External Memory Geometric Data Structures

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Summer School on Massive Datasets

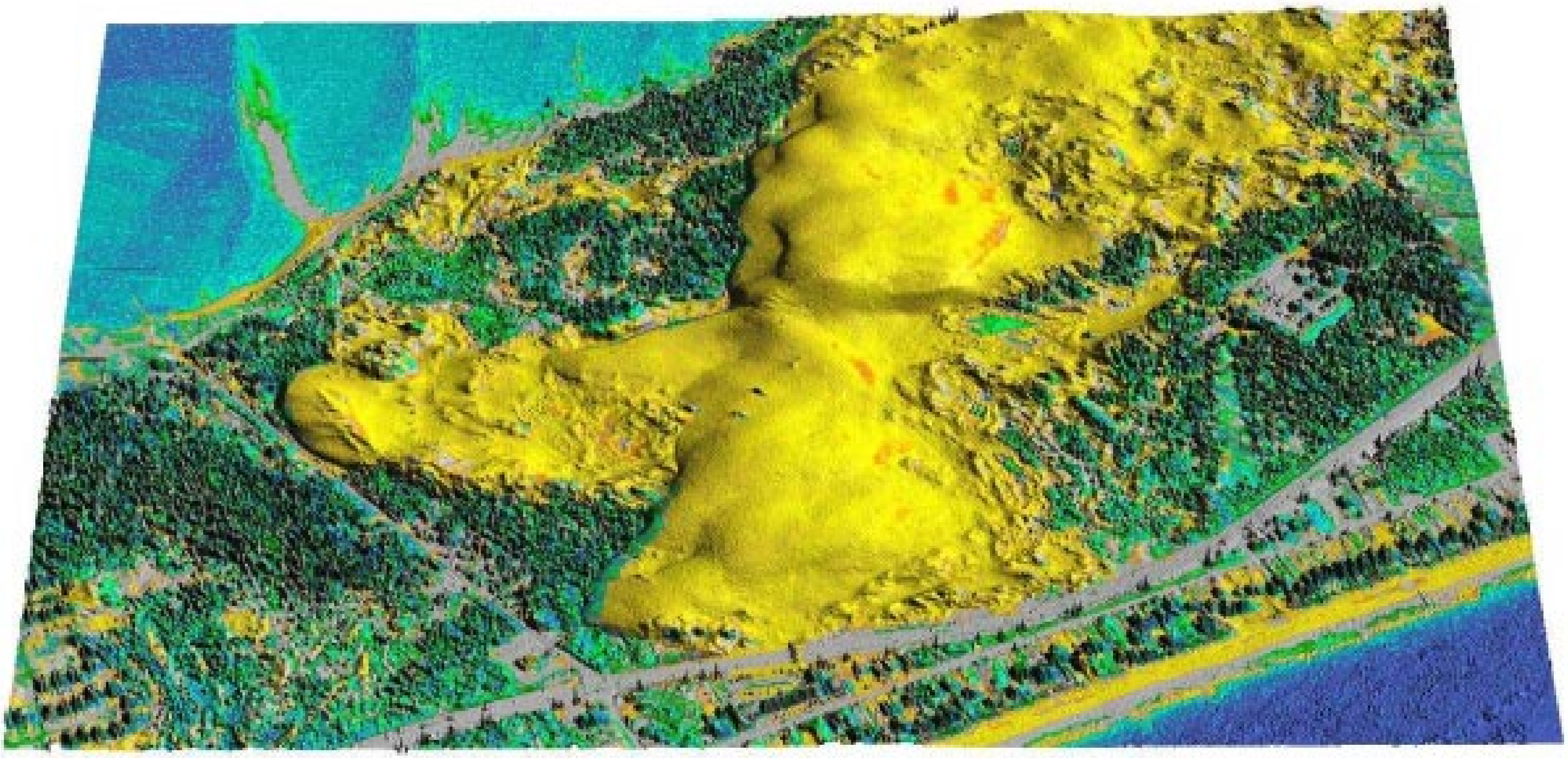
External Memory Geometric Data Structures

- Many massive dataset applications involve geometric data (or data that can be interpreted geometrically)
  - Points, lines, polygons
- Data need to be stored in data structures on external storage media such that on-line queries can be answered I/O-efficiently
- Data often need to be maintained during dynamic updates

- Examples:
  - Phone: Wireless tracking
  - Consumer: Buying patterns (supermarket checkout)
  - Geography: NASA satellites generate 1.2 TB per day

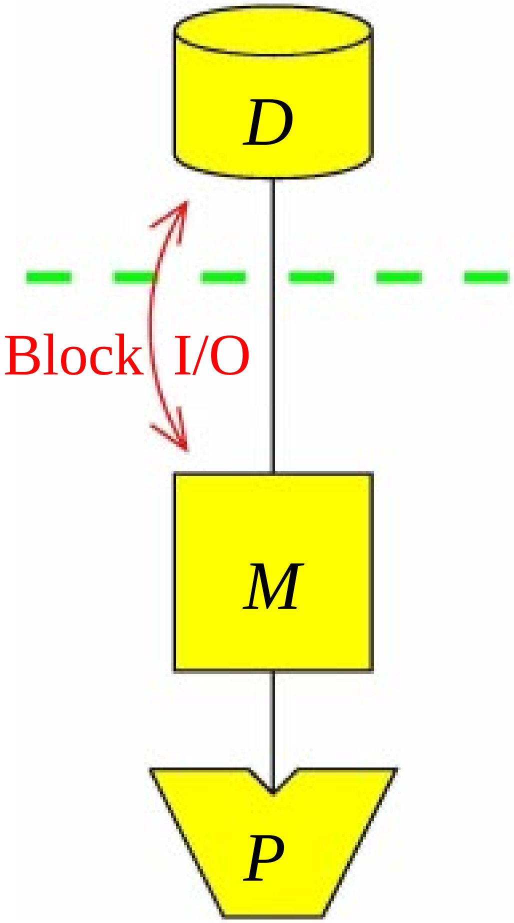
Example: LIDAR terrain data

- Massive (irregular) point sets (1-10m resolution)
- Appalachian Mountains (between 50GB and 5TB)
- Need to be queried and updated efficiently



Example: Jockey’s ridge (NC cost)

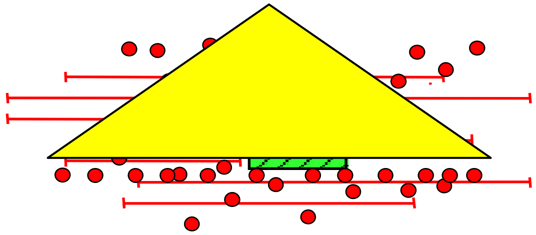
Model



- Model as previously
  - N : Elements in structure
  - B : Elements per block
  - M : Elements in main memory
  - T : Output size in searching problems
- Focus on
  - Worst-case structures
  - Dynamic structures
  - Fundamental structures
  - Fundamental design techniques

