Existence and computation of equilibria of first-price auctions with integral valuations and bids*

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Abstract

In classical auction theory, the existence of pure strategy Nash equilibria (PSNE) of first-price auctions has been established under a variety of mild conditions. Nevertheless, known general existence results do not apply to the case of valuations being distributed according to a discrete probability distribution. In this paper, we consider single-item, sealed-bid, first-price Bayesian auctions with independent, identically distributed private valuations from some finite distribution on the natural numbers and with bidding space also being the natural numbers. We consider two different standard ways of breaking ties between bidders: random tie-breaking, and tie-breaking by an auxiliary Vickrey auction. In the former case, we analytically characterize the bivalued distributions for which symmetric PSNE exist for the case of two bidders. In the latter case, we algorithmically characterize the distributions for which symmetric PSNE exist for *any* finite support size and *any* finite number of bidders. In particular, we show that for any distribution, there are either zero or two such equilibria. When two equilibria exist, exactly one is undominated. We give an efficient (linear time) procedure for computing these equilibria when they do exist. Finally, we consider relaxing the assumption of independent, identically distributed private valuations and show that for the most general way of doing this, the existence of a PSNE is an **NP**-hard problem.

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1 Introduction

Background and Motivation

We consider single-item, sealed-bid, first-price, Bayesian auctions between n risk-neutral bidders with independent, identically distributed, private valuations. A textbook example is the case of an auction with two bidders and valuations uniformly distributed in the real interval $[0; 1]$. In a first-price auction, bidders *underbid*. That is, they submit bids strictly smaller than their valuation. The central question to be answered is by how much they underbid. To provide an answer, auction theory (see [6, 10]) studies *pure strategy Nash equilibria* (PSNE) of such auctions. A PSNE is given by a family of maps, one for each bidder, mapping valuations to bids, so that each map is a best response to the other maps. For the above example, each bidder bidding half his valuation is the unique PSNE. In the past decades, PSNE of first-price auctions have been shown to exist under a variety of mild conditions. Early results were based on smoothness assumptions on the density function of the valuations and used tools from the theory of differential equations to analytically derive the functions mapping valuations to bids. Modern works, such as the seminal paper by Athey [1] extend much beyond the setting of private, independent valuations and prove the existence of PSNE under much weaker conditions on the distribution of valuations. However, to the best of our knowledge, all existence proofs of PSNE in the literature concern cases where distributions of valuations are *atomless*: All singletons must be assigned probability mass 0. Indeed, if the valuation distribution is discrete but the bidding space is the set of non-negative reals, it is easy to see that no PSNE for the auction exists [9].

In this paper, we deviate from the classical setting by studying existence of PSNE in auctions with discrete (in fact, *integral*) valuations *and* bids. In this case, PSNE may or may not exist, depending on the distribution. As a concrete example, consider the case of two bidders and independent private valuations, each distributed uniformly in $\{1, 5\}$. Each bidder must supply an integral bid. In case of a tie, the winner is chosen at random among the high bidders. It is straightforward to check that the symmetric profile $\pi = (\beta_1, \beta_2)$, with $\beta_1 = \beta_2 = \beta$, where $\beta(1) = 0$ and $\beta(5) = 2$, is a PSNE. On the other hand, if there are two bidders and valuations are distributed uniformly in $\{1, 100\}$, it can be easily verified that there is no PSNE. Indeed, a best reply β' to a given strategy β either has $\beta'(100) = \beta(100) + 1$ (if $\beta(100) \ll 100$) or $\beta'(100) = \beta(1) + 1$ (otherwise) so no equilibrium is possible.

The purpose of this paper is to gain a more general understanding of PSNE in single-item first-price auctions with integral valuations and bids. Our motivation is three-fold:

- The setting of integral valuations and bids seems very natural and has been considered in other contexts (e.g., [12, 4]). In particular, in realistic settings, bids *have* to be integral (since monetary amounts are integral!), so the classical assumption of a continuous bidding space is merely a convenient abstraction. Also, if valuations correspond to the prices that may be obtained by reselling the good (as they may), valuations are integral too. So, while we should expect the discrete-value, discrete-bid case to be less well-behaved than the continuous case, it still seems relevant to derive sufficient or necessary conditions for the existence of PSNE, also in the discrete case.

- Assuming integral valuations makes it possible to rigorously consider the problem of deciding whether a PSNE exists and computing such a PSNE from the point of view of the theory of discrete algorithms and computational complexity, as the distribution can then be given as input to a digital computer. In contrast, for the case of continuous valuations, the question about whether the PSNE can be computed is somewhat obfuscated by questions about how the input distribution is to be represented and about numerical issues concerning the computation itself. A discrete computational point of view on first-price auctions may increase in importance as we consider going beyond the setting of independent, private values. In the more general setup, the boundary between existence and non-existence of PSNE is complex and not completely understood. Moving to a discrete setting, one could hope to argue that this complexity is due to, say, **NP**-hardness of variants of the more general problem. We present some very preliminary results along this line below.

- Recent work by the second author [11] considers cryptographic implementation of first-price auctions *perturbed* by privacy concerns. The integral setting is necessary for ensuring that equilibria of the auctions survive the perturbation.

When valuations are sampled by discrete distributions, we expect the exact mechanism of tie-breaking to influence the equilibria of the auction, as ties will occur with non-zero probability. There are two standard ways of breaking ties, both of which will be considered in this paper:

- *Breaking ties at random:* If k bidders all submit the highest bid, we choose one of them as the winner in a (uniform) lottery. While this is a very natural rule, for the case of discrete valuations it has the disadvantage that even in the case of a symmetric equilibrium in monotonically increasing strategies, the equilibrium outcome may be an inefficient allocation with positive probability: The bidder who values the item the most does not necessarily get it, as different valuations may map to the same bid. Not only is this inefficiency somewhat undesirable from the point of view of economics; as we shall see below, it also makes the auction somewhat harder to analyze!

- *Breaking ties by a Vickrey auction:* All bidders submit an *auxiliary* bid in addition to their primary bid. In case of ties, a Vickrey (i.e., second price) auction is conducted among the bidders with the highest primary bid, using their auxiliary bids. Ties in the Vickrey auction can be broken arbitrarily. This tie-breaking rule was introduced by Maskin and Riley [7]. One advantage of this rule is that a monotone, symmetric PSNE obviously has an efficient allocation as outcome.

Our results

We summarize our results: For the random tie-breaking rule, we characterize the bivalued distributions for which symmetric PSNE exist for the case of two bidders. As the examples presented above suggest, the *gap* between the two possible valuations play an important role. To be precise, we show that for a distribution D with support $\{v_1, v_2\}$ with $v_1 < v_2$, v_1 occurring with probability $p > 0$ and v_2 occurring with probability $1 - p > 0$, the corresponding two-bidder auction with independent private values distributed according to D has a symmetric PSNE if and only if

$$v_2 - v_1 \leq g(p) = \left\lfloor \frac{1+p}{2p} \left\lfloor \frac{2}{1-p} \right\rfloor + 1 \right\rfloor .$$

For illustration, we plot the function $g(p)$ in Figure 1. The figure hints at the complexity and "messiness" of the auction with the random tie-breaking rule. Indeed, it seems quite tricky to extend the characterization to arbitrary support size and more bidders. One obstacle for improving our understanding is that when symmetric PSNE exist, they are not necessarily unique. For instance, for the case of the uniform distribution on $\{2, 6\}$, there are five symmetric PSNE, three of which are undominated, namely $(0, 2), (1, 2), (1, 3), (2, 3), (2, 4)$, where the first (resp., second) components denote the bids in case of valuation 2 (resp., 6).

For the case of Vickrey tie-breaking, the situation turns out to be *much* nicer. We show that for any finite discrete distribution on natural numbers and any finite number of bidders, the corresponding auction with independent private values has either zero or two symmetric PSNE. When two equilibria exist, exactly one of them is undominated, and this equilibrium may therefore arguably be considered a "canonical" solution to the auction. Further, we exhibit an efficient algorithm for computing (i.e., tabulating the maps from valuations to bids) the two equilibria when they do exist. The algorithm, which has a dynamic programming flavour, is given the density table of the distribution as input and runs in time linear in the support size of the distribution. We consider this result the main result of this paper.

We believe that this paper is merely scratching the surface of a rich constructive (and algorithmic) theory of first-price auctions with integral values and bids. In particular, as in classical auction theory, we should attempt to extend our understanding beyond the case of private, independent valuations. The most general setup for Bayesian single-item auctions is this: The Bayesian model is given as a set of states M of the world (in our discrete setup, this set should be finite). To each state i of a world and each bidder j is

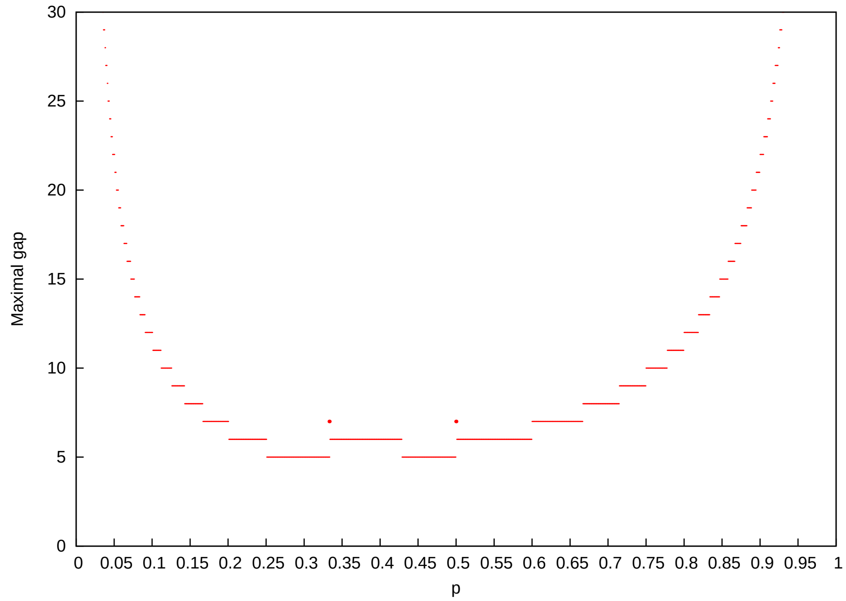

Figure 1: Maximal gap between valuations for which there is an equilibrium in the setting of two valuations, two bidders, and the random tie-breaking rule. Note that there are two isolated points, in $p = \frac{1}{3}$ and $p = \frac{1}{2}$.

