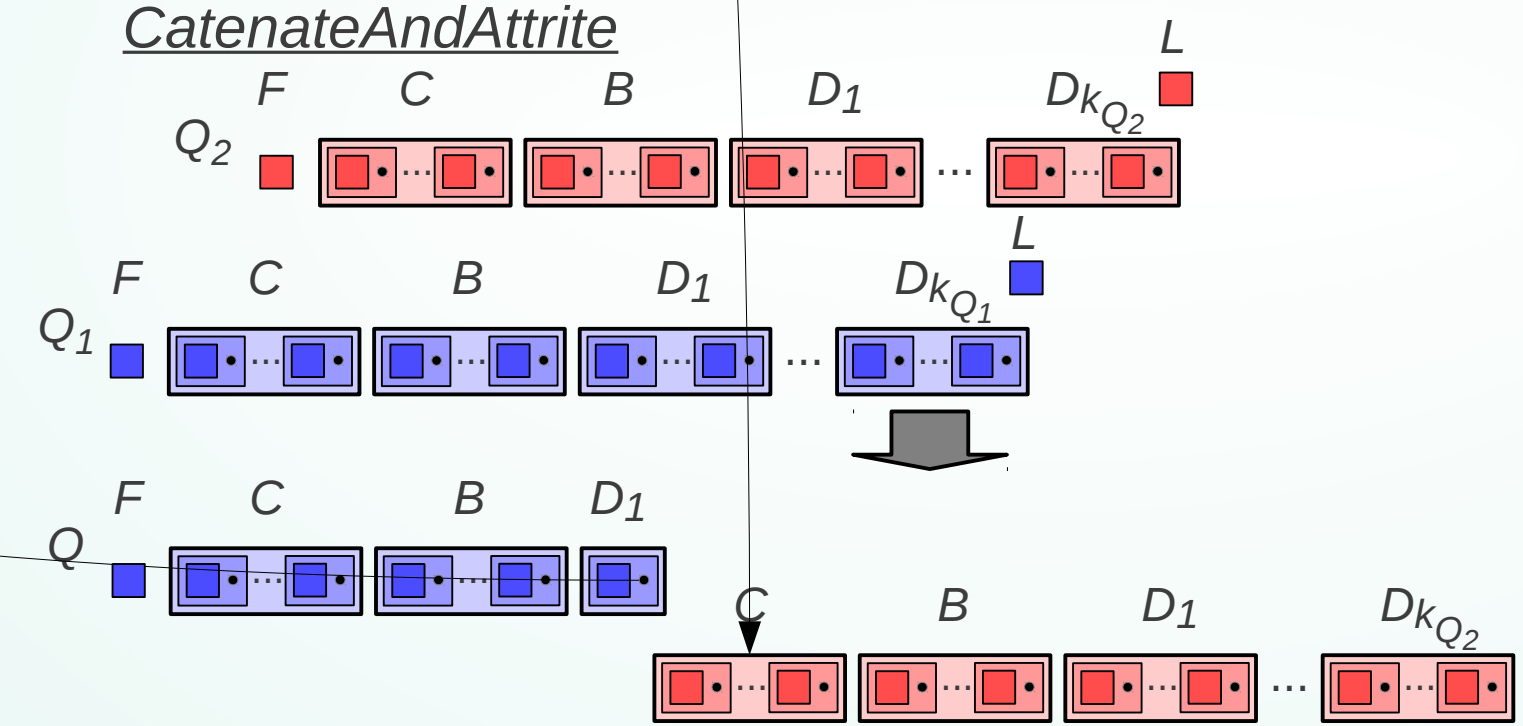

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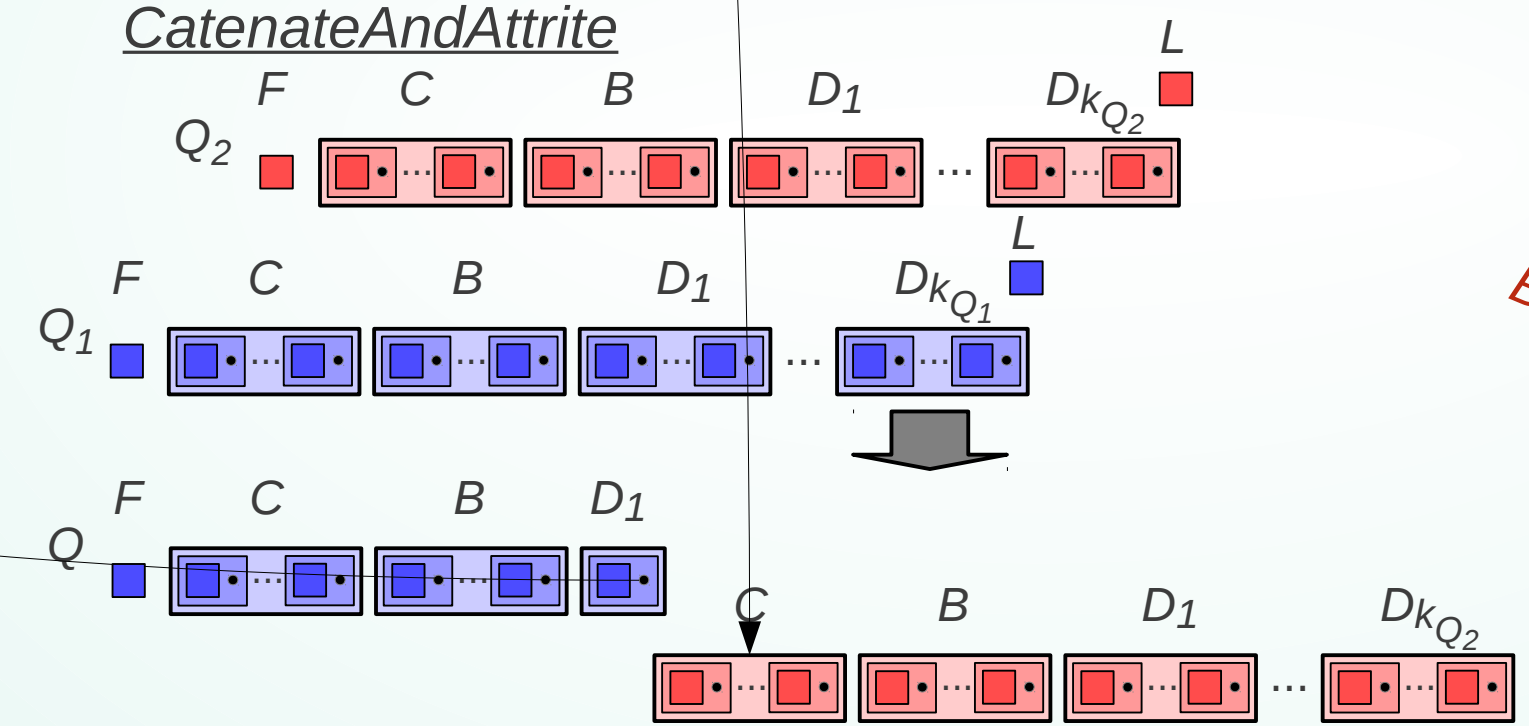