Outline

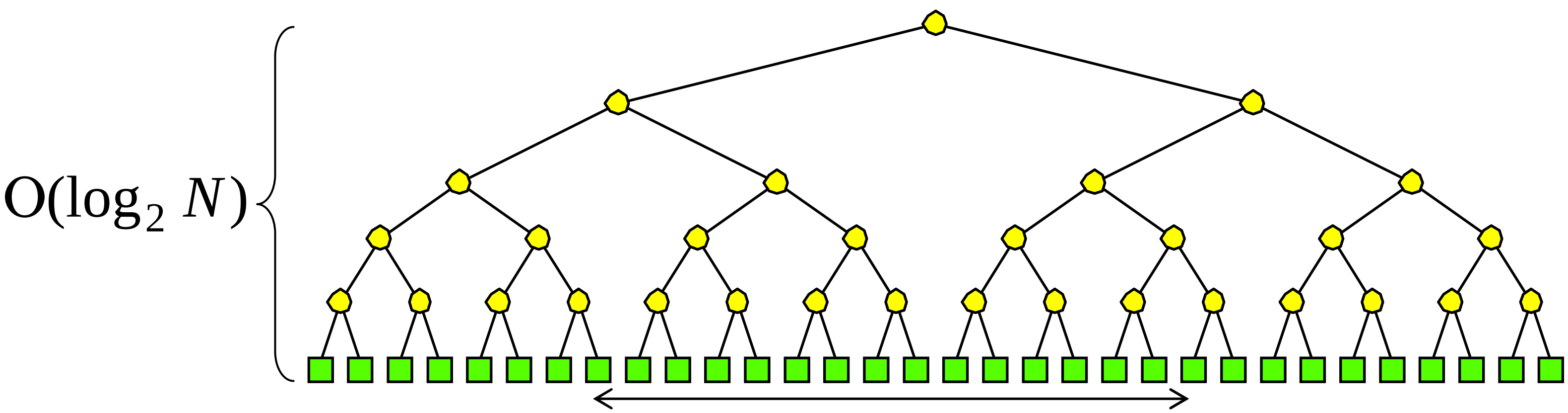
- Today: Dimension one
  – External search trees: B-trees
  – Techniques/tools
    * Persistent B-trees (search in the past)
    * Buffer trees (efficient construction)



- Tomorrow: "Dimension 1.5"
  – Handling intervals/segments (interval stabbing/point location)
  – Techniques/tools: Logarithmic method, weight-balanced B-trees, global rebuilding
- Saturday: Dimension two
  – Two-dimensional range searching

External Search Trees

- Binary search tree:
  - Standard method for search among N elements
  - We assume elements in leaves

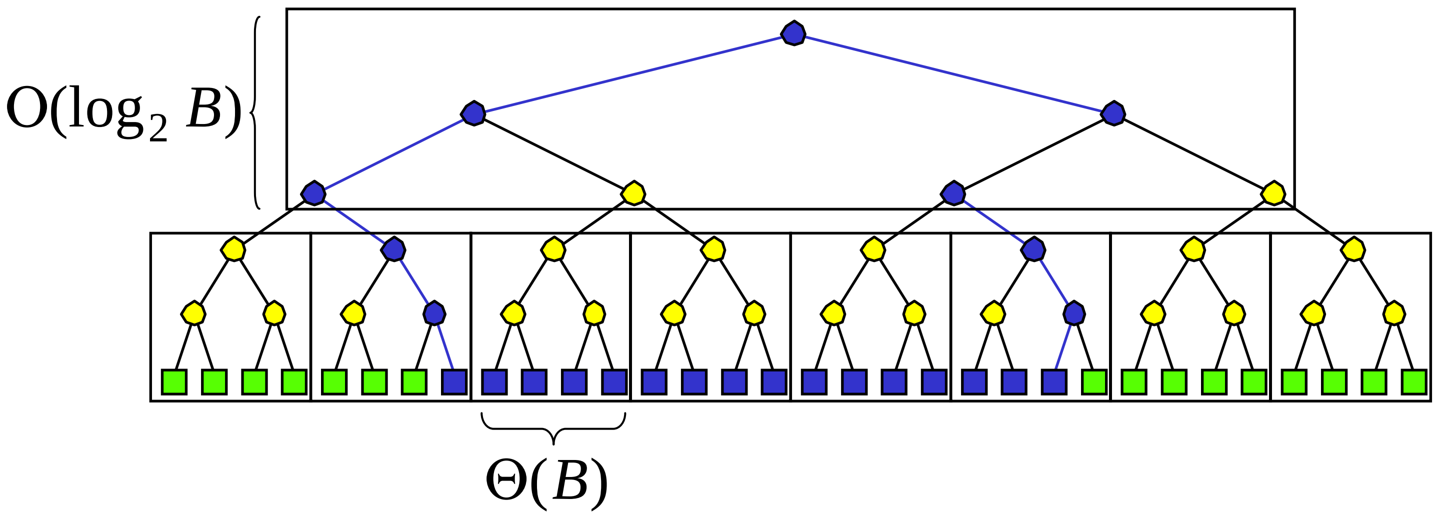


  - Search traces at least one root-leaf path
  - If nodes stored arbitrarily on disk

    $\Rightarrow$ Search in $O(\log_2 N)$ I/Os

    $\Rightarrow$ Rangesearch in $O(\log_2 N + T)$ I/Os

External Search Trees



- BFS blocking:
  - Block height $O(\log_2 N) / O(\log_2 B) = O(\log_B N)$
  - Output elements blocked