associated a valuation v_{ij} and a *signal* s_{ij} that bidder j receives and must base his bid upon. Also, each state i of the world occurs with some probability p_i, the resulting distribution being the *common prior* of the bidders. The case of independent, identically distributed, private valuations fit into this framework as follows: We let the set of states of the world be $M = V^n$ where n is the number of bidders and V the set of valuations, the probability distribution on M is a product distribution, and $s_{ij} = v_{ij}$ for all i, j, i.e., in each state of the world, each bidder learns his valuation and nothing else. We prove that the general case is *not* algorithmically tractable (unless **P**=**NP**): Given a model, it is **NP**-hard to decide if the corresponding first-price auction has a PSNE. Even so, the Bayesian models occurring in the hardness proof are not very natural and we believe that it would be very interesting to establish hardness or easiness of more natural restrictions of the general setup, in particular those encountered in classical auction theory.

Organization of paper

In Section 2 we set up the notation we will use for the remainder of the paper. In Section 3 we present our results on auctions using the random tie-breaking rule. In Section 4 we present our results on auction using the Vickrey tie-breaking rule and in particular our linear time algorithm for constructing the equilibrium, when it exists. In Section 5 we present our hardness result for the general Bayesian setup. Due to space constraints, many proofs have been placed in the appendices.

2 Definitions and notation

Throughout the paper, we consider single-item, sealed-bid, first-price auctions with $n \geq 2$ bidders and shall refer to these as just "auctions" from now on. For most of the paper, we consider the case of independent, identically distributed private valuations. For this case, which we shall refer to as a "p.i.i.d." auction, all valuations are independent samples from a probability distribution on $V = \{v_1, v_2, \dots, v_k\}$ with $v_1 < v_2 < \dots < v_k$ being integers greater than or equal to 1, and with probability p_i of v_i occurring. Abusing notation slightly, we shall refer to the distribution as well as its support by V. Each bidder i receives his valuation v_i as a private signal and no further information. We assume that the bidder never bids more than his valuation.

As explained above, we consider two different tie-breaking rules. We first set up the case of random tie-breaking and afterwards explain how to modify it for Vickrey tie-breaking. For random tie-breaking, each bidder i sends a single bid b_i. If there is a tie among highest bids, the winner is chosen uniformly at random among the high bidders. The winner pays his bid. Bidders are risk-neutral. That is, if they do not get the item, they get a payoff of 0 and if they do get the item, they get a payoff of their valuation of the item minus the price they pay. A pure strategy for a bidder is given by a map $\beta : V \to \mathbb{N}$ so that the bidder submits bid $b_j = \beta(v_j)$. A pure strategy is said to be monotonic if for all $v_j \leq v_i$, $\beta(v_j) \leq \beta(v_i)$, i.e., $b_j \leq b_i$.

The expected payoff of a player is the sum of the expected payoffs given his valuation, his bid and the strategies of the other players. A strategy profile is a family of strategies, one for each bidder. A pure strategy Nash equilibrium or PSNE is a family of strategy so that the strategy of each bidder maximizes his expected payoff, assuming other players stick to their strategies. When considering the case of independent, identically distributed, private valuations, we shall in particular be interested in *symmetric* strategy profiles and symmetric PSNE, i.e., profiles where all bidders use the same strategy β. We shall abuse notation and use β to denote the symmetric profile where all bidders use strategy β. Assuming that all bidders except bidder 1 stick to a symmetric profile β, the expected payoff of a player given his valuation v_i and bid $b_i = \beta'(v_i)$ is

$$E_\Pi(v_i, b_i, \beta) = P_{\text{WIN}}(b_i, \beta)(v_i - b_i),$$

where P_{WIN} is the probability of player 1 to win given his bid and the strategy profile β of the other players. In the symmetric profile β, if bidder 1 considers deviating to strategy β', the expected payoff is

$$E_\Pi(\beta', \beta) = \sum_{i=1}^{k} p_i E_\Pi(v_i, \beta'(v_i), \beta) = \sum_{i=1}^{k} p_i E_\Pi(v_i, b'_i, \beta).$$

Maximizing this general payoff is equivalent to maximizing each term of the sum. The symmetric profile β is a symmetric PSNE when $E_\Pi(\beta', \beta) \leq E_\Pi(\beta, \beta)$ for all strategies β', i.e., when $E_\Pi(v_i, b'_i, \beta) \leq E_\Pi(v_i, b_i, \beta)$ for all i and all b'_i.

For the case of Vickrey tie-breaking, each player submits two bids. The first bid is the bid that will be paid if the player is the unique highest bid in the first-price auction. In case of ties, a second price auction will be run with the second bid. However, given the standard properties of second-price auction [3], in any symmetric PSNE of an auction with independent, identically distributed valuations and Vickrey tie-breaking, all bidders submit their valuation as their second bid, so our analysis only needs to concern itself with the first bid.

We denote by P_i the probability that the highest valuation among all players except one (e.g., Player 1) is v_i. We have that

$$P_i = \left(\sum_{j=1}^{i} p_j\right)^{n-1} - \left(\sum_{j=1}^{i-1} p_j\right)^{n-1}.$$

Letting

- $W_j = v_i - b$ when $b_j < b$,
- $W_j = v_i - v_j$ when $b_j = b$ and $j \leq i$,
- $W_j = 0$ when ($b_j = b$ and $j > i$) or $b_j > b$,

the expected payoff for the case of Vickrey tie-breaking can now be computed by the following formula:

$$E_\Pi(v_i, b, \beta) = \sum_{j=1}^{k} P_j W_j.$$

With this modification of $E_\Pi(v_i, b_i, \beta)$, it holds, again, that $E_\Pi(\beta', \beta) = \sum_{i=1}^{k} p_i E_\Pi(v_i, b_i, \beta)$ and that β is a symmetric PSNE occurs when $E_\Pi(\beta', \beta) \leq E_\Pi(\beta, \beta)$ for all strategies β'.

3 Random tie-breaking rule

Lemma 1. *In a p.i.i.d. auction with random tie breaking, any symmetric PSNE β is monotonic.*

Proof. Suppose to the contrary that β is a symmetric PSNE with $\beta(v_i) = b_i > \beta(v_j) = b_j$ for $v_i < v_j$. Define β' with $\beta'(v_l) = \beta(v_l)$ for $l \neq i, l \neq j$, $\beta'(v_i) = \beta(v_j)$ and $\beta'(v_j) = \beta(v_i)$. We claim that $E_\Pi(\beta', \beta) > E_\Pi(\beta, \beta)$ which contradicts the equilibrium property. It is enough to show that $E_\Pi(v_j, b_j, \beta) + E_\Pi(v_i, b_i, \beta) < E_\Pi(v_j, b_i, \beta) + E_\Pi(v_i, b_j, \beta)$. Let $Y_{ij} = b_i - b_j$, $d_i = v_i - b_i$, $X_{ji} = v_j - v_i$. Note that $b_j < b_i \leq v_i < v_j$. Therefore, all the defined variables are non-negative. We have $E_\Pi(v_j, b_j, \beta) + E_\Pi(v_i, b_i, \beta) = P_{\text{WIN}}(b_j, \beta)(Y_{ij} + d_i + X_{ji}) + P_{\text{WIN}}(b_i)(d_i)$, and $E_\Pi(v_j, b_i, \beta) + E_\Pi(v_i, b_j, \beta) = P_{\text{WIN}}(b_i, \beta)(d_i + X_{ji}) + P_{\text{WIN}}(b_j)(Y_{ij} + d_i)$. Since $P_{\text{WIN}}(b_j) < P_{\text{WIN}}(b_i)$, we have $E_\Pi(v_j, b_j, \beta) + E_\Pi(v_i, b_i, \beta) < E_\Pi(v_j, b_i, \beta) + E_\Pi(v_i, b_j, \beta)$, as desired. □

We consider the case of two bidders and two valuations. Let $g(p) = \left\lfloor \frac{1+p}{2p} \left\lfloor \frac{2}{1-p} \right\rfloor \right\rfloor + 1$.

Theorem 1. *Let $V = \{v_1, v_2\}$ be a bivalued distribution with $p_1 = p > 0$ and $p_2 = 1 - p > 0$. Then, the corresponding p.i.i.d. auction with two bidders has a symmetric PSNE if and only if $v_2 - v_1 \leq g(p)$.*

Proof. **"only if direction":** Suppose that the auction has a symmetric PSNE β and let $b_1 = \beta(v_1)$ and $b_2 = \beta(v_2)$ and we shall show that $v_2 - v_1 \leq g(p)$. First, suppose that $b_1 = b_2$. Since β is an equilibrium, we have that bidder 1 switching to bidding $b_2 + 1$ rather than b_2 when he has valuation v_2 does not improve his conditional expected payoff. His conditional expected payoff if he sticks to β is $(v_2 - b_2)/2$ while his conditional expected payoff if he switches is $v_2 - b_2 - 1$, so we must have $v_2 - b_2 - 1 \leq (v_2 - b_2)/2$, i.e., $v_2 - b_2 \leq 2$. Since $b_1 \leq v_1$ and $b_1 = b_2$, we have $v_2 - v_1 \leq 2$. Since $g(p)$ is greater than 2 for all values of p, we are done for the case of $b_2 = b_1$.