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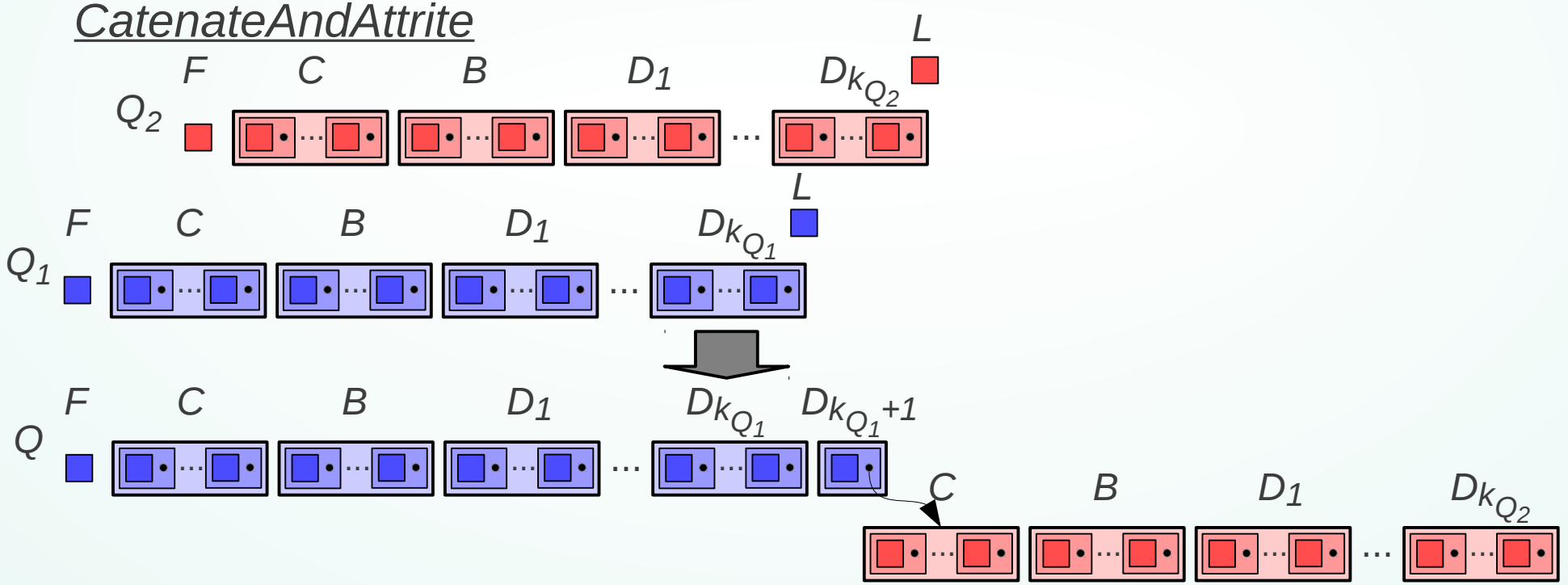

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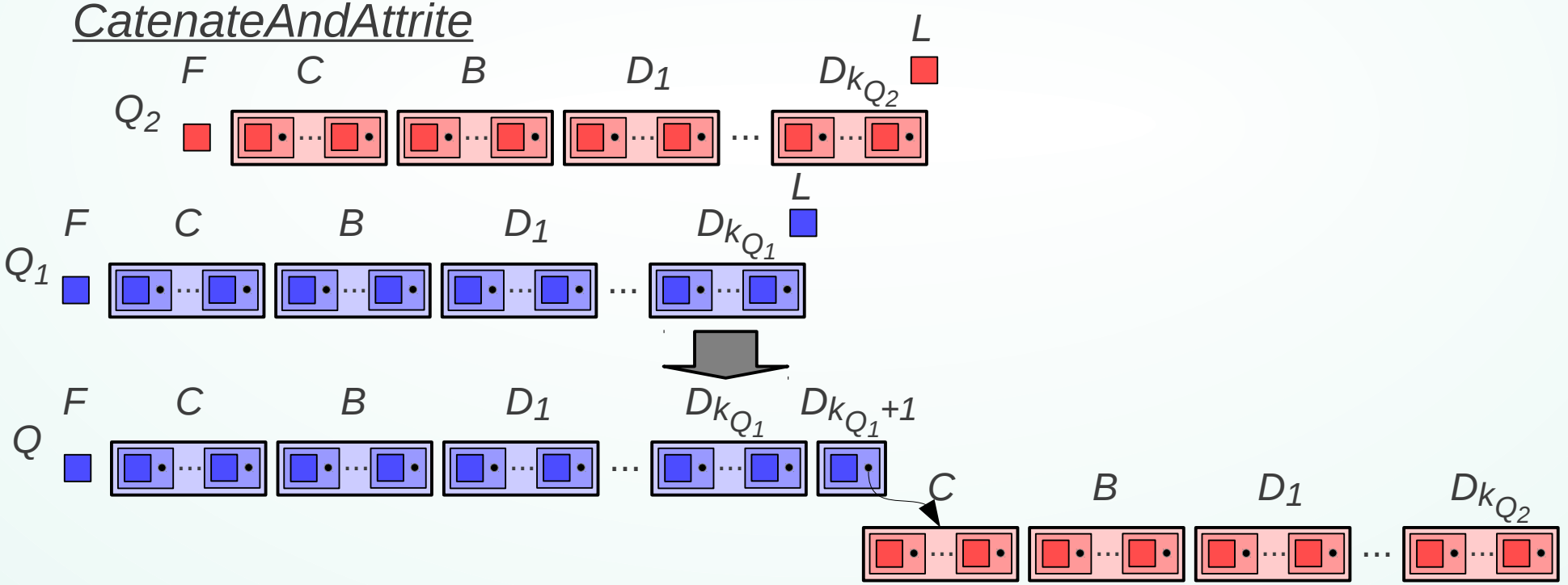

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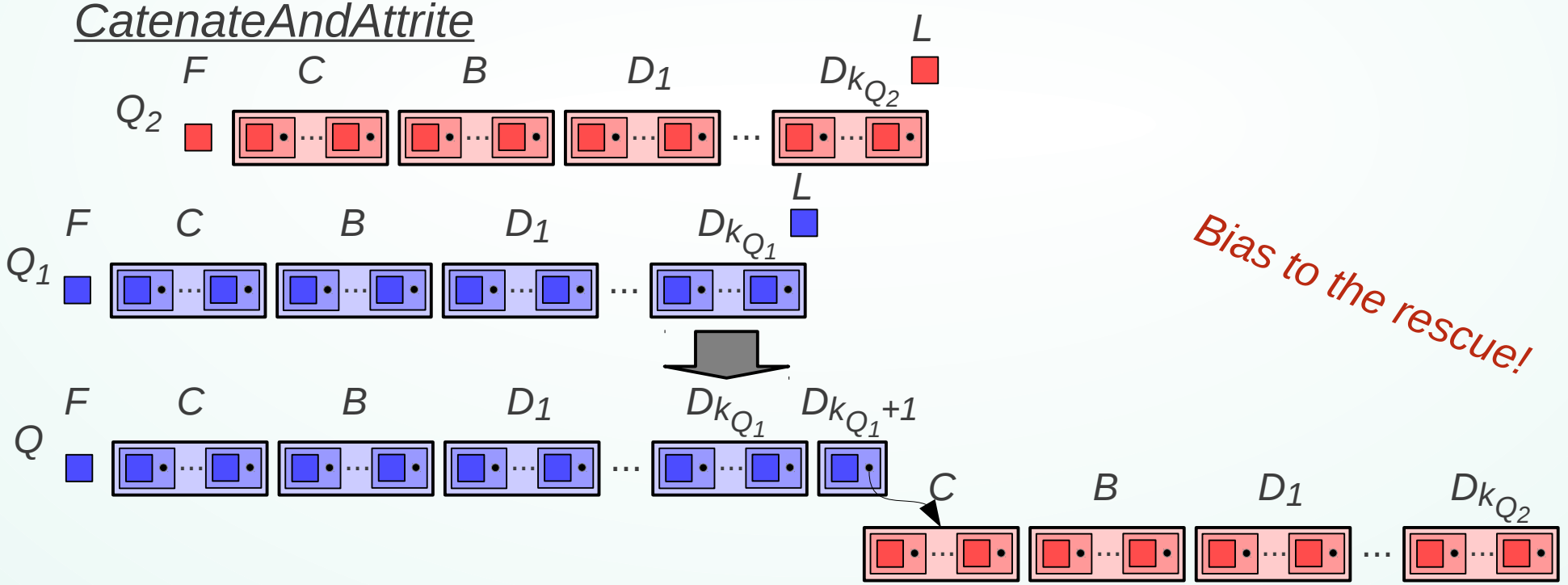