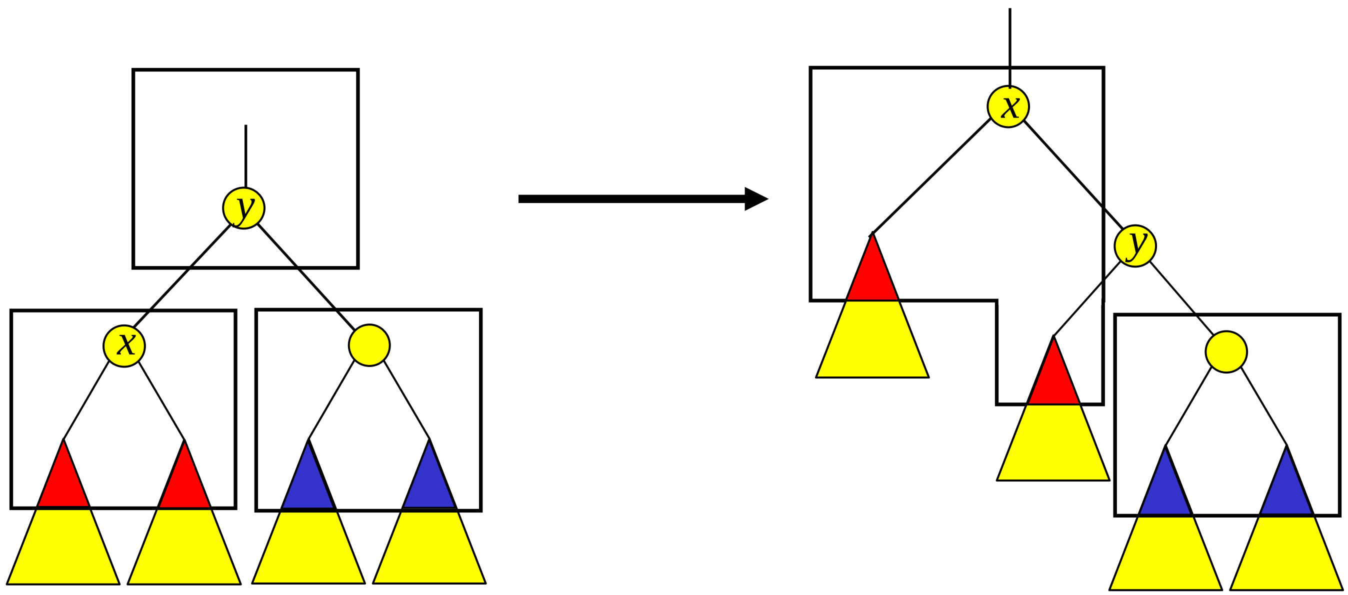
$\Downarrow$

Rangesearch in $O(\log_B N + T/B)$ I/Os

- Optimal: $O(N/B)$ space and $O(\log_B N + T/B)$ query

External Search Trees

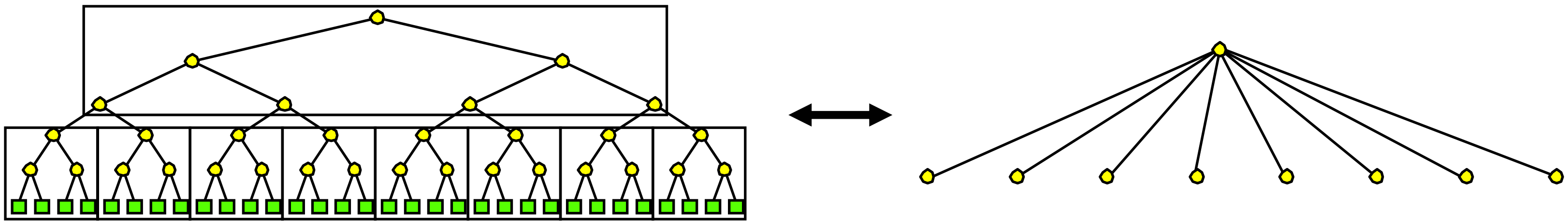
- Maintaining BFS blocking during updates?
  - Balance normally maintained in search trees using rotations



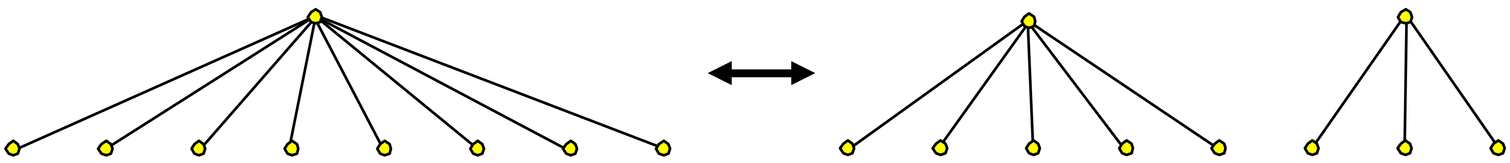
- Seems very difficult to maintain BFS blocking during rotation
  - Also need to make sure output (leaves) is blocked!

B-trees

- BFS-blocking naturally corresponds to tree with fan-out $\Theta(B)$

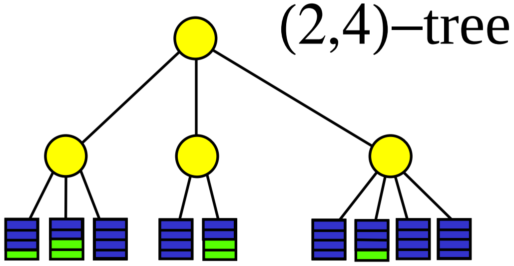


- B-trees balanced by allowing node degree to vary
  - Rebalancing performed by splitting and merging nodes



(a,b)-tree

- T is an (a,b)-tree ($a \geq 2$ and $b \geq 2a-1$)
  - All leaves on the same level (contain between a and b elements)
  - Except for the root, all nodes have degree between a and b
  - Root has degree between 2 and b



- (a,b)-tree uses linear space and has height $O(\log_a N)$

$\Downarrow$

Choosing $a,b = \Theta(B)$ each node/leaf stored in one disk block

$\Downarrow$

$O(N/B)$ space and $O(\log_B N + T/B)$ query

(*a*,*b*)-Tree Insert

- Insert:

Search and insert element in leaf *v*

DO *v* has *b+1* elements

Split *v*:

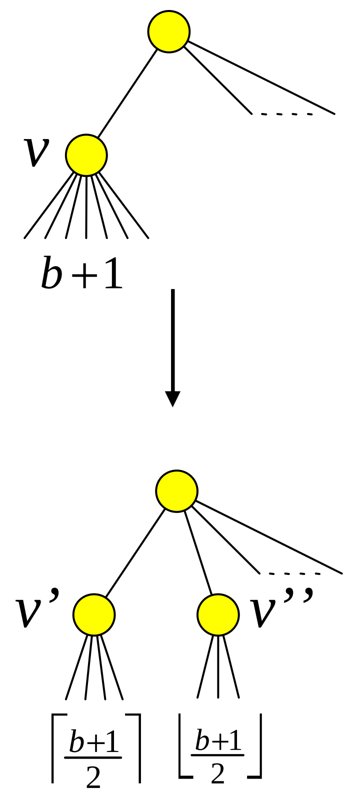
make nodes *v'* and *v''* with

$\left\lceil \frac{b+1}{2} \right\rceil \leq b$ and $\left\lfloor \frac{b+1}{2} \right\rfloor \geq a$ elements

insert element (ref) in *parent(v)*

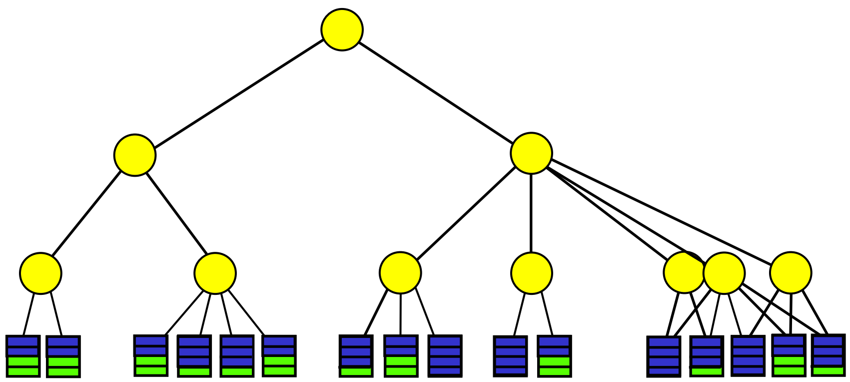
(make new root if necessary)

v=parent(v)



- Insert touch $O(\log_a N)$ nodes

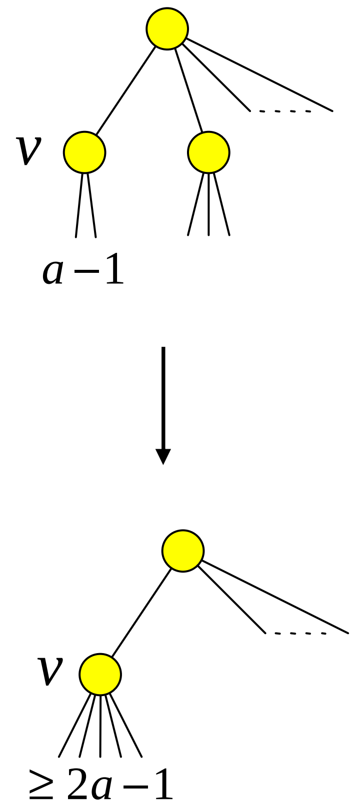
(*a*,*b*)-Tree Insert



(*a*,*b*)-Tree Delete

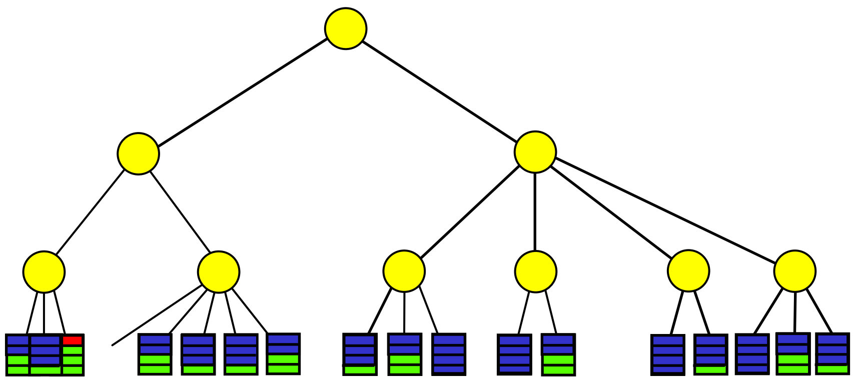
- Delete:

Search and delete element from leaf *v*
DO *v* has *a-1* children
  Fuse *v* with sibling *v*':
    move children of *v*' to *v*
    delete element (ref) from *parent(v)*
    (delete root if necessary)
  If *v* has >*b* (and $\leq a+b-1$) children split *v*
v=*parent(v)*



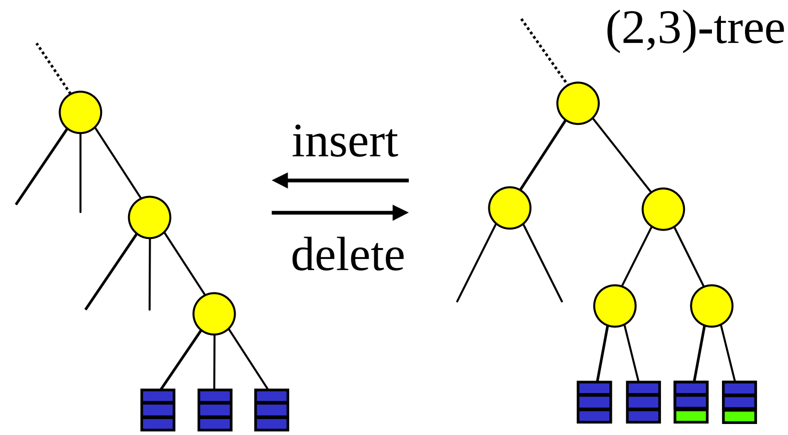
- Delete touch $O(\log_a N)$ nodes

(a,b)-Tree Delete



(a,b)-Tree

- (a,b)-tree properties:
  - If $b=2a-1$ one update can cause many rebalancing operations



  - If $b \geq 2a$ update only cause $O(1)$ rebalancing operations amortized
  - If $b>2a$ $O(\frac{1}{b/2-a}) = O(1/a)$ rebalancing operations amortized
    - Both somewhat hard to show
  - If $b=4a$ easy to show that update causes $O(\frac{1}{a}\log_a N)$ rebalance operations amortized
    - After split during insert a leaf contains $\simeq 4a/2=2a$ elements
    - After fuse (and possible split) during delete a leaf contains between $\simeq 2a$ and $\simeq \frac{5}{2}a$ elements

(*a*,*b*)-Tree

- (a,b)-tree with leaf parameters a_l,b_l ($b=4a$ and $b_l=4a_l$)
  - Height $O(\log_a \frac{N}{a_l})$
  - $O(1/a_l)$ amortized leaf rebalance operations
  - $O(\frac{1}{a \cdot a_l}\log_a N)$ amortized internal node rebalance operations

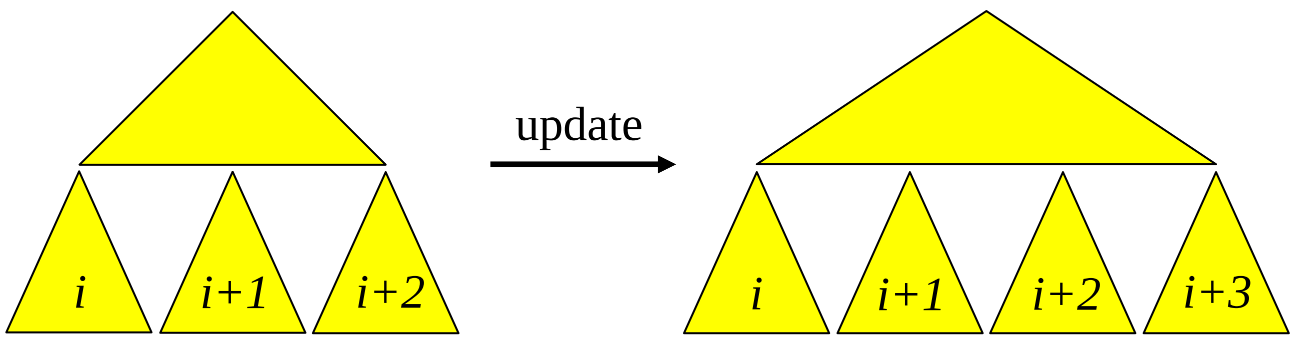
- B-trees: (a,b)-trees with $a,b = \Theta(B)$
  - B-trees with elements in the leaves sometimes called B^+-tree
- Fan-out k B-tree:
  - $(k/4,k)$-trees with leaf parameter $\Theta(B)$ and elements in leaves
- Fan-out $\Theta(B^{1/c})$ B-tree with $c \geq 1$
  - $O(N/B)$ space
  - $O(\log_{B^{1/c}} N + T/B) = O(\log_B N + T/B)$ query
  - $O(\log_B N)$ update

Persistent B-tree

- In some applications we are interested in being able to access previous versions of data structure
  - Databases
  - Geometric data structures (later)
- Partial persistence:
  - Update current version (getting new version)
  - Query all versions
- We would like to have partial persistent B-tree with
  - $O(N/B)$ space – N is number of updates performed
  - $O(\log_B N)$ update
  - $O(\log_B N + T/B)$ query in any version

Persistent B-tree

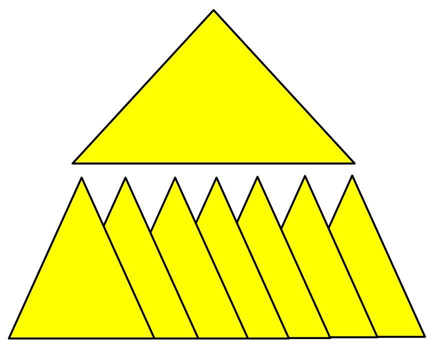
- East way to make B-tree partial persistent
  - Copy structure at each operation
  - Maintain "version-access" structure (B-tree)



- Good $O(\log_B N + T/B)$ query in any version, but
  - $O(N/B)$ I/O update
  - $O(N^2/B)$ space

Persistent B-tree

- Idea:
  - Elements augmented with "existence interval"
  - Augmented elements stored in one structure
  - Elements "alive" at "time" t (version t) form B-tree



  - Version access structure (B-tree) to access B-tree root at time t

Persistent B-tree

- Directed acyclic graph with elements in leaves (sinks)
  - Routing elements in internal nodes
- Each element (routing element) and node has existence interval
- Nodes alive at time t make up $(B/4,B)$-tree on alive elements
- B-tree on all roots (version access structure)

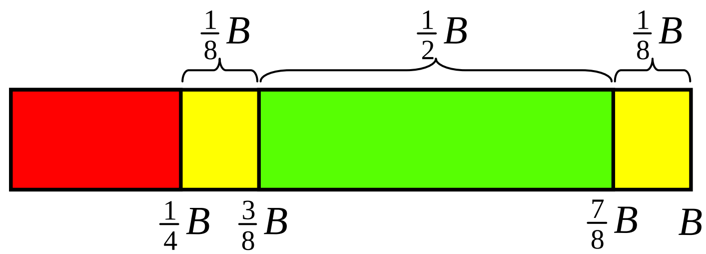
$\Downarrow$

Answer query at version t in $O(\log_B N + T/B)$ I/Os as in normal B-tree

- Additional invariant:
  - New node (only) contains between $\frac{3}{8}B$ and $\frac{7}{8}B$ live elements