Now suppose that $b_2 \neq b_1$. By Lemma 1, we in fact have $b_2 > b_1$. Since β is an equilibrium, we have that bidder 1 switching to bidding $b_2 + 1$ rather than b_2 when he has valuation v_2 does not improve his conditional expected payoff. Since $b_2 > b_1$, his conditional expected payoff if he sticks to β is $(p + (1-p)/2)(v_2 - b_2) = (1+p)(v_2 - b_2)/2$, while his conditional expected payoff if he switches is $v_2 - b_2 - 1$. Therefore, we must have $v_2 - b_2 - 1 \leq (1+p)(v_2 - b_2)/2$ which is equivalent to $v_2 - b_2 \leq \frac{2}{1-p}$. Since $v_2 - b_2$ is an integer, it is in fact equivalent to $v_2 - b_2 \leq \left\lfloor \frac{2}{1-p} \right\rfloor$.

To complete the analysis, we consider two subcases. First, assume that $b_2 = b_1 + 1$. Since $b_1 \leq v_1$ in any equilibrium, we have $v_2 - v_1 \leq v_2 - b_1 = v_2 - b_2 + 1 \leq \left\lfloor \frac{2}{1-p} \right\rfloor + 1 \leq g(p)$, as desired. Finally, assume that $b_2 > b_1 + 1$. Since β is an equilibrium, we have that bidder 1 switching to bidding $b_1 + 1$ rather than b_2 does not improve his conditional expected payoff. His conditional expected payoff if he sticks to β is $(1+p)(v_2 - b_2)/2$, while his conditional expected payoff if he switches is $p(v_2 - b_1 - 1)$. Therefore, we must have $p(v_2 - b_1 - 1) \leq (1+p)(v_2 - b_2)/2$ which implies that $v_2 - b_1 \leq \frac{1+p}{2p}(v_2 - b_2) + 1 \leq \frac{1+p}{2p} \left\lfloor \frac{2}{1-p} \right\rfloor + 1$. Also, since $v_2 - b_1$ is an integer, we have $v_2 - b_1 \leq \left\lfloor \frac{1+p}{2p} \left\lfloor \frac{2}{1-p} \right\rfloor \right\rfloor + 1 = g(p)$. Finally, since $b_1 \leq v_1$, we have $v_2 - v_1 \leq v_2 - b_1 \leq g(p)$.

"if direction": We must show that if $v_2 - v_1 \leq g(p)$, then the auction has a symmetric PSNE. First suppose that $v_2 = v_1 + 1$. It is easily checked that β with $\beta(v_2) = \beta(v_1) = 1$ is a PSNE. Thus, we can assume that $v_2 \geq v_1 + 2$ from now on, and we claim that in that case β with $b_1 = \beta(v_1) = v_1$ and $b_2 = \beta(v_2) = \max(v_1 + 1, v_2 - \left\lfloor \frac{2}{1-p} \right\rfloor)$ is a symmetric PSNE. It is clear that it is not possible to improve the conditional expected payoff against β when one has valuation v_1 (null is best achievable payoff in that), so to verify that the stated β is an equilibrium, we only have to show that it is not possible to improve the conditional expected payoff when one has valuation v_2 by deviating from bidding b_2. It is enough to consider deviations to bidding one of $b_1, b_1 + 1, b_2 + 1$ instead of b_2. Indeed, bidding b_1 dominates all smaller bids, bidding b_2+1 dominates all larger bids, and bidding b_1+1 dominates all bids between b_1+1 and b_2-1. We shall show that none of these deviations can improve the expected payoff, and we shall be done. As shown above in the "only if" part of the proof, the statement that the deviation to bidding b_2+1 does not improve payoff, is equivalent to the inequality $v_2 - b_2 \leq \left\lfloor \frac{2}{1-p} \right\rfloor$. By construction of β, this inequality is satisfied. If $b_2 = b_1 + 1$, we only have to check that the deviation to bidding b_1 does not improve payoff. If we bid b_1, our conditional expected payoff will be $(p/2)(v_2 - b_1)$. If we bid b_2, our conditional expected payoff is $((1+p)/2)(v_2 - b_2)$. So, we have to check that $(p/2)(v_2 - b_1) \leq ((1+p)/2)(v_2 - b_2)$ which is equivalent to $v_2 - b_2 \geq p$, which is true, since $v_2 - b_2 \geq 2$ by construction. The last remaining case if the case when $b_2 > b_1 + 1 = v_1 + 1$, i.e., $b_2 = \max(v_1 + 1, v_2 - \left\lfloor \frac{2}{1-p} \right\rfloor) = v_2 - \left\lfloor \frac{2}{1-p} \right\rfloor$. We should check that deviations to bidding either b_1 and $b_1 + 1$ does not improve payoff. In fact, since we have that $v_2 \geq v_1 + 2 = b_1 + 2$, we have that $p(v_2 - (b_1 + 1)) \geq \frac{p}{2}(v_2 - b_1)$, so deviating to bidding $b_1 + 1$ dominates deviating to bidding b_1, so we only have to check deviations to bidding $b_1 + 1$. As shown above, the statement that the deviation to bidding $b_1 + 1$ does not improve payoff is equivalent to the inequality $v_2 - b_1 \leq \frac{1+p}{2p}(v_2 - b_2) + 1$. But since $b_2 = v_2 - \left\lfloor \frac{2}{1-p} \right\rfloor$, this inequality follows from $v_2 - v_1 \leq g(p)$ and we are done. ☐

It is tricky to extend the above result to more valuations and bidders and we leave open the existence of a polynomial time algorithm to determine if a symmetric PSNE exists. To illustrate the difficulty, we present some partial results for the case of *flat* distributions, i.e., uniform distributions on some set of integers, but leave even this case open.

Assume that in a symmetric PSNE β, the valuation v_i maps to b_i. Let t_i be the number of bids in β that are equal to b_i and let b_{j_i} be the bid corresponding to the highest valuation such that $b_{j_i} < b_i$. The expected value given v_i is

$$E_\Pi(v_i, b_i, \beta) = \frac{j_i}{k}(v_i - b_i) + \frac{t_i}{2k}(v_i - b_i) = \frac{2j_i + t_i}{2k}(v_i - b_i). \tag{1}$$

Lemma 2. *Given a set of valuations $\{v_1, \ldots, v_k\}$, in a p.i.i.d auction with a flat distribution, any symmetric PSNE β with $\beta(v_i) = b_i$ satisfies $v_i - b_i \leq \frac{2i}{t_i}$.*

Proof. Suppose $v_i - b_i > \frac{2i}{t_i}$ for some i. Since $v_i - b_i > \frac{2i}{t_i}$ implies that $v_{i+1} - b_i > \frac{2(i+1)}{t_i}$ when $t_i \geq 2$, we can assume that without loss of generality that i is the largest index for which $\beta(v_i) = b_i$. That is, $i = j_i + t_i$. Note that $E_\Pi(v_i, b_i + 1, \beta) \geq \frac{j_i + t_i}{k}(v_i - b_i - 1)$ with the inequality being strict when there are no bids on tie in $b_i + 1$. Since β is an equilibrium, we have: $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_i + 1, \beta)$ and combining these two inequalities we have $\frac{2j_i + t_i}{2k}(v_i - b_i) \geq \frac{j_i + t_i}{k}(v_i - b_i - 1)$, i.e., $v_i - b_i \leq \frac{2j_i + 2t_i}{t_i} = \frac{2i}{t_i}$. ☐

Lemma 3. *In a p.i.i.d. auction with a flat distribution, the distances $b_i - b_{i-1}$ between bids in a symmetric PSNE are at most 2, except possibly for $i = 2$, where the distance $b_2 - b_1$ is at most* 3.

Proof. Let $Y_{i,i-1} = b_i - b_{i-1}$. Fix i such that $Y_{i,i-1} \geq 2$. Thus, i is the smallest index on a sequence of valuations that map to the same bid. This implies that the index j_i of Equation 1, is $j_i = i - 1$. Let t_i be the number of such valuations mapping to b_i. The expected payoff of v_i in $b_{i-1} + 1$ is

$$E_\Pi(v_i, b_{i-1} + 1, \beta) = \frac{i-1}{k}(v_i - b_{i-1} - 1).$$

Since β is an equilibrium, we know that $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_{i-1} + 1, \beta)$. That is,

$$\left(\frac{i-1}{k} + \frac{t_i}{2k} \right)(v_i - b_i) \geq \frac{i-1}{k}(v_i - b_{i-1} - 1.)$$

Rearranging, we have $Y_{i,i-1} \leq \frac{(i-1) + \frac{t_i}{2}(v_i - b_i)}{i-1} \leq \frac{(i-1) + \frac{t_i}{2}\frac{2i}{t_i}}{i-1} = 1 + \frac{i}{i-1}$. Since $Y_{i,i-1}$ is integer and $\frac{i}{i-1} < 2$ for $i \geq 3$, then $Y_{i,i-1} \leq 2$ except for $i \leq 2$ for which $\frac{i}{i-1} = 2$, so $Y_{2,1} \leq 3$. ☐

Lemma 2 and Lemma 3 restricts the possible bids in a symmetric PSNE significantly, but we know of no algorithm for determining whether a symmetric PSNE exists much better than doing a search with backtracking though all strategies consistent with these lemmas. Nevertheless, the heuristic presented as Algorithm 1 attempts to build an equilibrium from below and empirically often produces an equilibrium when one exists on small inputs (but may fail in producing one even when one exists and may produce a strategy profile which is not an equilibrium). Our (correct) algorithm for the case of Vickrey tie-breaking is based on a similar idea.