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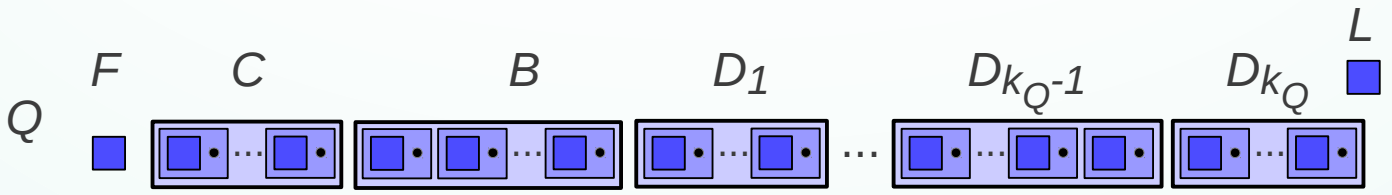

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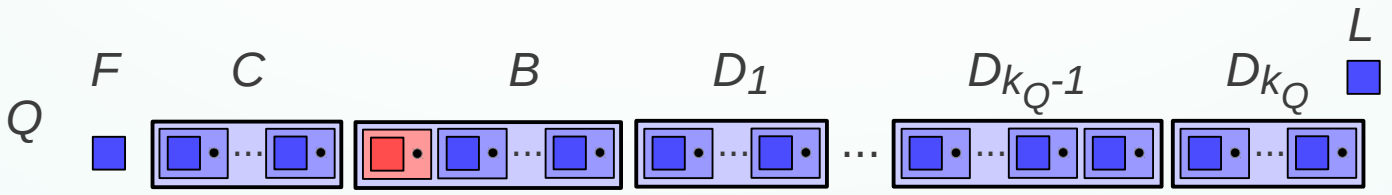

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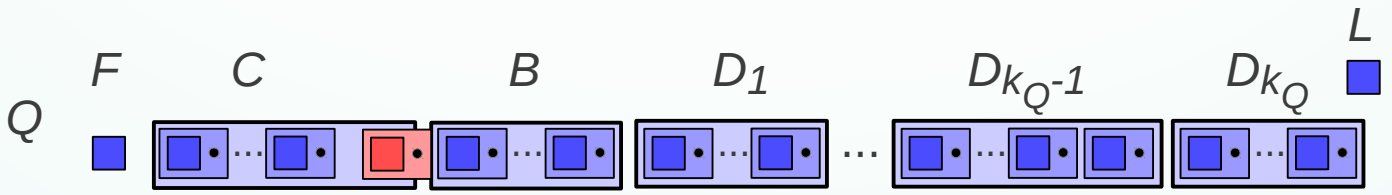

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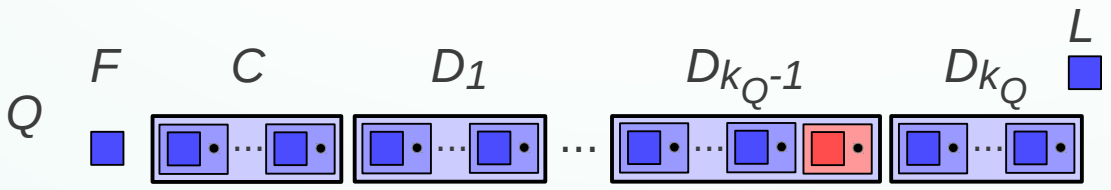

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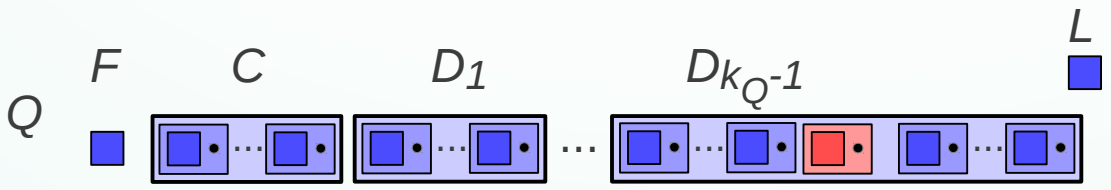

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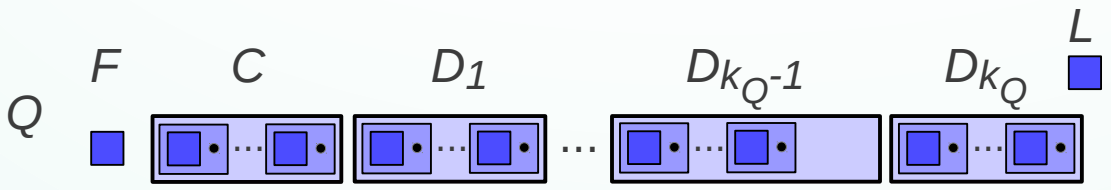

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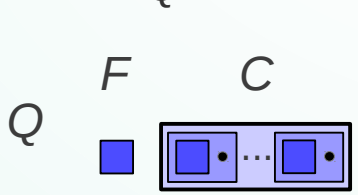

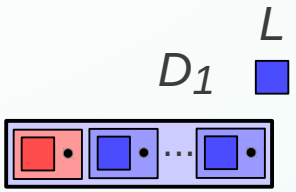

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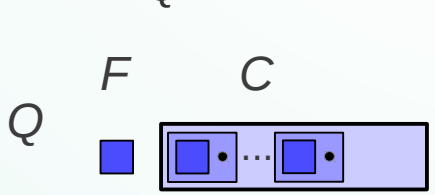

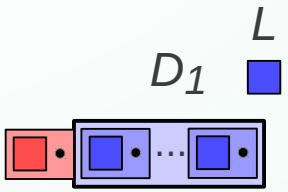

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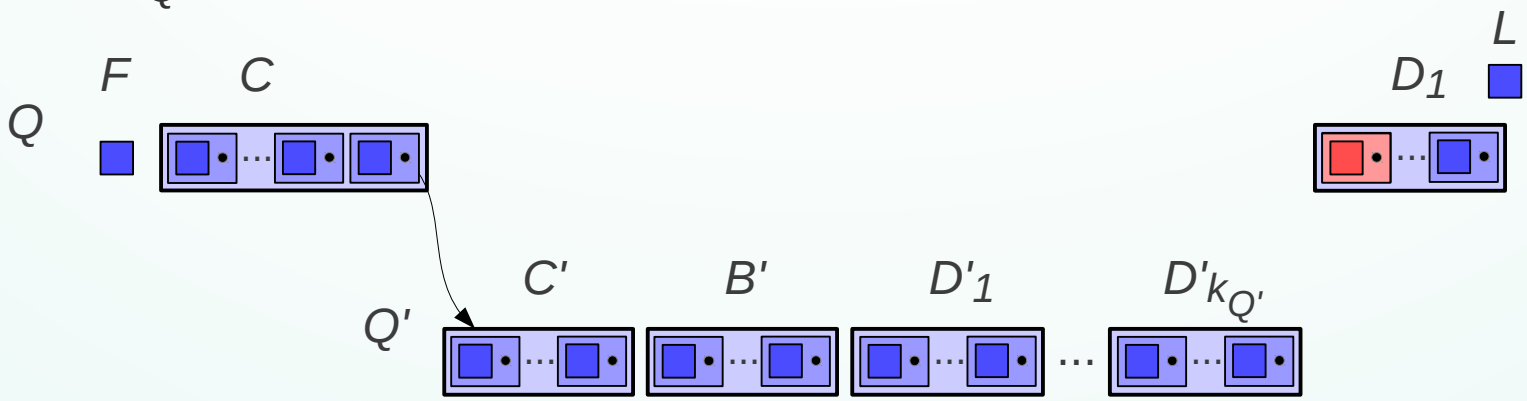


Outline

- Introduction to Algorithms and Data Structures
- Implicit Working-Set Dictionaries
- Skyline Queries
- Catenable Priority Queues with Attrition
- Conclusion

Conclusion

Conclusion

- We have seen how invariants are formed and used in dynamic data structures to give the three data structures:
  - Implicit Cache-Oblivious Working-Set Dictionaries
  - 2D Skyline Data Structures in External Memory
  - Catenable Priority Queues with Attrition in External Memory
- Open problems:
  - Can we change the insert(*e*) operation of the working set dictionary so that *e* gets a working set value of *n* instead of *0*?
  - In what other problems does *attrition* occur as a subproblem?
  - Can the PQA be modified to solve other skyline related problems like Top-k Domination and variants?
  - Using PQAs for High-dimensional Skyline Structures?