$\Downarrow$

$O(N/B)$ blocks

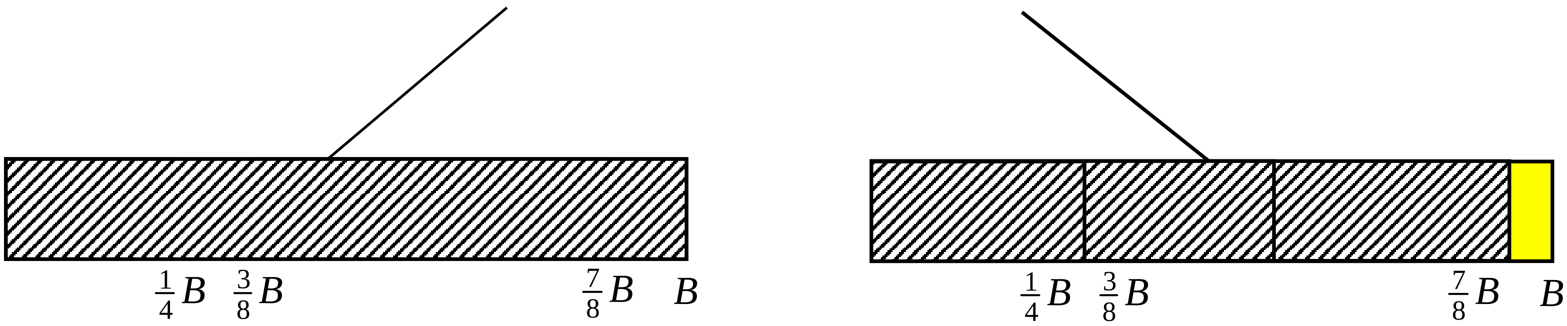


Persistent B-tree Insert

- Search for relevant leaf *l* and insert new element
- If *l* contains $x > B$ elements: Block overflow
  - Version split:

    Mark *l* dead and create new node *v* with *x* alive element
  - If $x > \frac{7}{8}B$: Strong overflow
  - If $x < \frac{3}{8}B$: Strong underflow
  - If $\frac{3}{8}B \leq x \leq \frac{7}{8}B$ then recursively update *parent*(*l*):

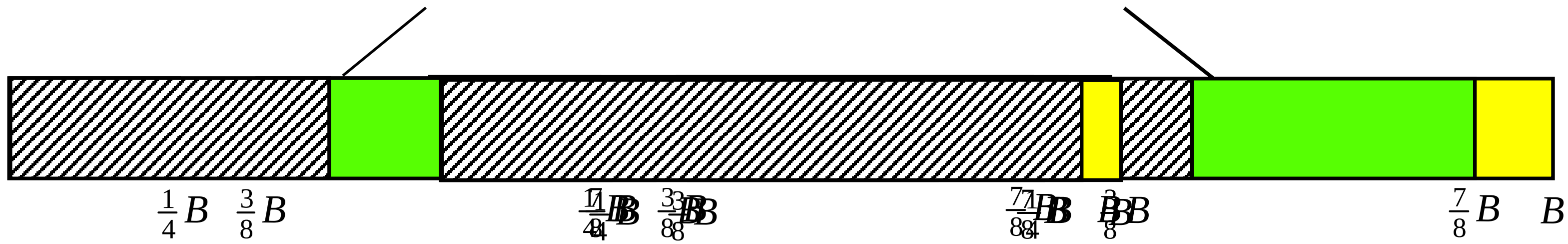
    Delete reference to *l* and insert reference to *v*



Persistent B-tree Insert

- Strong overflow ($x > \frac{7}{8}B$)
  - Split v into v' and v' with $\frac{x}{2}$ elements each ($\frac{3}{8}B < \frac{x}{2} \leq \frac{1}{2}B$)
  - Recursively update *parent*(l):

    Delete reference to l and insert reference to v' and v''



- Strong underflow ($x < \frac{3}{8}B$)
  - Merge x elements with y live elements obtained by version split on sibling ($x + y \geq \frac{1}{2}B$)
  - If $x + y \geq \frac{7}{8}B$ then (strong overflow) perform split
  - Recursively update *parent*(l):

    Delete two references insert one or two references

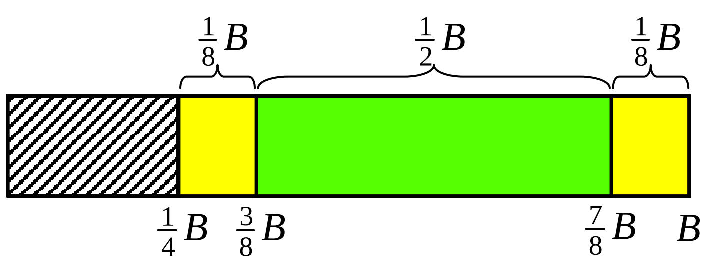
Persistent B-tree Delete

- Search for relevant leaf l and mark element dead
- If l contains $x < {}^1\!/_4 B$ alive elements: Block underflow
  - Version split:

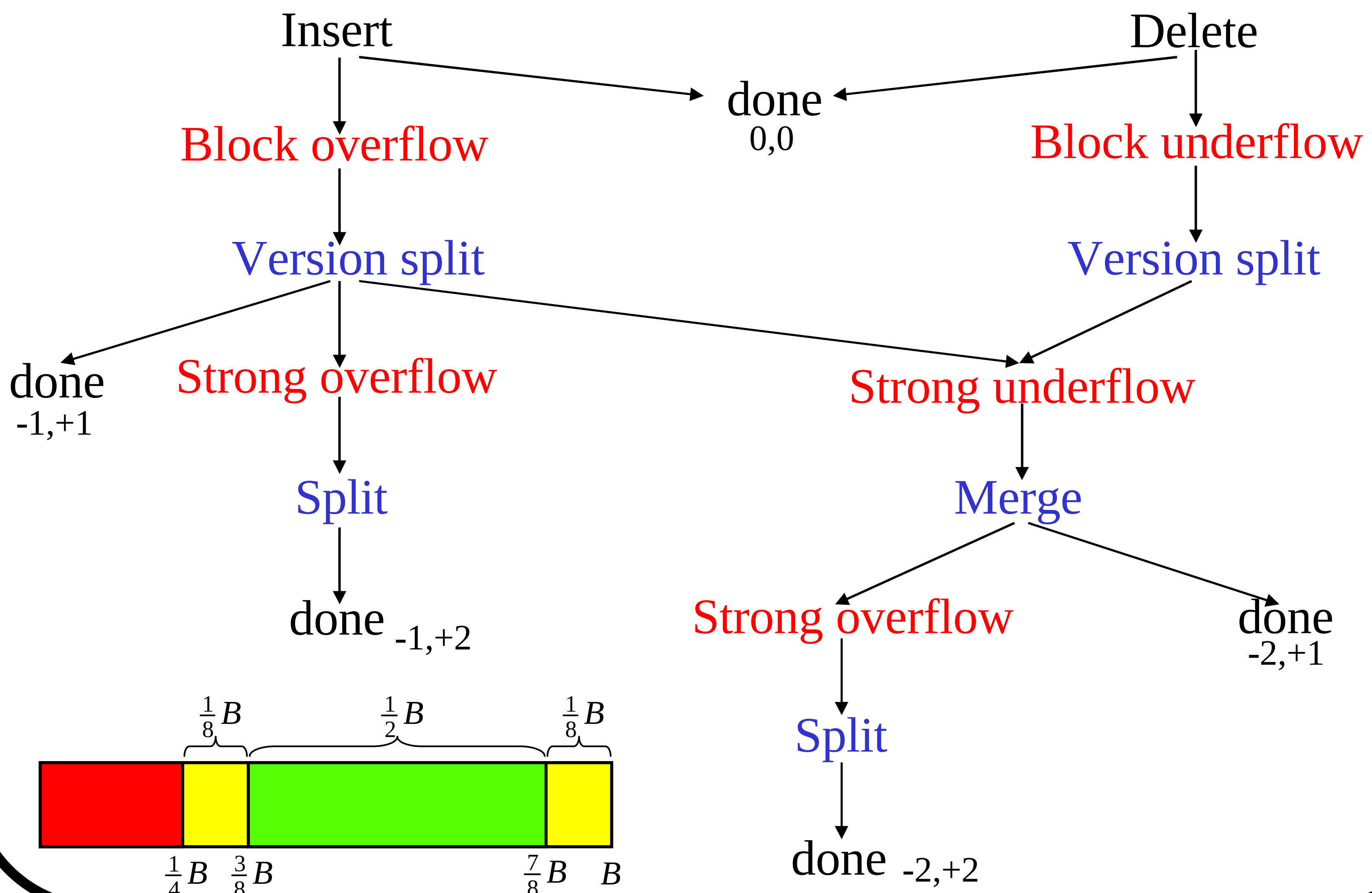
    Mark l dead and create new node v with x alive element
  - Strong underflow ($x < {}^3\!/_8 B$):