Algorithm 1 Heuristic for producing an equilibrium for flat distributions, two bidders, and random tie-breaking

```
start = 2
b_1 ← v_1
if v_2 − v_1 = 7 then
    b_2 ← v_1 + 3
    start = 3
end if
for i = start to k do
    t ← number of bids that are equal to b_{i−1}
    if v_i − b_{i−1} > 2i + 2 then
        return Fail
    else
        if v_i − b_{i−1} ≥ 2i then
            b_i = b_{i−1} + 2
        else
            if v_i − b_{i−1} ≥ ((2i−1)/(t+1)) then
                b_i = b_{i−1} + 1
            else
                b_i = b_{i−1}
            end if
        end if
    end if
end for
```

We can prove that the heuristic is correct in a very special case.

Theorem 2. *For any flat distribution of valuations, if we have $v_{i+1} - v_i \geq 4$ for every i, then Algorithm 1 outputs Fail if there is no symmetric PSNE for the p.i.i.d auction and otherwise outputs such a symmetric PSNE.*

4 Vickrey tie-breaking rule

The following theorem is the analogue of Lemma 1, but trickier to prove.

Theorem 3. *A symmetric PSNE for a p.i.i.d auction with Vickrey tie-breaking is monotonic.*

Lemma 4. *In a symmetric PSNE for a p.i.i.d auction with Vickrey tie-breaking, we have $b_{i+1} = b_i$ or $b_{i+1} = b_i + 1$ for every $i \leq k - 1$.*

Proof. We assume to the contrary that we have a symmetric PSNE β with $b_{i+1} \geq b_i + 2$ for a given $i \leq k-1$. Since we have monotonicity by Theorem 3, we have:

$$E_\Pi(v_{i+1}, b_i + 1, \beta) = \sum_{j=1}^{i} P_j(v_{i+1} - (b_i + 1)),$$

$$E_\Pi(v_{i+1}, b_{i+1}, \beta) = \sum_{j=1}^{i} P_j(v_{i+1} - b_{i+1}).$$

Since $b_{i+1} > b_i + 1$, we have $E_\Pi(v_{i+1}, b_i + 1, \beta) > E_\Pi(v_{i+1}, b_{i+1}, \beta)$. In a PSNE, we would have $E_\Pi(v_{i+1}, b_i + 1, \beta) \leq E_\Pi(v_{i+1}, b_{i+1}, \beta)$. So β is not a PSNE. So in a PSNE, we have $b_{i+1} \leq b_i + 1$ for every $i \geq k - 1$. Because of Theorem 3, we cannot have $b_{i+1} < b_i$. This ends the proof of the Lemma 4. □

Let β_i denote the restriction of β to the i lowest valuations, with $i > 0$.

Lemma 5. *For a given p.i.i.d auction with Vickrey tie-breaking, if we have two symmetric PSNE β and β', then*

$$\beta_i = \beta'_i \Rightarrow \beta(v_{i+1}) = \beta'(v_{i+1}).$$

This lemma is crucial for constructing our algorithm for the case of the Vickrey tie-breaking rule as it enables an equilibrium to be constructed by dynamic programming. We can see that the corresponding statement doesn't hold for the random tie-breaking auction: Valuations (2,7) with equal probability, have equilibria with range $B = (2, 3)$ and $B = (2, 4)$. If we add valuation $v_3 = 9$, the only surviving equilibrium has range $B = (2, 3, 5)$. On the other hand, if $v_3 = 12$, the only possible equilibrium has range $B = (2, 4, 6)$.

Theorem 4. *A p.i.i.d auction with Vickrey tie-breaking either has no symmetric PSNE or exactly two which differ only by their bid for the lowest valuation.*

We are now ready to present our main algorithm. We have a set of valuations $V = \{v_1, v_2 \ldots, v_k\}$ and its associated probability distribution $P = \{p_1, p_2, \ldots, p_k\}$. We want to find a symmetric PSNE β for V and P. From now on, we denote $b_j = \beta(v_j)$ for $1 \leq j \leq k$. When we have a set of valuations V, a probability distribution P and a strategy β, we define an array A with for every $1 \leq i \leq k$:

$$A(i) = (A_1(i), A_2(i), A_3(i))$$

with

$$A_1(i) = \sum_{j \backslash b_j < b_i} P_j,$$

$$A_2(i) = \sum_{j \backslash b_j = b_i, j \leq i} (v_i - v_j) P_j,$$

$$A_3(i) = \sum_{j=1}^{i} P_j.$$

Suppose we have a strategy β_i defined on the i first valuations. Then we have $E_\Pi(v_i, b_i, \beta_i) = \sum_{j \backslash b_j < b_i} (P_j(v_i - b_i)) + \sum_{j \backslash b_j = b_i, j \leq i} (P_j(v_j - v_i)) = A_1(i)(v_i - b_i) + A_2(i)$ and $E_\Pi(v_i, b_i + 1, \beta_i) = \sum_{j=1}^{i} (P_j(v_i - (b_i + 1))) = A_3(i)(v_i - (b_i + 1))$. So, if we know β_i and $[A(1), A(2), \ldots, A(i)]$, we can compute in constant time the values of $E_\Pi(v_i, b_i, \beta_i)$ and $E_\Pi(v_i, b_i + 1, \beta_i)$. This makes Algorithm 2 well-defined.

Proposition 1. *Algorithm 2 outputs a symmetric PSNE for a given p.i.i.d auction with Vickrey tie-breaking if one exists.*

It is easy to see that Algorithm 2 is linear-time: All computations from line 1 to line 8 can be done in a constant time. From line 9 to the end, the algorithm is only composed of one loop, which begin at 3 to stop at k (line 9). So the number of iterations is $O(k)$. In one iteration, we have assignments to values in A, which can be done in a constant time, and computations of two payoffs (line 14). We have seen previously that these payoffs can be computed in a constant time, assuming we know the i first values of A, which is the case at this point. So every iteration can be done in $O(1)$.

Note that the algorithm produces a strategy profile even if there is no equilibrium. Unfortunately, testing if the output is an equilibrium is slightly more expensive than producing it!

Proposition 2. *A given strategy β can be verified to be a symmetric PSNE in a p.i.i.d auction with Vickrey tie-breaking in quadratic time.*

Algorithm 2 Algorithm for producing equilibrium for Vickrey tie-breaking auction

$b_1 \leftarrow v_1 - 1$
$A_1(1) \leftarrow 0$
$A_2(1) \leftarrow 0$
$A_3(1) \leftarrow P_1$
$b_2 \leftarrow v_1$
$A_1(2) \leftarrow P_1$
$A_2(2) \leftarrow 0$
$A_3(2) \leftarrow A_3(1) + P_2$
for $i = 3$ to k **do**
    $b_i \leftarrow b_{i-1}$
    $A_1(i) \leftarrow A_1(i-1)$
    $A_2(i) \leftarrow A_2(i-1) + (A_3(i-1) - A_1(i-1))(v_i - v_{i-1})$
    $A_3(i) \leftarrow A_3(i-1) + P_i$
    if $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_i + 1, \beta)$ **then**
        $b_i \leftarrow b_{i-1} + 1$
        $A_1(i) \leftarrow A_3(i-1)$
        $A_2(i) \leftarrow 0$
    end if
end for

5 NP-hardness of the general Bayesian auction

We present our hardness result. We note that similar hardness results for the existence of PSNE in incomplete information games where shown by Blair *et al.* [2] and Conitzer and Sandholm [5] using similar techniques, but these earlier results do not apply directly to first-price auctions. The result is a "proof-of-concept" as the auctions used to prove hardness are not very natural. It would be very interesting to show hardness for more natural subclasses of the general Bayesian setup.

Theorem 5. *Given an auction with random tie-breaking rule, it is* **NP***-hard to decide if it has a PSNE.*

Proof. We make a reduction from SUBSET SUM, i.e., the problem of deciding, given a collection $z_1, z_2, \ldots, z_n$ of positive integers, if they can be divided in two groups so that the integers in each group add up to precisely half the total sum. Given an instance of SUBSET SUM, we construct a two-bidder Bayesian auction which has a PSNE if and only if the instance is a positive instance.

For the proof, we need the following "gadget": a two-bidder p.i.i.d auction without a pure equilibrium and with "well-controlled mixed ones". Let D be the distribution on 1,8 with $p_D(1) = 4/7$ and $p_D(8) = 3/7$. The p.i.i.d auction A with two bidders, valuations distributed according to D and random tie-breaking has the following mixed equilibrium E: In case of valuation 1, the bid is 0. In case of valuation 8, the bid is 1 with probability 34/111, 2 with probability 21/111, and 3 with probability 56/111. Furthermore, this is the only equilbrium that has deterministic behavior on valuation 1 and puts more than half the probability mass on one particular bid when the valuation is 8.