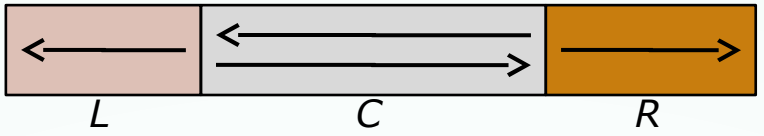
Thank you :)

References

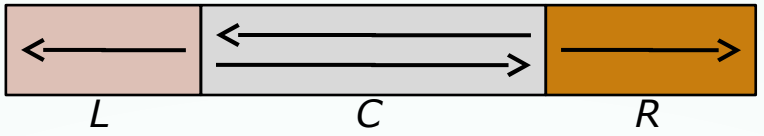
1) **A Cache-Oblivious Implicit Dictionary with the Working Set Property**
   - Gerth Stlting Brodal, Casper Kejlberg-Rasmussen, Jakob Truelsen
   - ISAAC 2010
   - *Available at http://dx.doi.org/10.1007/978-3-642-17514-5_4*
2) **Cache-Oblivious Implicit Predecessor Dictionaries with the Working-Set Property**
   - Gerth Stølting Brodal, Casper Kejlberg-Rasmussen
   - STACS 2012
   - *Available at http://dx.doi.org/10.4230/LIPIcs.STACS.2012.112*
3) **I/O-Efficient Planar Range Skyline and Attrition Priority Queues**
   - Casper Kejlberg-Rasmussen, Yufei Tao, Konstantinos Tsakalidis, Kostas Tsichlas, Jeonghun Yoon
   - PODS 2013
   - *Available at http://doi.acm.org/10.1145/2463664.2465225*

Extra Slides

Implicit Moveable Dictionaries

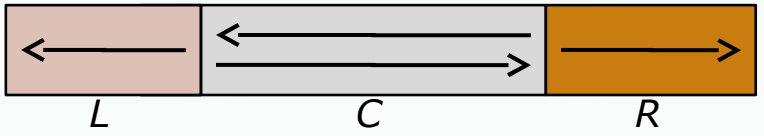


Implicit Moveable Dictionaries



- Uses O(*1*) FG dictionaries as black boxes
- *Recall the FG interface:*
  - Insert-right(*e*): insert element *e* into the dictionary which grows to the right
  - Delete-right(*e*): delete element *e* from the dictionary which shrinks from the right
  - Search(*e*): finds the address of *e* if *e* is in the dictionary
  - Predecessor(*e*): finds the address of the predecessor of *e*
  - Successor(*e*): finds the address of the successor of *e*

Implicit Moveable Dictionaries



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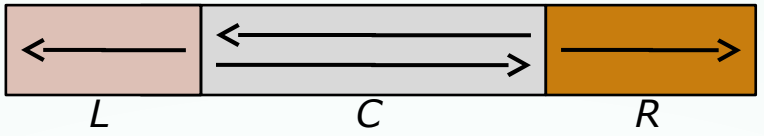

- *L* and *R* will shrink and grow over time
  - *L*/*R* might get too small or
  - *L*/*R* might get too large compared to *C*
- We introduce the notion of *jobs*
  - Grow-left/right – Counters when *L*/*R* gets too small
  - Shrink-left/right – Counters when *L*/*R* gets too large
  - Jobs run O(*1*) steps every operation: searches, updates

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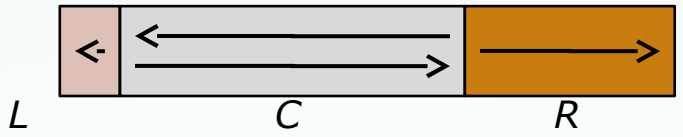

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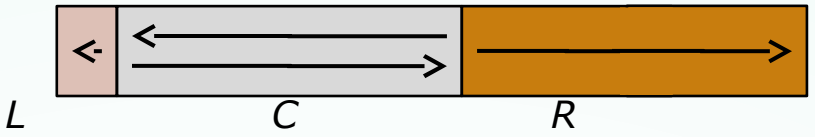

- L and R will shrink and grow over time
  - L/R might get too small or
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- We introduce the notion of *jobs*
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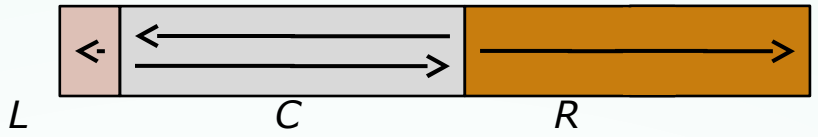

- *L* and *R*
  - *L*/*R*
  - *L*/*R*
- We intro
  - Grow-left/right – Counters when *L*/*R* gets too small
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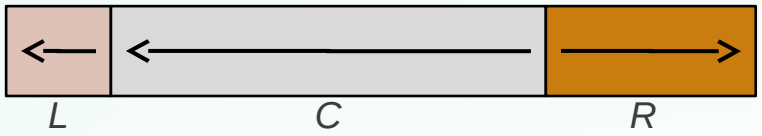
  - Jobs run O(*1*) steps every operation: searches, updates

Summary of Invariants

- We ensure that both L and R are not too small/large
- We have queued at most 2 jobs
- All jobs finish before their deadline

Implicit Moveable Dictionaries



Grow-left

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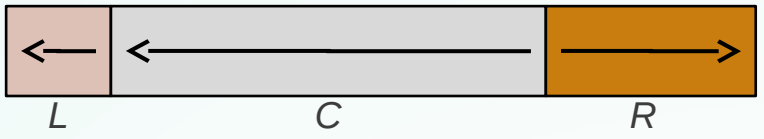

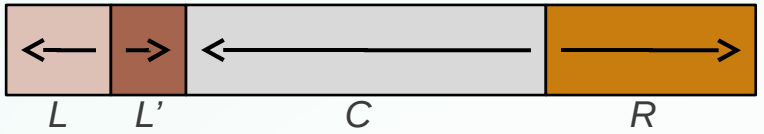

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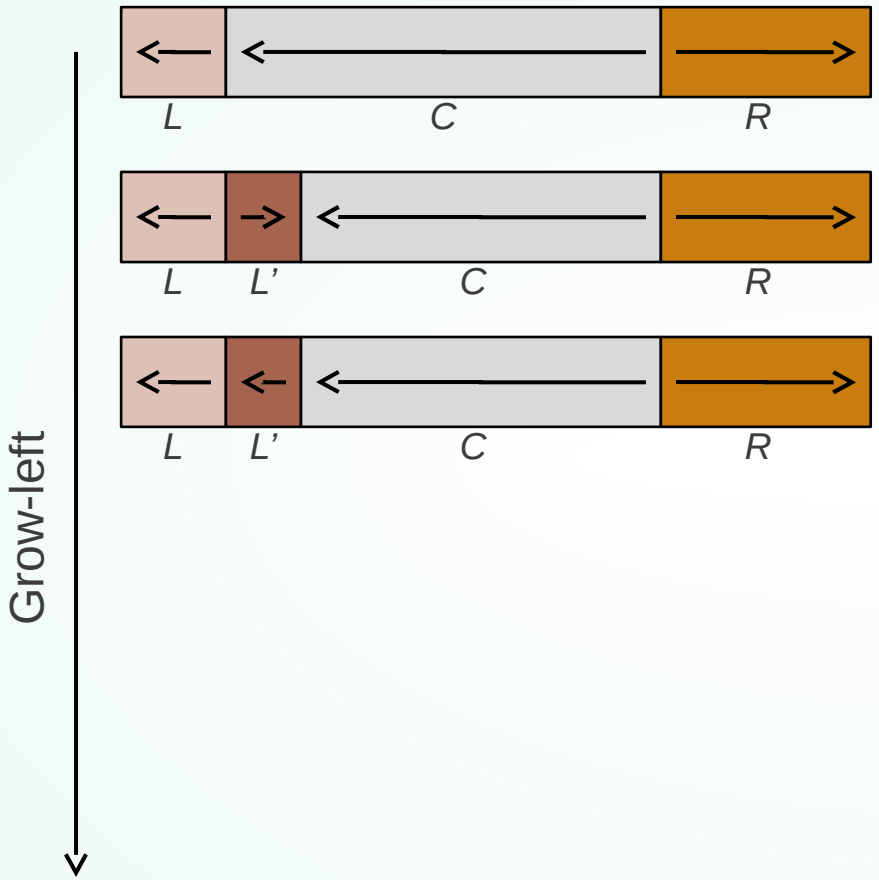