    Merge (version split) and possibly split (strong overflow)
  - Recursively update *parent*(l):

    Delete two references insert one or two references



Persistent B-tree



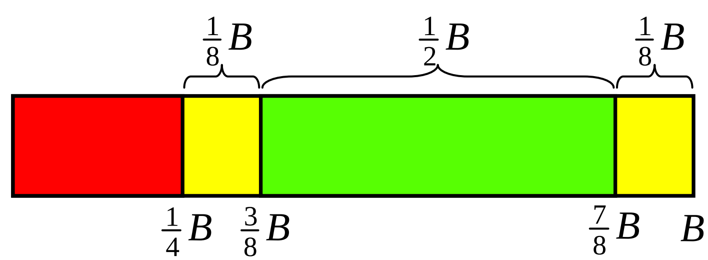
Persistent B-tree Analysis

- Update: $O(\log_B N)$
  - Search and “rebalance” on one root-leaf path
- Space: $O(N/B)$
  - At least $\frac{1}{8}B$ updates in leaf in existence interval
  - When leaf l die
    - At most two other nodes are created
    - At most one block over/underflow one level up (in *parent*(l))

$\Downarrow$

  - During N updates we create:
    - $O(N/B)$ leaves
    - $O(N/B^i)$ nodes i levels up

$\Rightarrow$ Space: $\sum_i O(N/B^i) = O(N/B)$



Summary: B-trees

- Problem: Maintaining N elements dynamically

- Fan-out $\Theta(B^{1/c})$ B-tree ($c \geq 1$)
  - Degree balanced tree with each node/leaf in $O(1)$ blocks
  - $O(N/B)$ space
  - $O(\log_B N + T/B)$ I/O query
  - $O(\log_B N)$ I/O update
- Space and query optimal in comparison model

- Persistent B-tree
  - Update current version
  - Query all previous versions

Other B-tree Variants

- Weight-balanced B-trees
  - Weight instead of degree constraint
  - Nodes high in the tree do not split very often
  - Used when secondary structures are used

  More later!
- Level-balanced B-trees
  - Global instead of local balancing strategy
  - Whole subtrees rebuilt when too many nodes on a level
  - Used when parent pointers and divide/merge operations needed
- String B-trees
  - Used to maintain and search (variable length) strings

  More later (Paolo)

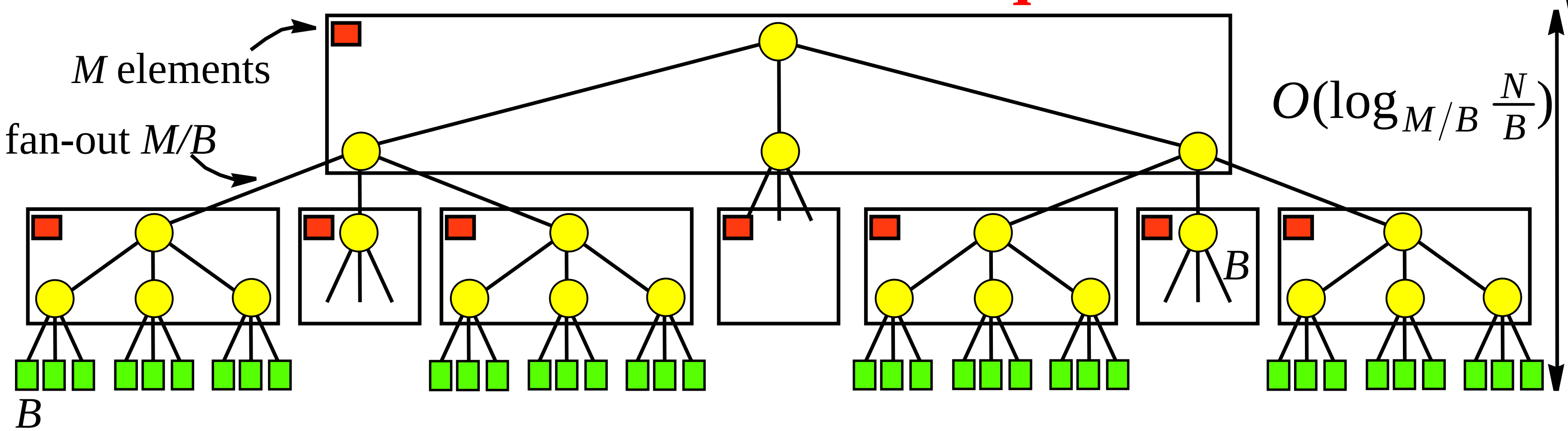
B-tree Construction

- In internal memory we can sort N elements in $O(N \log N)$ time using a balanced search tree:
  - Insert all elements one-by-one (construct tree)
  - Output in sorted order using in-order traversal

- Same algorithm using B-tree use $O(N \log_B N)$ I/Os
  - A factor of $O(B \frac{\log \frac{M}{B}}{\log B})$ non-optimal

- We could of course build B-tree bottom-up in $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/Os
  - But what about persistent B-tree?
  - In general we would like to have dynamic data structure to use in $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ algorithms $\Rightarrow O(\frac{1}{B} \log_{M/B} \frac{N}{B})$ I/O operations

Buffer-tree Technique



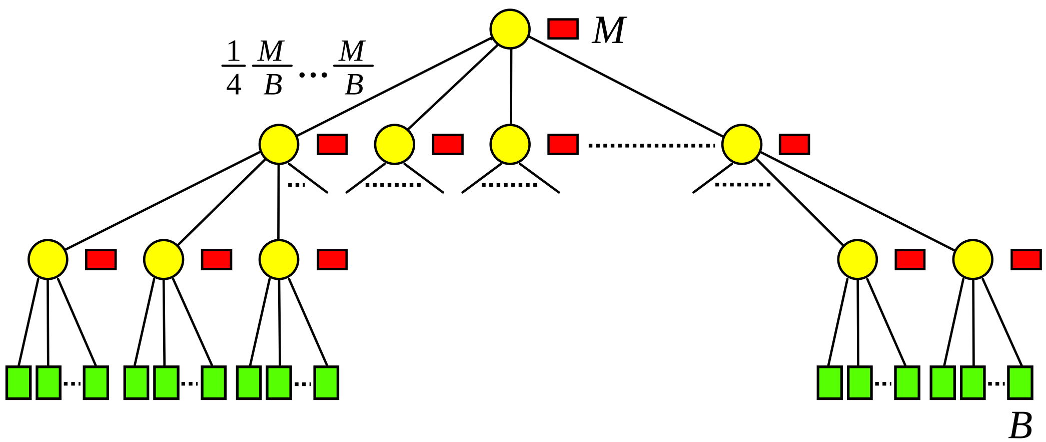
- Main idea: Logically group nodes together and add buffers
  - Insertions done in a "lazy" way – elements inserted in buffers.
  - When a buffer runs full elements are pushed one level down.
  - Buffer-emptying in $O(M/B)$ I/Os

    $\Rightarrow$ every *block* touched constant number of times on each level

    $\Rightarrow$ inserting N elements (N/B blocks) costs $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/Os.