Now, given an instance $z = (z_1, z_2, \ldots, z_n)$ of SUBSET SUM, we consider the following Bayesian auction A_z. The states of the world are given by

$$M = \{(v_1, \sigma_1, v_2, \sigma_2) | v_1, v_2 \in \{1, 8\}, \sigma_1 \in \{0, 1, 2, 3\}, \sigma_2 \in \{a, b, c, -1, -2, \ldots, -n\}\}$$

with the following probability distribution:

$$p(v_1, \sigma_1, v_2, \sigma_2) = p_D(v_1) g(v_1, \sigma_1) p_D(v_2) h(v_2, \sigma_2)$$

where $g(1, 0) = 1$ and $g(1, s) = 0$ for all other values of s, $g(8, 0) = 0, g(8, 1) = 34/111, g(8, 2) = 21/111,$ $g(8, 3) = 56/111$, while $h(1, 0) = 1$, $h(1, s) = 0$ for $s \neq 0$, Let $\epsilon = 56/111 - 1/2$. $h(8, a) = 56/111 - \epsilon = 1/2,$

$h(8,b) = 34/111$, $h(8,c) = 21/111 - \epsilon$. $h(8,-i) = (2\epsilon)z_i(\sum_j z_j)^{-1}$ for $i \in \{1,\ldots,n\}$and $h(8,s) = 0$ for other values of s. In state $(v_1, \sigma_1, v_2, \sigma_2)$ of the world, bidder 1 has the valuation v_1 and gets the signal σ_1 while bidder 2 has the valuation v_2 and gets the signal σ_2.

We claim that A_z has a PSNE if and only if z is a positive instance of SUBSET SUM. For one direction, suppose that z is a positive instance of SUBSET SUM. and let I_1, I_2 be a partition of the indices $\{1,2,\ldots,n\}$ so that $\sum_{i \in I_1} z_i = \sum_{i \in I_2} z_i$. We claim that the strategy profile where bidder 1 bids σ_1 (his signal) and bidder 2 bids σ_2 if $\sigma_2 \in \{1,3\}$, bids 2 if $-\sigma_2 \in I_1$ and bids 3 if $-\sigma_2 \in I_2$ is a PSNE. Indeed, this follows directly from the fact that E is a mixed equilibrium of the auction A.

For the other direction, suppose that A_z has a PSNE E'. Let $\beta_1 : \{0,1,2,3\} \to \mathbb{N}$ be the bidding function of bidder 1 and let $\beta_2 : \{0,1,3,-1,\ldots,-n\} \to \mathbb{N}$ be the bidding function of bidder 2. Since the two signals received by the bidders are independent and each determine the valuation of the bidder, E' induces a mixed equilibrium E'' in the auction A when the signals are interpreted as a source of randomness. As bidder 2 receives the signal a with probability $1/2$, this mixed equilibrium must be E. But this is only possible if bidder 2 bids 3 on signal a, 1 on signal b, 2 on signal c, 3 on some subset I_1 of indices $\{-1,-2,\ldots,-n\}$, so that $\sum_{-i \in I_1}(2\epsilon)z_i(\sum_{j\in\{1,\ldots,n\}} z_j)^{-1} = \epsilon$ and 2 on the remaining indices. That is, the SUBSET SUM instance must be positive. □

6 Conclusion

The present paper barely scratches the surface of the topic of single-item first-price auctions with discrete valuations and bids. Open problems include:

- Maskin and Riley [8] has shown that under certain smoothness conditions, p.i.i.d. auctions with continuously distributed valuations do not have asymmetric equilibria. Are there p.i.i.d auctions with integral valuations and integral bids which have asymmetric PSNE under the random tie-breaking rule? We have no examples of such auctions. In contrast, it is easy to find examples of asymmetric equilibria in p.i.i.d auctions with Vickrey tie-breaking by "plugging in" asymmetric equilibria [3] of the auxiliary Vickrey auction.

- Given a distribution on valuations, can it be decided in polynomial time whether the corresponding p.i.i.d. auction under the random tie-breaking rule has PSNE or is this problem **NP**-hard? We show in this paper that the case of Vickrey tie-breaking is polynomial time decidable.

- We have shown that for the case of Vickrey tie-breaking, we can in linear time construct the two possible candidates for PSNE. To test that the output is an equilibrium we unfortunately need quadratic time. Is there an alternative way of checking if a given p.i.i.d auction with Vickrey tie-breaking has an equilibrium which only needs linear time in the number of valuations?

- Given a *non-product* distribution on valuation profiles, can it be decided in polynomial time whether the corresponding auction with private values under Vickrey tie-breaking rule has PSNE?

- Can the **NP**-hardness result of this paper be extended to auctions with more natural signals?

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A Proof of Theorem 2

Lemma 6. *When the Algorithm 1 returns a strategy β where $b_{i+1} = b_i + 2$ for every i, then β is a pure-strategy Nash equilibrium.*

Proof. To prove it, we have to show $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and b.

For a given i, we have

$$E_\Pi(v_i, b_i, \beta) = \frac{2i-1}{2k}(v_i - b_i).$$

We can notice b is a dominated bid if $b < b_1$ (because in this case we have $E_\Pi(v_i, b, \beta) = 0$ for any i), or if $b > b_k + 1$ (because in this case we have $E_\Pi(v_i, b, \beta) < E_\Pi(v_i, b_k + 1, \beta)$ for any i).

If b is an undominated bid, for any given i, we have $b = b_i + q$ (q can be negative) and:

- if q is even:

  $b = b_{i+\frac{q}{2}}$, so $P_{\text{WIN}}(b) = P_{\text{WIN}}(b_{i+\frac{q}{2}}) = \frac{2i+q-1}{2k} = \frac{2i-1+q}{2k}$
- if q is odd:

  $b = b_{i+\frac{q-1}{2}} + 1$, so $P_{\text{WIN}}(b) = P_{\text{WIN}}(b_{i+\frac{q-1}{2}} + 1) = \frac{2i+q-1-1+1}{2k} = \frac{2i-1+q}{2k}$

For a given integer $q > 0$ such that $b_i + q$ is an undominated bid, we have

$$E_\Pi(v_i, b_i + q, \beta) = \frac{2i-1+q}{2k}(v_i - b_i - q).$$

So

$$\begin{aligned}
E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_i + q, \beta) &\Leftrightarrow \frac{2i-1}{2k}(v_i - b_i) \geq \frac{2i-1+q}{2k}(v_i - b_i - q) \\
&\Leftrightarrow (2i-1)(v_i - b_i) \geq (2i-1+q)(v_i - b_i - q) \\
&\Leftrightarrow (2i-1)(v_i - b_i) \geq (2i-1)(v_i - b_i) + q(v_i - b_i) - q(2i-1) - q^2 \\
&\Leftrightarrow 0 \geq (v_i - b_i) - (2i-1) - q \\
&\Leftrightarrow v_i - b_i \leq 2i - 1 + q.
\end{aligned}$$

The algorithm sets b_i at $b_{i-1} + 2$. It implies that $v_i - b_i \leq 2i$. Since $q \geq 1$, we have $v_i - b_i \leq 2i - 1 + q$. So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and any $b > b_i$.

For a given integer $q > 0$ such that $b_i + q$ is an undominated bid, we have

$$E_\Pi(v_i, b_i - q, \beta) = \frac{2i-1-q}{2k}(v_i - b_i + q).$$

So

$$\begin{aligned}
E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_i - q, \beta) &\Leftrightarrow \frac{2i-1}{2k}(v_i - b_i) \geq \frac{2i-1-q}{2k}(v_i - b_i + q) \\
&\Leftrightarrow (2i-1)(v_i - b_i) \geq (2i-1-q)(v_i - b_i + q) \\
&\Leftrightarrow (2i-1)(v_i - b_i) \geq (2i-1)(v_i - b_i) - q(v_i - b_i) + q(2i-1) - q^2 \\
&\Leftrightarrow 0 \geq -(v_i - b_i) + (2i-1) - q \\
&\Leftrightarrow v_i - b_i \geq 2i - 1 - q.
\end{aligned}$$

The algorithm sets b_i at $b_{i-1}+2$. It implies we have $v_i - b_i \geq 2i-2$. Since $q \geq 1$, we have $v_i - b_i \geq 2i-1-q$. So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and any $b < b_i$.

So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and any b. Thus, β is a pure-strategy Nash equilibrium. □

Lemma 7. *When the Algorithm 1 returns a strategy β where $b_{i+1} = b_i + 2$ for every $i \geq 2$ and $b_2 = b_1 + 3$, then β is a pure-strategy Nash equilibrium.*

Proof. To prove it, we have to show $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and b. But actually, we only need to show $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1, \beta)$ and $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1+1, \beta)$ for any i, since all other inequalities are shown in the proof of Lemma 6.

For a given i, we have:

$$E_\Pi(v_i, b_i, \beta) = \frac{2i-1}{2k}(v_i - b_i).$$

So $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1, \beta)$. So $\frac{2i-1}{2k}(v_i - b_i) \geq \frac{1}{2k}(v_i - b_1)$. So $(2i-1)(v_i - b_i) \geq v_i - b_1$. So $(2i-1)(v_i - b_i) \geq (v_i - b_i) + (b_i - b_1)$. So $(2i-2)(v_i - b_i) \geq b_i - b_1$. So $(2i-2)(v_i - b_i) \geq 2i - 1$.

For $i \geq 3$ the algorithm sets b_i at $b_{i-1} + 2$. It implies we have $v_i - b_i \geq 2i - 2$. Moreover, since $i \geq 3$, we have $(2i-2)(2i-2) \geq 2i-1$. So $(2i-2)(v_i - b_i) \geq 2i-1$ for $i \geq 3$. So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1, \beta)$ for any $i \geq 3$. For $i = 2$, we have $v_i - b_i = 4$ from the algorithm, which implies the inequality.

We also have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1 + 1, \beta)$. So $\frac{2i-1}{2k}(v_i - b_i) \geq \frac{2}{2k}(v_i - b_1 - 1)$. So $(2i-1)(v_i - b_i) \geq 2(v_i - b_1 - 1)$. So $(2i-1)(v_i - b_i) \geq 2(v_i - b_i) + 2(b_i - b_1 - 1)$. So $(2i-3)(v_i - b_i) \geq 2(b_i - b_1 - 1)$. So $(2i-3)(v_i - b_i) \geq 2(2i-2)$.