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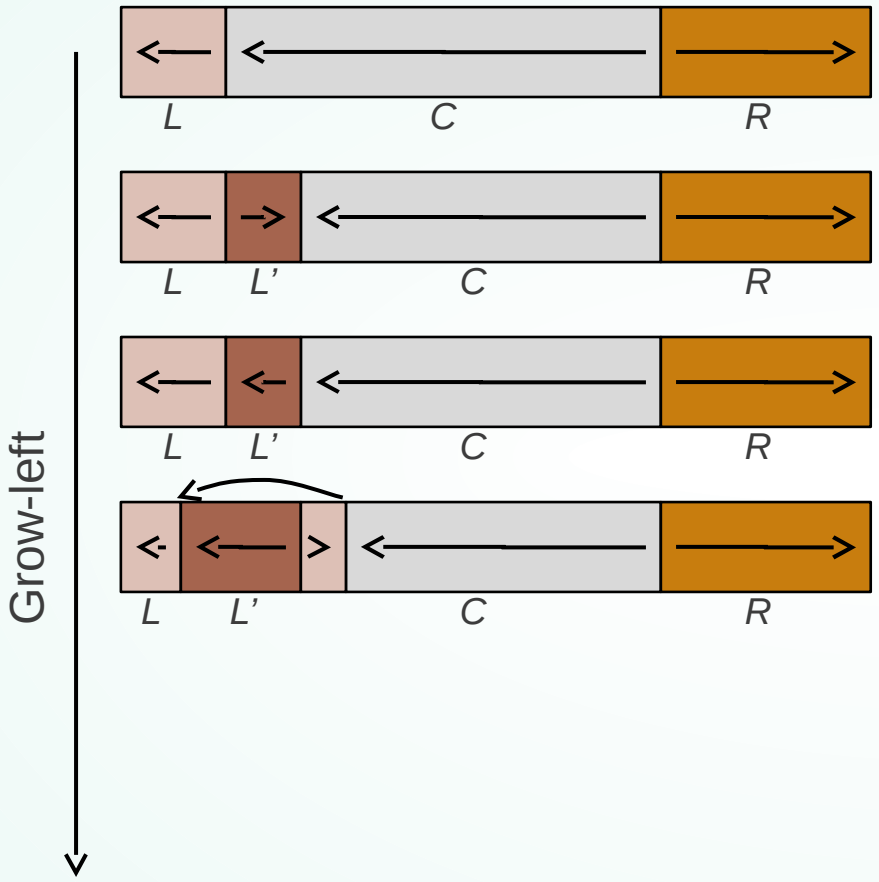

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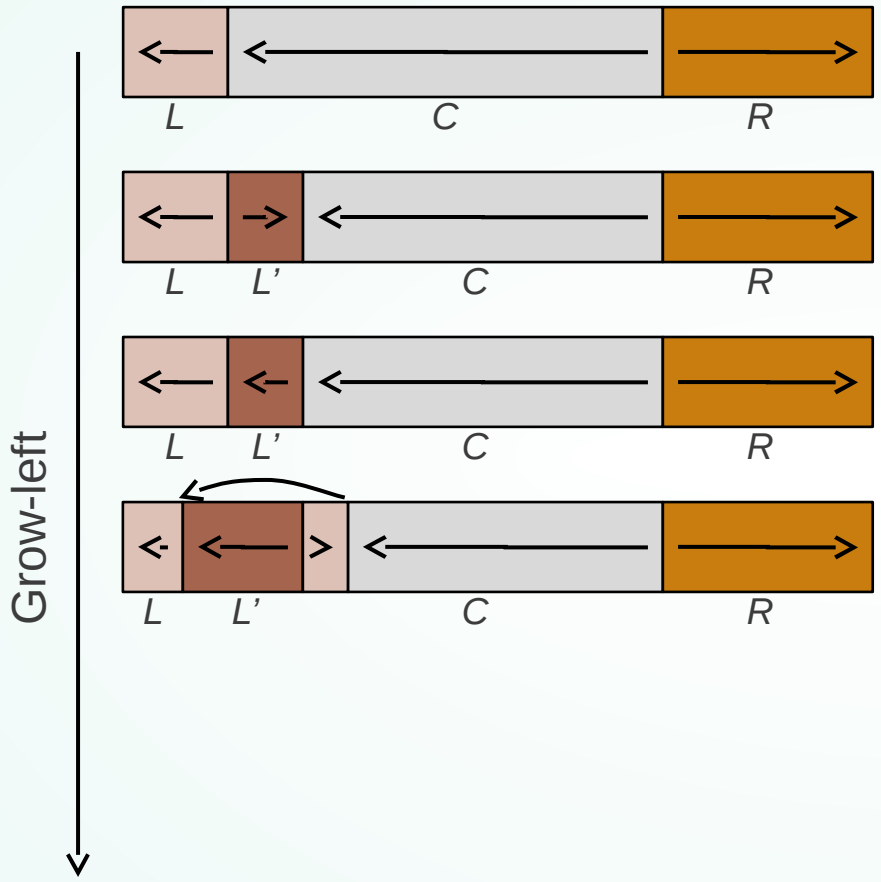