Basic Buffer-tree

- Definition:
  - Fan-out $\frac{M}{B}$ B-tree — $(\frac{1}{4}\frac{M}{B},\frac{M}{B})$-tree with size B leaves
  - Size M buffer in each internal node

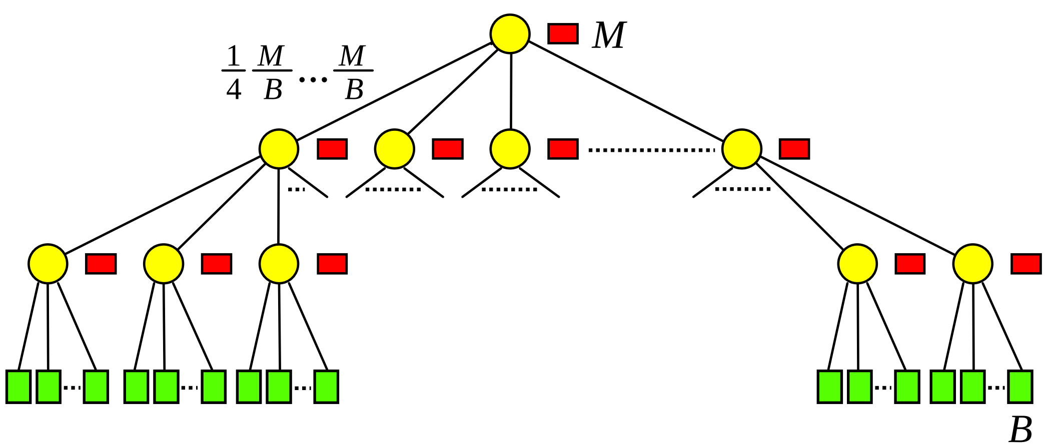


- Updates:
  - Add time-stamp to insert/delete element
  - Collect B elements in memory before inserting in root buffer
  - Perform buffer-emptying when buffer runs full

Basic Buffer-tree

- Note:
  - Buffer can be larger than M during recursive buffer-emptying
    - * Elements distributed in sorted order

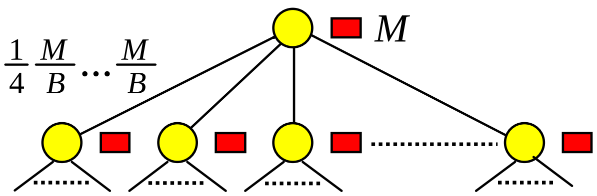
      $\Rightarrow$ at most M elements in buffer unsorted
  - Rebalancing needed when “leaf-node” buffer emptied
    - * Leaf-node buffer-emptying only performed after all full internal node buffers are emptied



Basic Buffer-tree

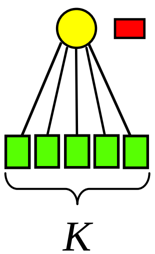
- Internal node buffer-empty:

  – Load first M (unsorted) elements into memory and sort them
  – Merge elements in memory with rest of (already sorted) elements
  – Scan through sorted list while
    * Removing "matching" insert/deletes
    * Distribute elements to child buffers
  – Recursively empty full child buffers



- Emptying buffer of size X takes $O(X/B+M/B)$=O(X/B) I/Os

Basic Buffer-tree



- Buffer-empty of leaf node with K elements in leaves

  – Sort buffer as previously

  – Merge buffer elements with elements in leaves

  – Remove "matching" insert/deletes obtaining K' elements

  – If $K'<K$ then

    * Add K-K' "dummy" elements and insert in "dummy" leaves

    Otherwise

    * Place K elements in leaves
    * Repeatedly insert block of elements in leaves and rebalance

- Delete dummy leaves and rebalance when all full buffers emptied

Basic Buffer-tree

- Invariant:

  Buffers of nodes on path from root to emptied leaf-node are empty

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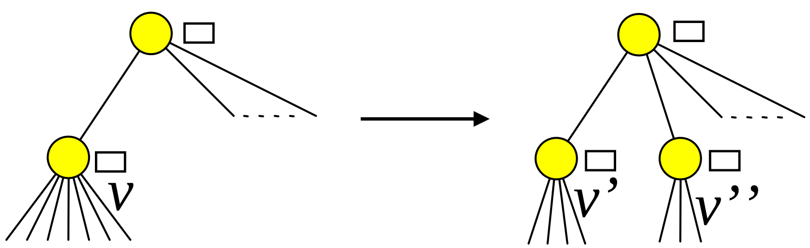
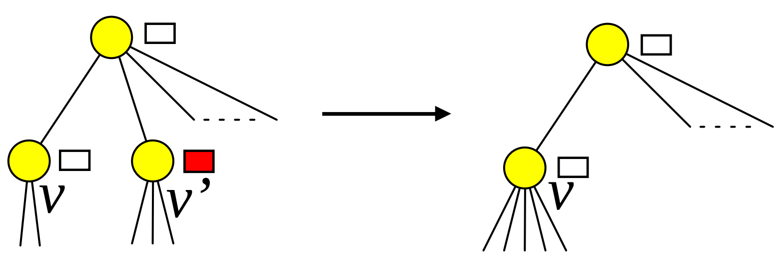
- Insert rebalancing (splits) performed as in normal B-tree



- Delete rebalancing: *v'* buffer emptied before fuse of *v*
  - Necessary buffer emptyings performed before next dummy-block delete
  - Invariant maintained



Basic Buffer-tree

- Analysis:
  - Not counting rebalancing, a buffer-emptying of node with $X \geq M$ elements (full) takes $O(X/B)$ I/Os

    $\Rightarrow$ total full node emptying cost $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
  - Delete rebalancing buffer-emptying (non-full) takes $O(M/B)$ I/Os

    $\Rightarrow$ cost of one split/fuse $O(M/B)$ I/Os
  - During N updates
    * $O(N/B)$ leaf split/fuse
    * $O(\frac{N/B}{M/B}\log_{M/B}\frac{N}{B})$ internal node split/fuse

    $\Downarrow$

    Total cost of N operations: $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