For $i \geq 3$ the algorithm sets b_i at $b_{i-1}+2$. It implies we have $v_i - b_i \geq 2i-2$. Moreover, since $i \geq 3$, we have $(2i-3)(2i-2) \geq 2(2i-2)$. So $(2i-3)(v_i - b_i) \geq 2(2i-2)$ for $i \geq 3$. So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_1+1, \beta)$ for any $i \geq 3$. For $i = 2$, we have $v_i - b_i = 4$ from the algorithm, which implies the inequality.

So we have $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for any i and any b. So β is a pure-strategy Nash equilibrium. □

We are now ready to prove Theorem 2. We first assume the Algorithm 1 sets $b_{i+1} < b_i + 2$ during its execution. We look at the first index i it does it. So $v_{i+1} - b_i = (v_{i+1} - v_1) - (b_i - v_1)$. But we have $b_2 - v_1 \leq 3$, $b_{i+1} - b_i \leq 1$, $b_j - b_{j-1} \leq 2$ and $v_{j+1} - v_j \geq 4$ for every $j \leq i$. So $(v_{i+1} - v_1) - (b_i - v_1) \geq 4i - (2i-2) = 2(i+1)$. So we have $v_{i+1} - b_i \geq 2(i+1)$. In this case, the algorithm sets $b_{i+1} = b_i + 2$. So, by contraposition, we have the Algorithm 1 never sets $b_{i+1} = b_i$ or $b_{i+1} = b_i + 1$. So, we have two cases:

1. The algorithm returns a strategy β: By the result we just proved, we know $b_{i+1} - b_i \geq 2$ for every i. By lemmas 6 and 7, we know this strategy is an equilibrium.

2. The algorithm returns "Fail": There exists $2 \leq i \leq k-1$ such that the algorithm assigns a value to b_i but returns "Fail" before assigning a value to b_{i+1}. So, we have $v_{i+1} - b_i > 2i + 4$ (line 9). From the Lemma 2, if there exists an equilibrium β', then we have $v_{i+1} - b'_{i+1} \leq 2i + 2$. From Lemma 3, we have $v_{i+1} - b'_i \leq 2i + 4$. This means that $b'_i > b_i$. But from Lemma 3, we have $b'_i - v_1 = (\sum_{j=3}^{i}(b'_i - b'_{i-1})) + (b'_2 - v_1) \leq (i-2) * 2 + 3 = 2i - 1$. So, $b_i - v_1 < 2i - 1$. We now distinguish two cases:

   - $b_2 - b_1 = 3$: Then $b_i - v_1 = (\sum_{j=3}^{i}(b_i - b_{i-1})) + (b_2 - v_1) = (i-2)*2+3 = 2i-1$. So $b'_i - v_1 > 2i-1$, and so from the Lemma 3, we have β' is not an equilibrium. So by contraposition, there is no equilibrium and the algorithm has given the correct output.

   - $b_2 - b_1 = 2$: Then, $b_i - v_1 = (\sum_{j=3}^{i}(b_i - b_{i-1})) + (b_2 - v_1) = (i-2) * 2 + 2 = 2i - 2$. So $b'_i = v_1 + 2i - 1$. From Lemma 3, we have also $b'_j = v_1 + 2j - 1$ for every $2 \leq j \leq i$. So $b'_2 = v_1 + 3 = b'_1 + 3$. But since the algorithm sets b_2 at $b_1 + 2$, we have $v_2 - v_1 \leq 6$. So $E_\Pi(v_2, v_1 + 1, \beta') = \frac{2}{2k}(v_2 - (v_1 + 1)) > \frac{3}{2k}(v_2 - (v_1 + 3)) = E_\Pi(v_2, b'_2, \beta')$. So β' is not an equilibrium. So, by contraposition, there is no equilibrium. So, the algorithm has given the correct output.

B Proof of Theorem 3

We assume we have a symmetric PSNE β which is non-monotonic. So there is an $i \leq k-1$ such that $\beta(v_{i+1}) < \beta(v_i)$. From now on, we denote $b_j = \beta(v_j)$ for $1 \leq j \leq k$. We also denote:

- $Q_1(b) = \sum_{j \backslash b_j < b} P_j$
- $Q_2(b, v_{j'}) = \sum_{j \backslash b_j = b, j \leq j'} ((v - v_j)P_j)$
- $Q_3(b, j') = \sum_{j \backslash b_j = b, j \leq j'} P_j$

We have

- $E_\Pi(v_i, b_i, \beta) = Q_1(b_i)(v_i - b_i) + Q_2(b_i, v_i)$
- $E_\Pi(v_i, b_{i+1}, \beta) = Q_1(b_{i+1})(v_i - b_{i+1}) + Q_2(b_{i+1}, v_i)$
- $E_\Pi(v_{i+1}, b_i, \beta) = Q_1(b_i)(v_{i+1} - b_i) + Q_2(b_i, v_{i+1})$
- $E_\Pi(v_{i+1}, b_{i+1}, \beta) = Q_1(b_{i+1})(v_{i+1} - b_{i+1}) + Q_2(b_{i+1}, v_{i+1})$

Since β is a PSNE, we have

- $E_\Pi(v_i, b_{i+1}, \beta) \leq E_\Pi(v_i, b_i, \beta)$
- $E_\Pi(v_{i+1}, b_{i+1}, \beta) \geq E_\Pi(v_{i+1}, b_i, \beta)$

$$\Downarrow$$

- $E_\Pi(v_i, b_{i+1}, \beta) - E_\Pi(v_i, b_i, \beta) \leq 0$
- $E_\Pi(v_{i+1}, b_{i+1}, \beta) - E_\Pi(v_{i+1}, b_i, \beta) \geq 0$

So,

$$E_\Pi(v_i, b_{i+1}, \beta) - E_\Pi(v_i, b_i, \beta) \leq E_\Pi(v_{i+1}, b_{i+1}, \beta) - E_\Pi(v_{i+1}, b_i, \beta)$$

$$\Downarrow$$

$$\begin{gathered} Q_1(b_{i+1})(v_i - b_{i+1}) + Q_2(b_{i+1}, v_i) - (Q_1(b_i)(v_i - b_i) + Q_2(b_i, v_i)) \\ \leq Q_1(b_{i+1})(v_{i+1} - b_{i+1}) + Q_2(b_{i+1}, v_{i+1}) - (Q_1(b_i)(v_{i+1} - b_i) + Q_2(b_i, v_{i+1})) \end{gathered}$$

$$\Downarrow$$

$$\begin{gathered} 0 + Q_2(b_{i+1}, v_i) - 0 - Q_2(b_i, v_i) \\ \leq Q_1(b_{i+1})(v_{i+1} - v_i) + Q_2(b_{i+1}, v_{i+1}) - Q_1(b_i)(v_{i+1} - v_i) - Q_2(b_i, v_{i+1}) \end{gathered}$$

$$\Downarrow$$

$$0 \leq (Q_1(b_{i+1}) - Q_1(b_i))(v_{i+1} - v_i) + Q_2(b_{i+1}, v_{i+1}) - Q_2(b_{i+1}, v_i) + Q_2(b_i, v_i) - Q_2(b_i, v_{i+1}).$$

We have

$$\begin{aligned} Q_2(b_{i+1}, v_{i+1}) - Q_2(b_{i+1}, v_i) &= \sum_{j \backslash b_j = b_{i+1}} ((v_{i+1} - v_j)P_j) - \sum_{j \backslash b_j = b_{i+1}} ((v_i - v_j)P_j) \\ &= \sum_{j \backslash b_j = b_{i+1}} (v_{i+1} - v_i)P_j \\ &= Q_3(b_{i+1}, i)(v_{i+1} - v_i) \end{aligned}$$

So,

$$(Q_1(b_{i+1}) - Q_1(b_i))(v_{i+1} - v_i) + Q_3(b_{i+1}, i)(v_{i+1} - v_i) + Q_2(b_i, v_i) - Q_2(b_i, v_{i+1}) \geq 0.$$

This implies:

$$(Q_1(b_{i+1}) - Q_1(b_i) + Q_3(b_{i+1}, i))(v_{i+1} - v_i) + Q_2(b_i, v_i) - Q_2(b_i, v_{i+1}) \geq 0.$$

We have $Q_1(b_i) = \sum_{b_j<b_i} Q_j = Q_1(b_{i+1}) + Q_3(b_{i+1}, i) + \sum_{b_j=b_{i+1},j>i} Q_j + \sum_{b_{i+1}<b_j<b_i} Q_j$. So, $Q_1(b_i) \geq Q_1(b_{i+1}) + Q_3(b_{i+1}, i)$ and so $Q_1(b_{i+1}) - Q_1(b_i) + Q_3(b_{i+1}, i) \leq 0$. This implies $Q_2(b_i, v_i) - Q_2(b_i, v_{i+1}) \geq 0$. But we have $Q_2(b_i, v_{i+1}) = Q_2(b_i, v_i) + Q_3(b_i, i)(v_{i+1} - v_i)$ with $Q_3(b_i, i) > 0$. So $Q_2(b_i, v_i) - Q_2(b_i, v_{i+1}) < 0$. So, $0 > 0$ which is false. That is, the hypothesis of non-monotonicity is false. So every symmetric PSNE is monotonic.