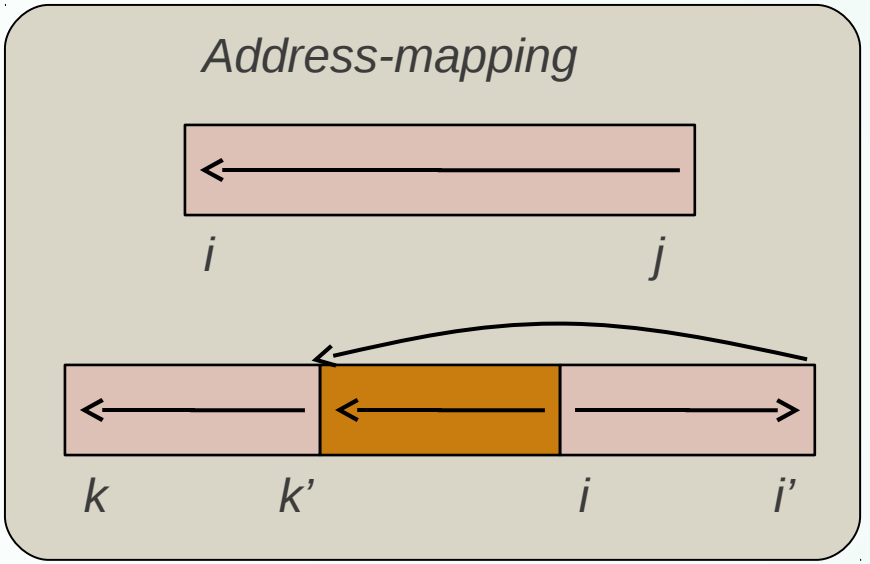

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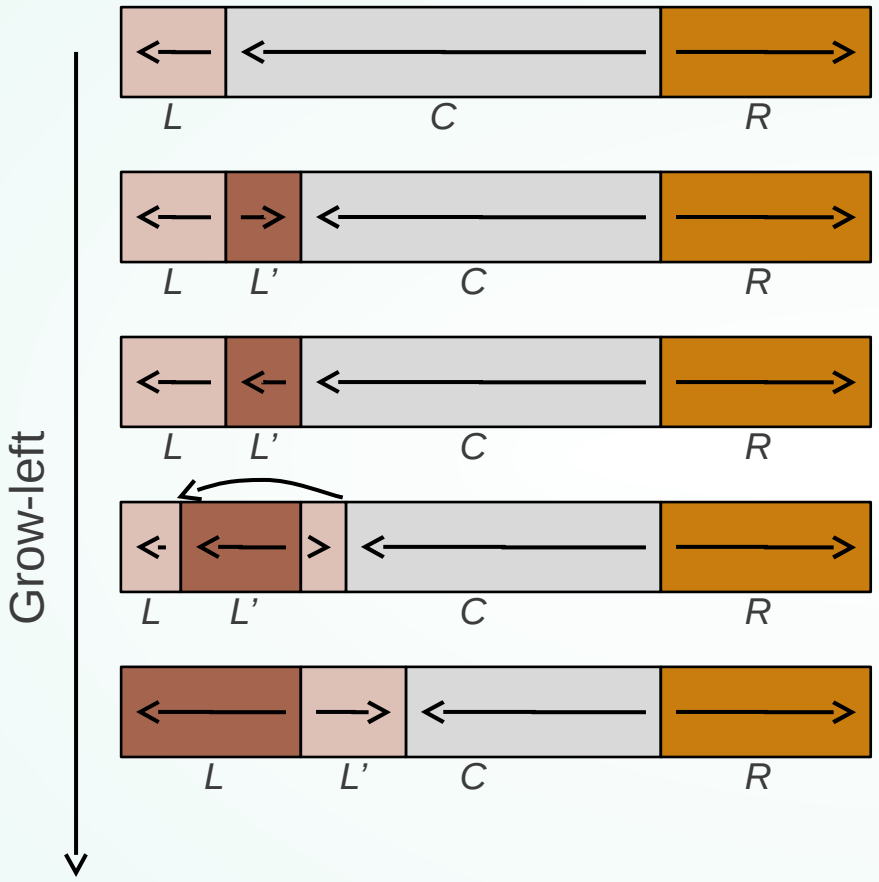
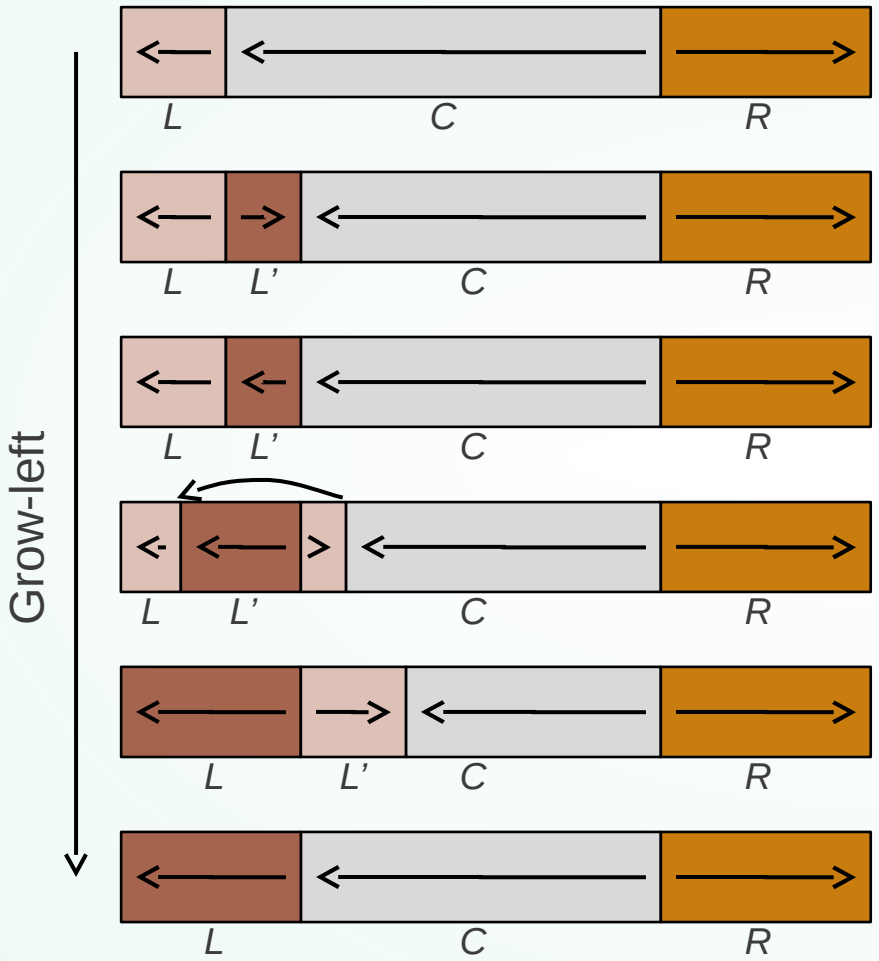


Implicit Moveable Dictionaries



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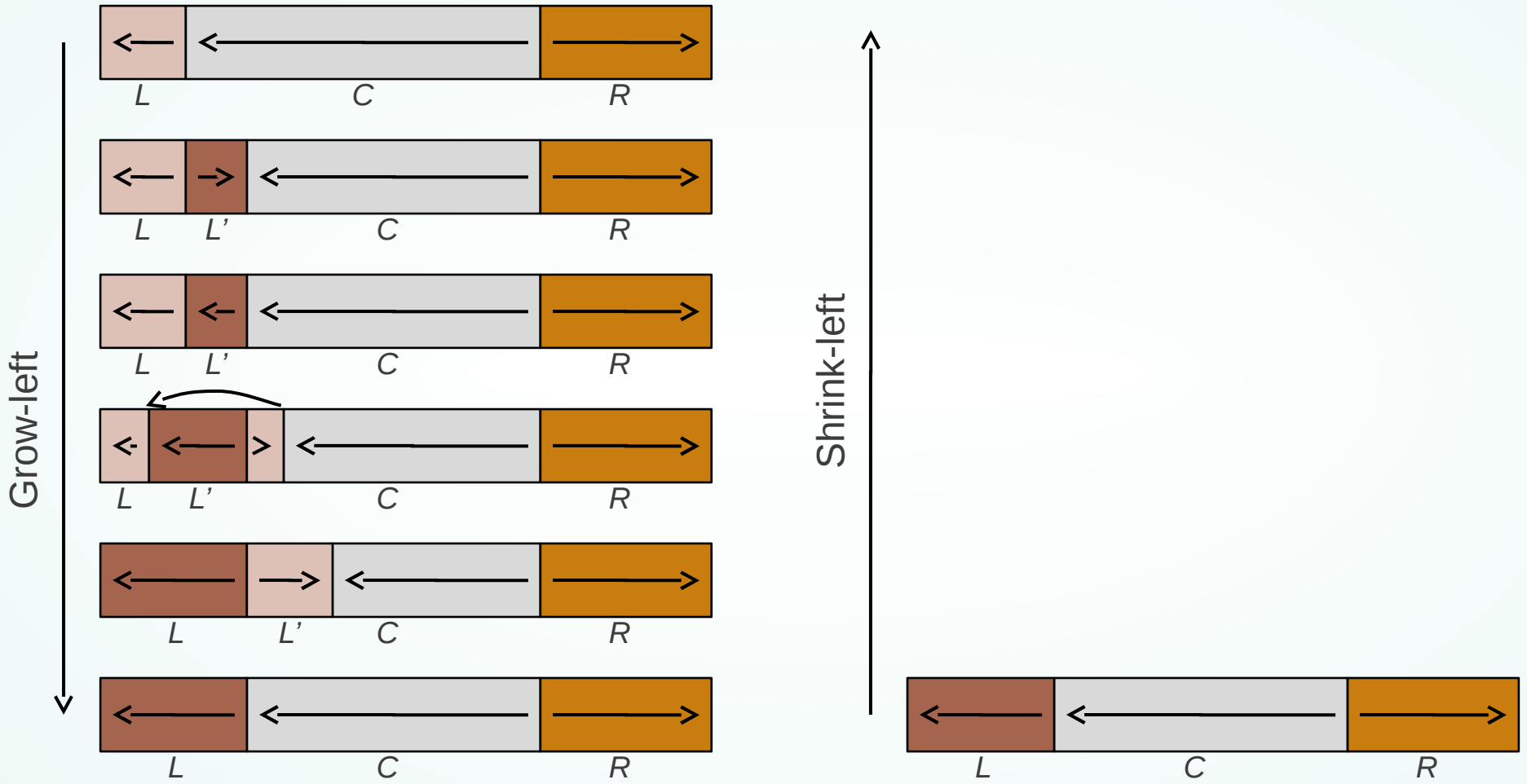