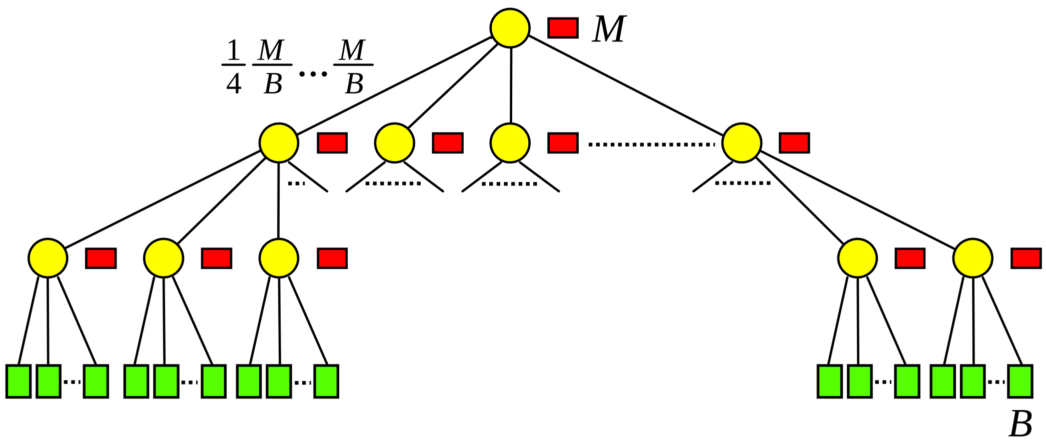
Basic Buffer-tree

- Emptying all buffers after N insertions:

  Perform buffer-emptying on all nodes in BFS-order

  $\Rightarrow$ resulting full-buffer emptyings cost $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

  empty $O(\frac{N/B}{M/B})$ non-full buffers using $O(M/B) \Rightarrow O(N/B)$ I/Os

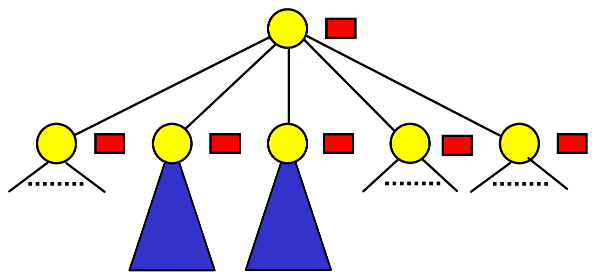


$\Downarrow$

- N elements can be sorted using buffer tree in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

Buffer-tree Technique

- Insert and deletes on buffer-tree takes $O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ I/Os amortized
  - Alternative rebalancing algorithms possible (e.g. top-down)
- One-dim. rangesearch operations can also be supported in $O(\frac{1}{B}\log_{M/B}\frac{N}{B}+\frac{T}{B})$ I/Os amortized
  - Search elements handle lazily like updates
  - All elements in relevant sub-trees reported during buffer-emptying
  - Buffer-emptying in $O(X/B+T'/B)$, where T' is reported elements

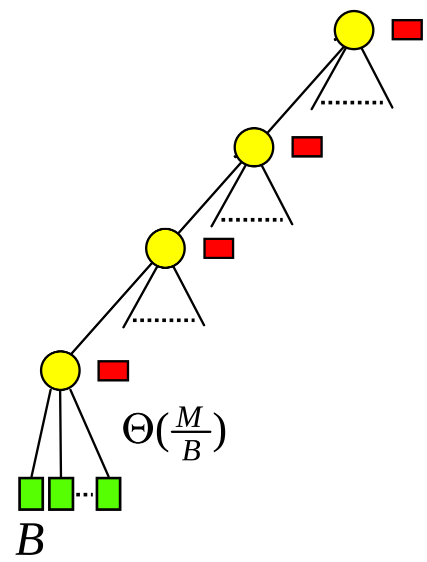


- Buffer-tree can e.g. be use in standard plane-sweep algorithms for orthogonal line segment intersection (alternative to distribution sweeping)

Buffered Priority Queue

- Basic buffer tree can be used in external priority queue
- To delete minimal element:

  - Empty all buffers on leftmost path
  - Delete $\frac{1}{4}M$ elements in leftmost leaf and keep in memory
  - Deletion of next M minimal elements free
  - Inserted elements checked against minimal elements in memory



- $O(\frac{M}{B}\log_{M/B}\frac{N}{B})$ I/Os every $O(M)$ delete $\Rightarrow O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ amortized

Other External Priority Queues

- External priority queue has been used in the development of many I/O-efficient graph algorithms

- Buffer technique can be used on other priority queue structure
  - Heap
  - Tournament tree
- Priority queue supporting update often used in graph algorithms
  - $O(\frac{1}{B}\log_2 \frac{N}{B})$ on tournament tree
  - Major open problem to do it in $O(\frac{1}{B}\log_{M/B} \frac{N}{B})$ I/Os
- Worst case efficient priority queue has also been developed
  - B operations require $O(\log_{M/B} \frac{N}{B})$ I/Os

Other Buffer-tree Technique Results

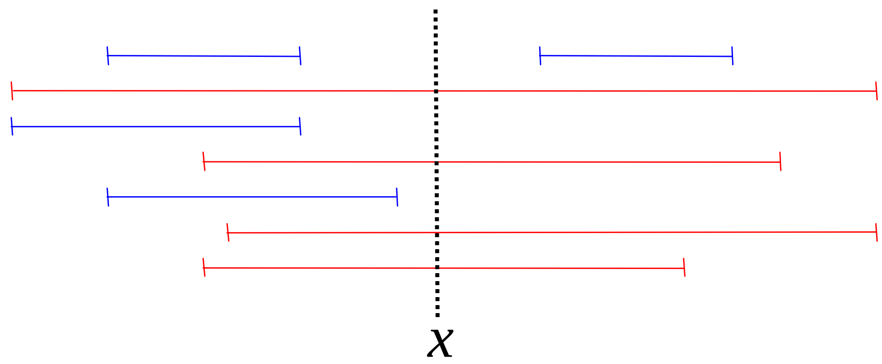
- Attaching $\Theta(B)$ size buffers to normal B-tree can also be use to improve update bound
- Buffered segment tree
  - Has been used in batched range searching and rectangle intersection algorithm
- Can normally be modified to work in D-disk model using D-disk merging and distribution
- Has been used on String B-tree to obtain I/O-efficient string sorting algorithms
- Can be used to construct (bulk load) many data structures, e.g:
  - R-trees
  - Persistent B-trees

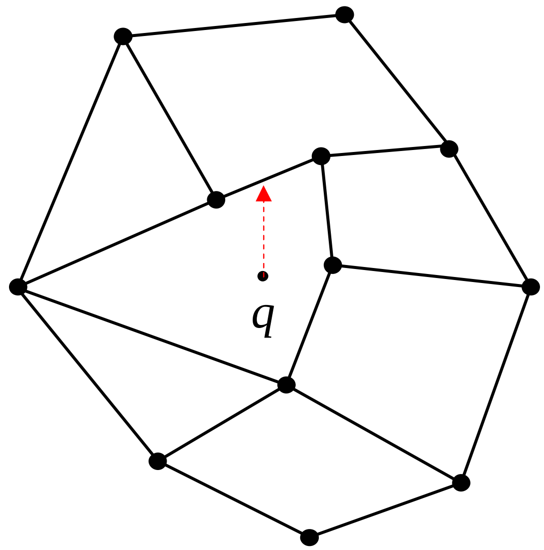
Summary

- Fan-out $\Theta(B^{1/c})$ B-tree $(c \geq 1)$
  - Degree balanced tree with each node/leaf in $O(1)$ blocks
  - $O(N/B)$ space
  - $O(\log_B N + T/B)$ I/O query
  - $O(\log_B N)$ I/O update
- Persistent B-tree
  - Update current version, query all previous versions
  - B-tree bounds with N number of operations performed
- Buffer tree technique
  - Lazy update/queries using buffers attached to each node
  - $O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ amortized bounds
  - E.g. used to construct structures in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

Tomorrow

- “Dimension 1.5” problems: Interval stabbing and point location





- Use of tools/techniques discussed today as well as
  - Logarithmic method
  - Weight-balanced B-trees
  - Global rebuilding