C Proof of Theorem 4

Lemma 5. *For any p.i.i.d auction with Vickrey tie-breaking, if we have two symmetric PSNE β and β', then*

$$\beta_i = \beta'_i \Rightarrow \beta(v_{i+1}) = \beta'(v_{i+1}).$$

Proof. We suppose we have two symmetric PSNE β and β' with $\beta_i = \beta'_i$ and $\beta(v_{i+1}) \neq \beta'(v_{i+1})$. From now on, we denote b_j the value of $\beta(v_j)$ and b'_j the value of $\beta'(v_j)$. Since $b_{i+1} \neq b'_{i+1}$ and by Lemma 4, we have one of the two values equal to $b_i = b'_i$, and the other equal to $b_i + 1 = b'_i + 1$. We can assume w.l.o.g. $b_{i+1} = b_i$ and $b'_{i+1} = b'_i + 1$. We denote $Q_1 = \sum_{j \backslash b_j < b_i} p_j$ and $Q_2 = \sum_{j \backslash b_j = b_i, j<i} (p_j(v_{i+1} - v_j))$. We have:

$$E_\Pi(v_{i+1}, b_i, \beta) = Q_1(v_{i+1} - b_i) + Q_2 + P_i(v_{i+1} - v_i),$$

$$E_\Pi(v_{i+1}, b_i + 1, \beta) \geq (\sum_{j=1}^{i+1} P_j)(v_{i+1} - b_{i+1}).$$

The inequality is strict when we have $b_{i+2} = b_{i+1} = b_i$.

$$E_\Pi(v_{i+1}, b_i, \beta') = Q_1(v_{i+1} - b_i) + Q_2 + P_i(v_{i+1} - v_i),$$

$$E_\Pi(v_{i+1}, b_i + 1, \beta') = (\sum_{j=1}^{i} P_j)(v_{i+1} - b_{i+1}).$$

Since β is an equilibrium, we have:

$$E_\Pi(v_{i+1}, b_i, \beta) \geq E_\Pi(v_{i+1}, b_i + 1, \beta)$$

and

$$E_\Pi(v_{i+1}, b_i + 1, \beta') \geq E_\Pi(v_{i+1}, b_i, \beta').$$

We can also notice

$$\begin{aligned} E_\Pi(v_{i+1}, b_i, \beta) = E_\Pi(v_{i+1}, b_i, \beta') &\Leftrightarrow E_\Pi(v_{i+1}, b_i + 1, \beta') \geq E_\Pi(v_{i+1}, b_i + 1, \beta) \\ &\Leftrightarrow (\sum_{j=1}^{i} P_j)(v_{i+1} - b_{i+1}) \geq (\sum_{j=1}^{i+1} P_j)(v_{i+1} - b_{i+1}) \\ &\Leftrightarrow 0 \geq P_{i+1}(v_{i+1} - b_{i+1}) \end{aligned}$$

which is not possible, since probabilities are strictly positive. This finishes the proof of Lemma 5. □

Corollary 1. *For a given set of valuations V and its associated probability distribution P, if we have two symmetric PSNE β and β', then*

$$\beta(v_1) = \beta'(v_1) \Rightarrow \beta = \beta'.$$

Proof. By induction and with the Lemma 5, we have the proof of Corollary 1. □

Lemma 8. *For any set of valuations V and its associated probability distribution P, if there is a symmetric PSNE β, then $\beta(v_1) = v_1 - 1$ or $\beta(v_1) = v_1$.*

Proof. We suppose we have a symmetric PSNE β with $\beta(v_1) \leq v_1 - 2$. From now on, we denote $b_j = \beta(v_j)$ for $1 \leq j \leq k$.

We have

$$E_\Pi(v_1, b_1, \beta) = \sum_{j=1}^{k} P_j W_j$$

with

1. $W_j = v_1 - b_1$ when $b_j < b_1$,
2. $W_j = v_1 - v_j$ when $b_j = b_1$ and $j \leq 1$,
3. $W_j = 0$ when ($b_j = b_1$ and $j > 1$) or $b_j > b_1$.

Because of Theorem 3, b_1 is the smallest bid. So, the first case never occurs. When the second case occurs, we have $j = 1$. So $W_j = W_1 = v_1 - v_1 = 0$. So, we have $W_j = 0$ for $1 \leq j \leq k$. So, $E_\Pi(v_1, b_1, \beta) = 0$. We have also $E_\Pi(v_1, v_1 - 1, \beta) = \sum_{j=1}^{k} P_j W'_j$ with

1. $W'_j = v_1 - (v_1 - 1)$ when $b_j < v_1 - 1$
2. $W'_j = v_1 - v_j$ when $b_j = v_1 - 1$ and $j \leq 1$
3. $W'_j = 0$ when ($b_j = v_1 - 1$ and $j > 1$) or $b_j > v_1 - 1$

We have $b_1 < v_1 - 1$. So $W'_1 = v_1 - (v_1 - 1) = 1 > 0$. When the second case occurs, we have $j = 1$, so $W'_j = v_1 - v_1 = 0$. So, all the W'_j are positive and $W'_1 > 0$. So $\sum_{j=1}^{k} P_j W'_j > 0$. So

$$E_\Pi(v_1, v_1 - 1, \beta) > 0 = E_\Pi(v_1, b_1, \beta).$$

So, β is not a PSNE. So, in a symmetric PSNE β, we have $\beta(v_1) \geq v_1 - 1$. Since the bids are always smaller than their associated valuations, we have $\beta(v_1) = v_1 - 1$ or $\beta(v_1) = v_1$ for every PSNE β. So we have the proof of Lemma 8. □

Corollary 2. *For a given p.i.i.d. auction, there are at most two symmetric PSNE.*

Proof. We assume we have three distinct symmetric PSNE. Because of Lemma 8, there are only two possible values for the first bid. So, two PSNE have the same value for their first bid. Because of Corollary 1, they are equal. So, by contraposition, we have the Corollary 2. □

Lemma 9. *For a given p.i.i.d auction, if it has one symmetric PSNE, it has two. Moreover, the only difference between them is the value of the first bid.*

Proof. For a given strategy β with $\beta(v_1) = v_1 - 1$, we denote β' the strategy defined by:

- $\beta'(v_1) = v_1$
- $\beta'(v_j) = \beta(v_j)$ for $2 \leq j \leq k$

From now on, we denote b_j the value of $\beta(v_j)$ and b'_j the value of $\beta'(v_j)$.

Lemma 10. *For any $1 \leq i \leq k$ and any $b \neq v_1 - 1$, we have*

$$E_\Pi(v_i, b, \beta) = E_\Pi(v_i, b, \beta').$$

Proof. For $1 \leq i \leq k$, we have

$$E_\Pi(v_i, b, \beta) = \sum_{j=1}^{k} P_j W_j$$

with

- $W_j = v_i - b$ when $b_j < b$,
- $W_j = v_i - v_j$ when $b_j = b$ and $j \leq i$,
- $W_j = 0$ when ($b_j = b$ and $j > i$) or $b_j > b$.

and

$$E_\Pi(v_i, b, \beta') = \sum_{j=1}^{k} P_j W'_j$$

with

- $W'_j = v_i - b$ when $b'_j < b$,
- $W'_j = v_i - v_j$ when $b'_j = b$ and $j \leq i$,
- $W'_j = 0$ when ($b'_j = b$ and $j > i$) or $b'_j > b$.

For $j \geq 2$, we have $b_j = b'_j$. So $W_j = W'_j$ for $j \geq 2$. When $j = 1$:

- $b > v_1$: So $b_1 < b$ and $b'_1 < b$. So $W_1 = W'_1 = v_i - b$.
- $b < v_1 - 1$: So $b_1 > b$ and $b'_1 > b$. So $W_1 = W'_1 = 0$.
- $b = v_1$: So $b_1 < b$ and $b'_1 = b$ with $1 \leq i$. So $W_1 = v_1 - b$ and $W'_1 = v_i - v_1$. Since $b = v_1$, we have $W_1 = W'_1$.

So for every $1 \leq j \leq k$, if $b \neq v_1 - 1$, then we have $W_j = W'_j$. So for $b \neq v_1 - 1$, we have:

$$E_\Pi(v_i, b, \beta) = E_\Pi(v_i, b, \beta').$$

□

Lemma 11. *For every $1 \leq i \leq k$, we have*

$$E_\Pi(v_i, b_i, \beta) = E_\Pi(v_i, b'_i, \beta').$$

Proof. If $i \neq 1$, we have $b_i = b'_i \neq v_1 - 1$. So $E_\Pi(v_i, b_i, \beta) = E_\Pi(v_i, b'_i, \beta')$ if $i \neq 1$.

If $i = 1$, we have $E_\Pi(v_i, \beta(v_i), \beta) = E_\Pi(v_1, b_1, \beta) = \sum_{j=1}^{k} P_j W_j = P_1 W_1 + \sum_{j=1}^{k} P_j W_j = P_1(v_1 - v_1) + \sum_{j=1}^{k} P_j \cdot 0 = 0$ and $E_\Pi(v_i, \beta'(v_i), \beta') = E_\Pi(v_1, b'_1, \beta') = \sum_{j=1}^{k} P_j W'_j = P_1 W'_1 + \sum_{j=1}^{k} P_j W'_j = P_1(v_1 - v_1) + \sum_{j=1}^{k} P_j \cdot 0 = 0$. So we have $E_\Pi(v_1, \beta(v_1), \beta) = E_\Pi(v_1, \beta'(v_1), \beta')$ and thereby $E_\Pi(v_i, b_i, \beta) = E_\Pi(v_i, b'_i, \beta')$ for every i. □

Lemma 12. *For any $1 \leq i \leq k$, we have*

$$E_\Pi(v_i, v_1 - 1, \beta) \leq E_\Pi(v_i, v_1, \beta)$$

and

$$E_\Pi(v_i, v_1 - 1, \beta') \leq E_\Pi(v_i, v_1, \beta').$$

Proof. We have

$$E_\Pi(v_1, v_1 - 1, \beta) = \sum_{j=1}^{k} P_j W_j$$

with

1. $W_j = v_i - (v_1 - 1)$ when $b_j < v_1 - 1$,
2. $W_j = v_i - v_j$ when $b_j = v_1 - 1$ and $j \leq i$,
3. $W_j = 0$ when ($b_j = v_1 - 1$ and $j > i$) or $b_j > v_1 - 1$.