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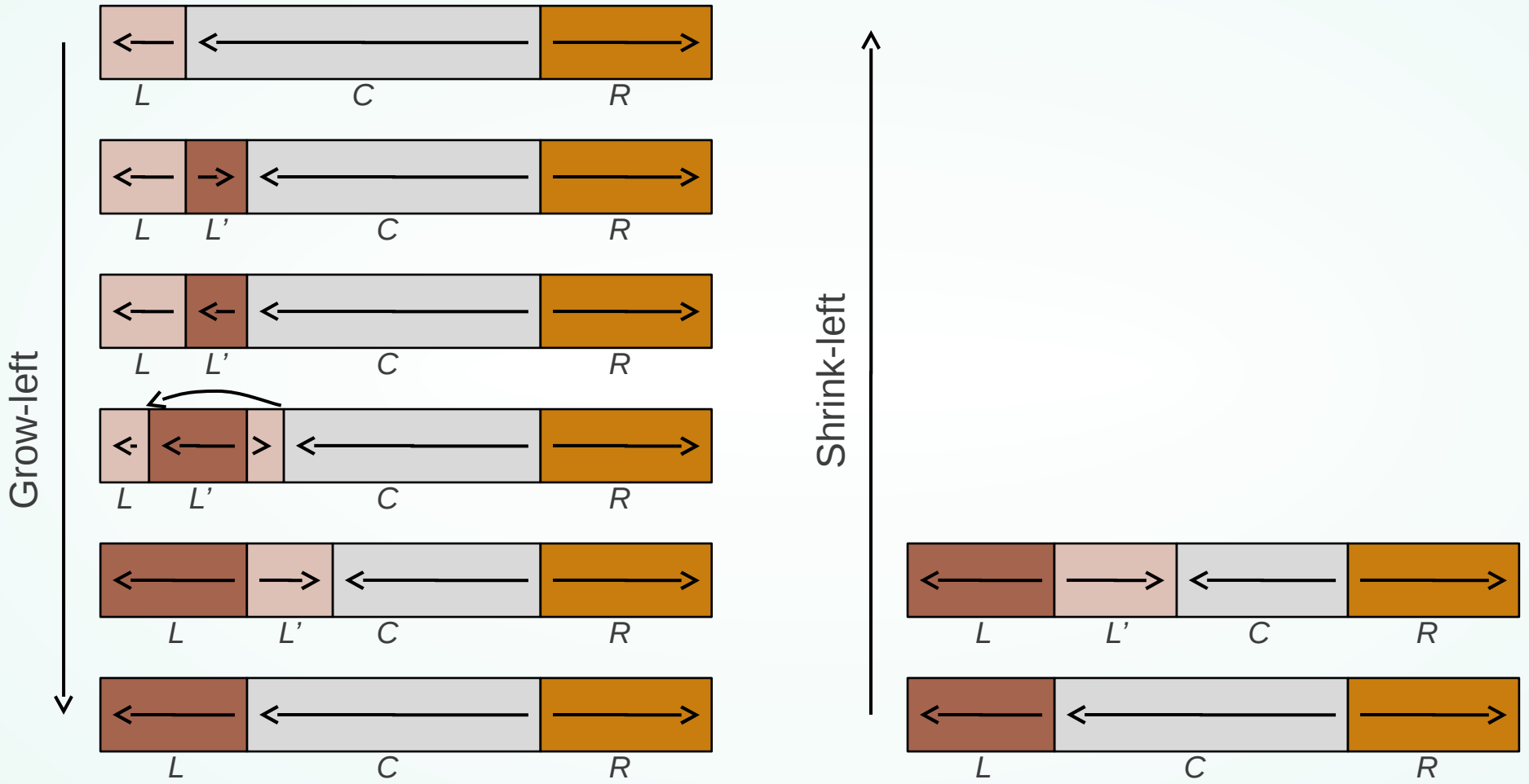

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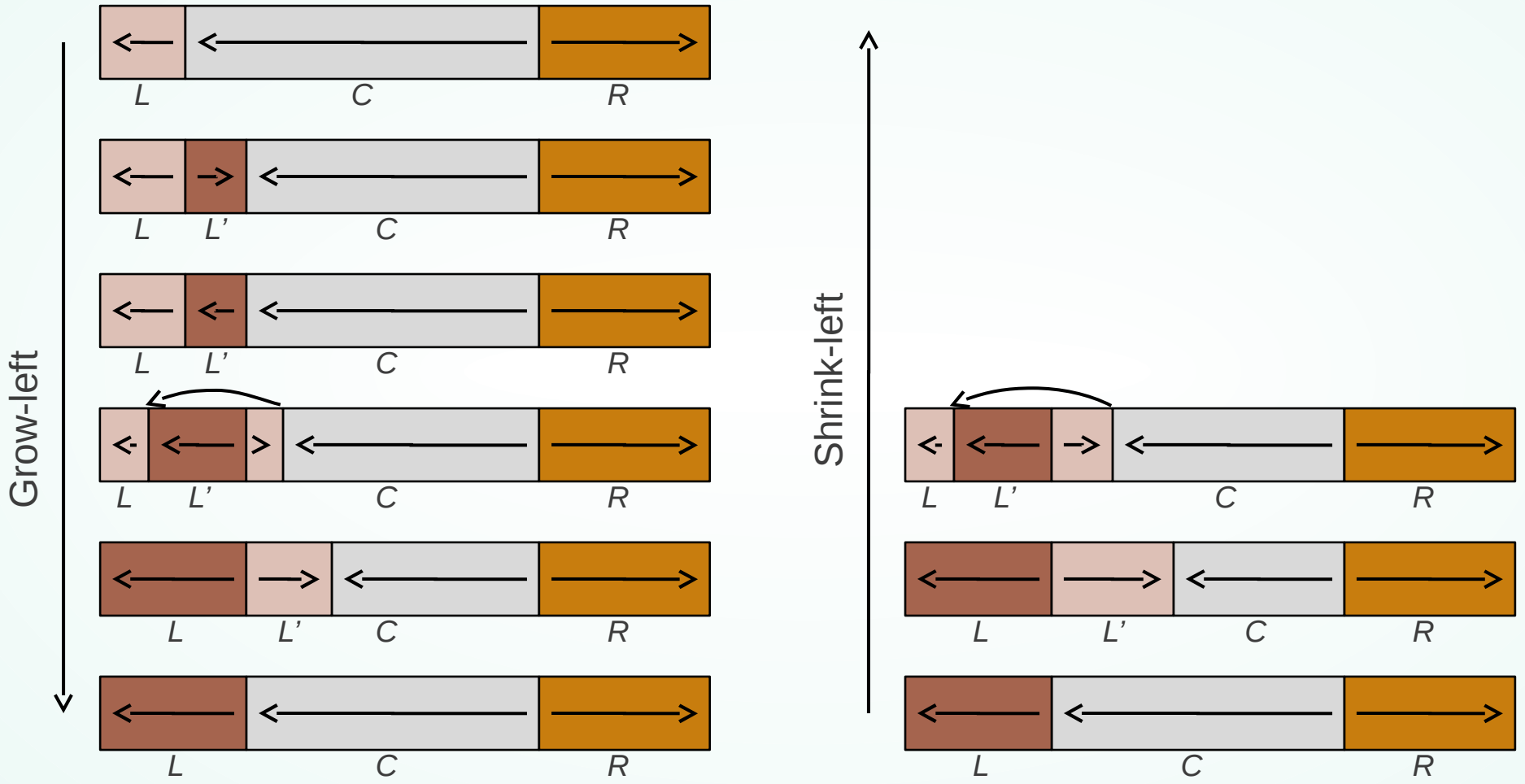

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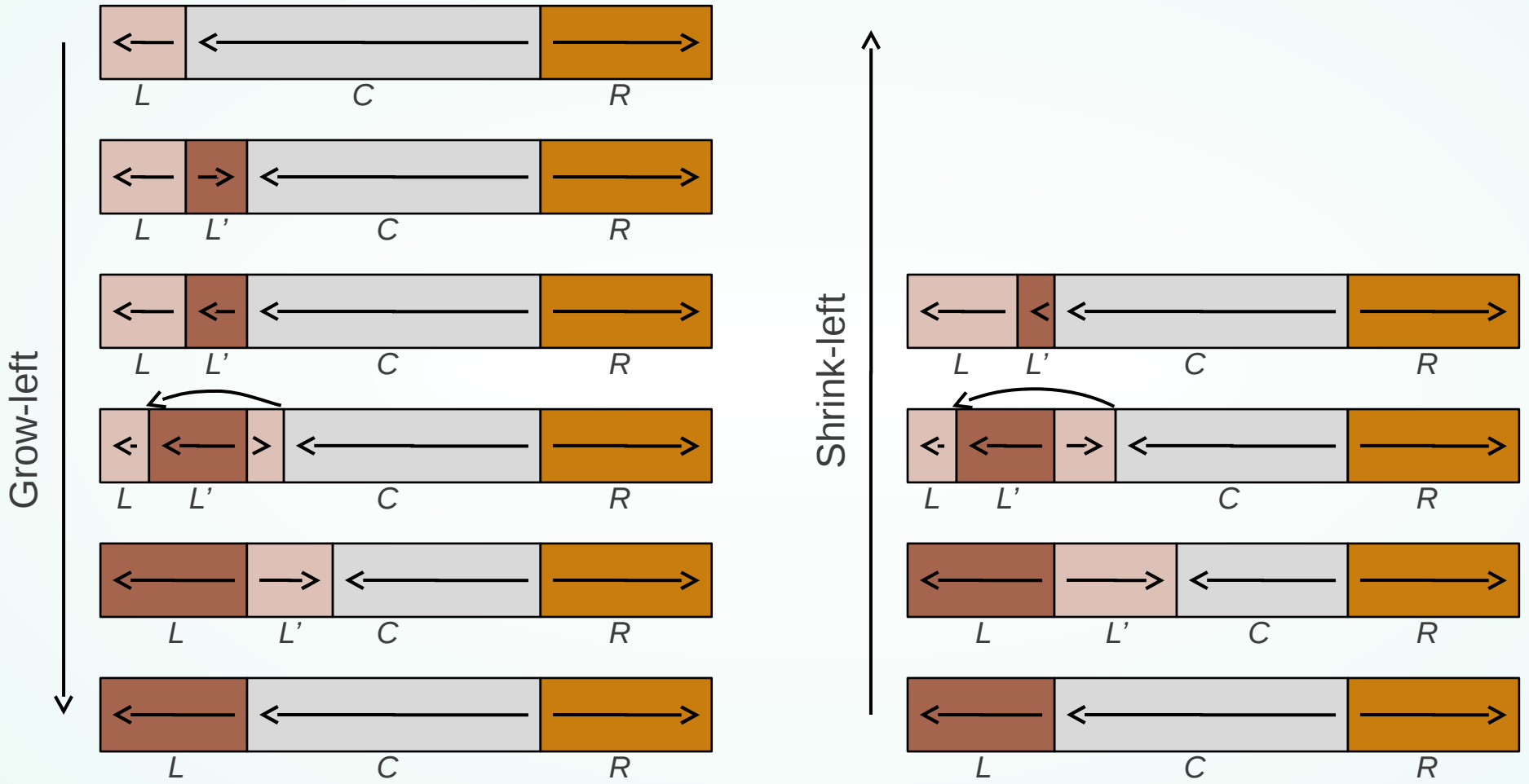

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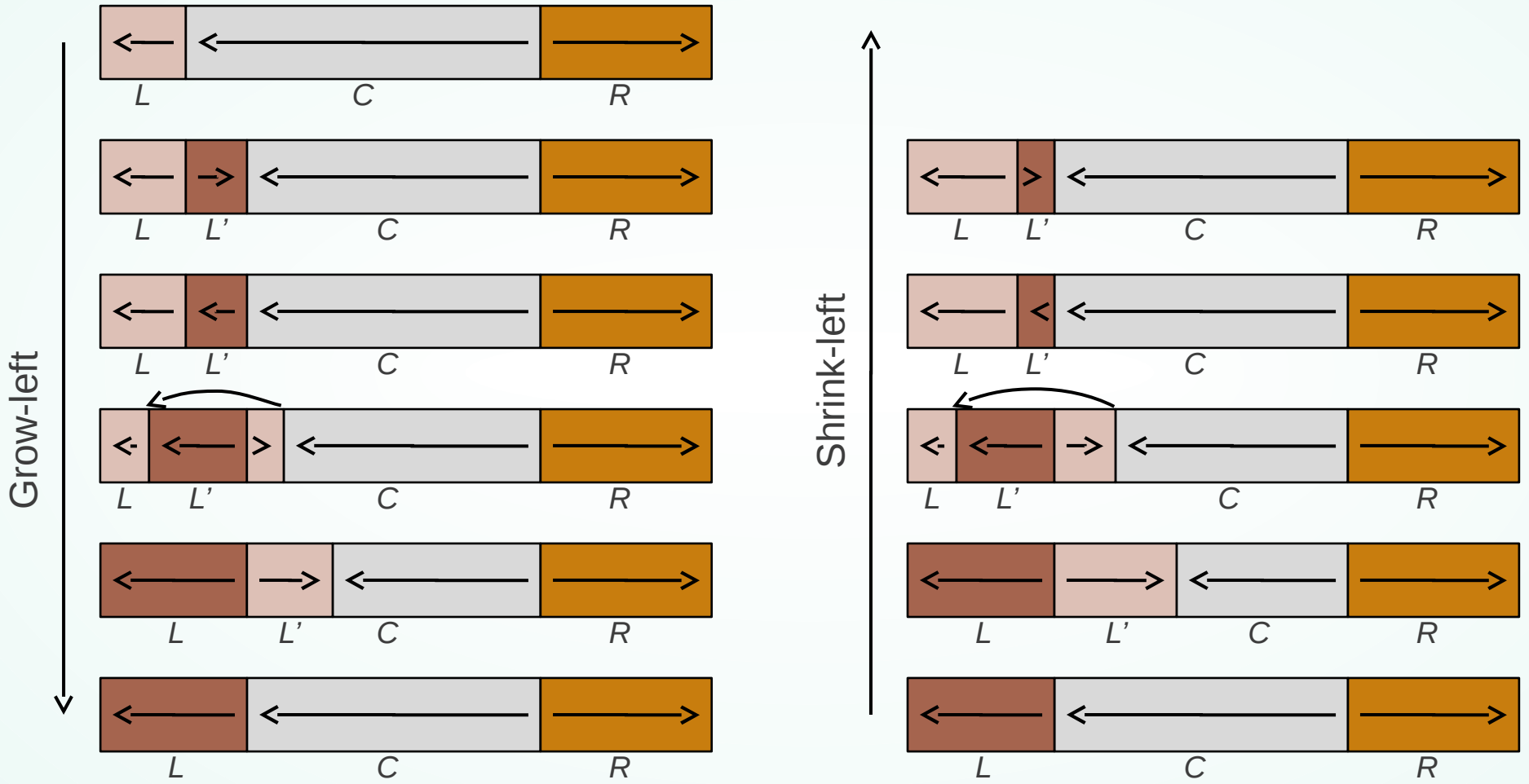

Implicit Moveable Dictionaries