We are always in the third case, except for $j = 1$, where we are in the second case with $W_j = W_1 = v_i - v_1$. So all the W_j are equal to 0 except the first one. So

$$E_\Pi(v_1, v_1 - 1, \beta) = P_1(v_i - v_1).$$

We have also

$$E_\Pi(v_1, v_1, \beta) = \sum_{j=1}^{k} P_j W_j$$

with

1. $W_j = v_i - v_1$ when $b_j < v_1$
2. $W_j = v_i - v_j$ when $b_j = v_1$ and $j \leq i$
3. $W_j = 0$ when ($b_j = v_1$ and $j > i$) or $b_j > v_1$.

If we are in the second case, then $j \leq i$, and so $v_j \leq v_i$. So, all the W_j are positive. We have $b_1 - v_1$. So $W_1 = v_i - v_1$. So, since all other W_j are positive, we have

$$E_\Pi(v_1, v_1, \beta) \geq P_1(v_i - v_1).$$

So,

$$E_\Pi(v_1, v_1 - 1, \beta) \leq E_\Pi(v_1, v_1, \beta).$$

We have

$$E_\Pi(v_1, v_1 - 1, \beta') = \sum_{j=1}^{k} P_j W'_j$$

with

1. $W'_j = v_i - (v_1 - 1)$ when $b'_j < v_1 - 1$,
2. $W'_j = v_i - v_j$ when $b'_j = v_1 - 1$ and $j \leq i$,
3. $W'_j = 0$ when ($b'_j = v_1 - 1$ and $j > i$) or $b'_j > v_1 - 1$.

We have $b'_j > v_1 - 1$ for $1 \leq j \leq k$. So we are always in the third case. So, $E_\Pi(v_1, v_1 - 1, \beta') = 0$. We have also

$$E_\Pi(v_1, v_1, \beta') = \sum_{j=1}^{k} P_j W'_j$$

with

1. $W'_j = v_i - v_1$ when $b'_j < v_1$,

2. $W'_j = v_i - v_j$ when $b'_j = v_1$ and $j \leq i$,

3. $W'_j = 0$ when ($b'_j = v_1$ and $j > i$) or $b'_j > v_1$.

If we are in the second case, then $j \leq i$, and so $v_j \leq v_i$. So, all the W_j are positive. That is, $E_\Pi(v_1, v_1, \beta') \geq 0$. So, $E_\Pi(v_1, v_1 - 1, \beta') \leq E_\Pi(v_1, v_1, \beta')$. This completes the proof of Lemma 12. □

We assume one of the two strategies, let's call it β_1, is a symmetric PSNE, and the other strategy, β_2, is not. So there exist i and b with

$$E_\Pi(v_i, \beta_1(v_i), \beta_1) \geq E_\Pi(v_i, b, \beta_1)$$

and

$$E_\Pi(v_i, \beta_2(v_i), \beta_2) < E_\Pi(v_i, b, \beta_2).$$

By Lemma 11, we have $E_\Pi(v_i, \beta_1(v_i), \beta_1) = E_\Pi(v_i, \beta_2(v_i), \beta_2)$. So $E_\Pi(v_i, b, \beta_1) \neq E_\Pi(v_i, b, \beta_2)$. So, by Lemma 10 we have $b = v_1 - 1$. By Lemma 12, we have $E_\Pi(v_i, v_1 - 1, \beta_2) \leq E_\Pi(v_i, v_1, \beta_2)$. So $E_\Pi(v_i, \beta_2(v_i), \beta_2) < E_\Pi(v_i, v_1, \beta_2)$ By Lemma 10, we have $E_\Pi(v_i, v_1, \beta_1) = E_\Pi(v_i, v_1, \beta_2)$. So $E_\Pi(v_i, \beta_1(v_i), \beta_1) < E_\Pi(v_i, v_1, \beta_1)$, which is not possible, since β_1 is a PSNE. So, if we have one symmetric PSNE, we also have a second symmetric PSNE. Moreover, the only difference between the two is the value of their first bid. This concludes the proof of Lemma 9. □

By combining Lemma 9 with Corollary 2, we have completed the proof of Theorem 4.

D Proof of Proposition 1

We first show that the A_i arrays are computed correctly by the algorithm. Indeed, we have $A_1(1) = 0$, $A_2(1) = 0$ and $A_3(1) = P_1$. Then, for any i, by induction:

- If $b_{i+1} = b_i$:

$$A_1(i+1) = \sum_{j \backslash b_j < b_{i+1}} P_j = \sum_{j \backslash b_j < b_i} P_j = A_1(i)$$

$$\begin{aligned} A_2(i+1) &= \sum_{j \backslash b_j = b_{i+1}, j \leq i+1} (P_j(v_{i+1} - v_j)) \\ &= \sum_{j \backslash b_j = b_i, j \leq i} (P_j((v_{i+1} - v_i) + (v_i - v_j))) \\ &= (\sum_{j \backslash b_j = b_i, j \leq i} P_j)(v_{i+1} - v_i) + (\sum_{j \backslash b_j = b_i, j \leq i} P_j(v_i - v_j)) \\ &= (A_3(i) - A_1(i))(v_{i+1} - v_i) + A_2(i) \end{aligned}$$

$$A_3(i+1) = \sum_{j=1}^{i+1} P_j = (\sum_{j=1}^{i} P_j) + P_{i+1} = A_3(i) + P_{i+1}$$

- If $b_{i+1} > b_i$:

$$A_1(i+1) = \sum_{j \backslash b_j < b_{i+1}} P_j = \sum_{j=1}^{i} P_j = A_3(i)$$

$$A_2(i+1) = \sum_{j \backslash b_j = b_{i+1}, j \leq i+1} (P_j(v_{i+1} - v_j)) = P_{i+1}(v_{i+1} - v_{i+1}) = 0$$

$$A_3(i+1) = \sum_{j=1}^{i+1} P_j = (\sum_{j=1}^{i} P_j) + P_{i+1} = A_3(i) + P_{i+1}$$

From Lemma 9 and Lemma 8, if there is a symmetric PSNE, then there is one that has $b_1 = v_1 - 1$. So the algorithm begins by setting b_1 at $v_1 - 1$ (line 1). From Theorem 4, we have the value of b_2 which is the same for the two possible symmetric PSNE. Since these PSNE start with $b_1 = v_1 - 1$ for one and $b_1 = v_1$ for the other, and from Lemma 4, we have $b_2 = v_1$ for a PSNE. It is what the algorithm sets (line 5). Then, for each i from 3 to k (line 9), from Theorem 4, there is only one b_i which leads to a symmetric PSNE. If after setting b_i at b_{i-1} (line 10), we have $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_i + 1, \beta)$, then $b_i = b_{i-1}$ does not lead to a symmetric PSNE. So, from Lemma 4, the only way to get a symmetric PSNE is to set $b_i = b_{i-1} + 1$, which indeed is what the algorithm does (line 15). If we have not $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_i + 1, \beta)$, then from the proof of Lemma 5, after setting $b_i = b_{i-1} + 1$, we would have $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_i - 1, \beta)$. So $b_i = b_{i-1} + 1$ does not lead to a symmetric PSNE and we have to keep $b_i = b_{i-1}$ (line 14). So if there is a symmetric PSNE, the algorithm gives one.

E Proof of Proposition 2

Once we have β, for a given $1 \leq i \leq k$ and a given b, we set $J = \max\{j \backslash b_j \leq b\}$. Depending on the value of b, b_J and b_i, we can compute $E_\Pi(v_i, b, \beta)$:

- $b_J < b$

$$\begin{aligned} E_\Pi(v_i, b, \beta) &= \sum_{j=1}^{J} (P_j(v_i - b)) \\ &= A_3(J)(v_i - b) \end{aligned}$$

- $b_J = b$
  - $b_i < b$

$$\begin{aligned} E_\Pi(v_i, b, \beta) &= \sum_{j \backslash b_j < b} (P_j(v_i - b)) \\ &= A_1(J)(v_i - b) \end{aligned}$$

  - $b_i = b$

$$\begin{aligned} E_\Pi(v_i, b, \beta) &= \sum_{j \backslash b_j < b} (P_j(v_i - b)) + \sum_{j \backslash b_j = b, j \leq i} (P_j(v_i - v_j)) \\ &= A_1(i)(v_i - b) + A_2(i) \end{aligned}$$

  - $b_i > b$

$$\begin{aligned} E_\Pi(v_i, b, \beta) &= \sum_{j \backslash b_j < b} (P_j(v_i - b)) + \sum_{j \backslash b_j = b} (P_j(v_i - v_j)) \\ &= A_1(J)(v_i - b) + \sum_{j \backslash b_j = b} (P_j((v_i - v_J) + (v_J - v_j))) \\ &= A_1(J)(v_i - b) + (\sum_{j \backslash b_j = b} P_j)(v_i - v_J) + \sum_{j \backslash b_j = b} (v_J - v_j) \\ &= A_1(J)(v_i - b) + (A_3(J) - A_1(J))(v_i - v_J) + A_2(J) \end{aligned}$$

So, if we know β and A, we can compute any expected payoff in a constant time.

Proposition 3. *Algorithm 3 indicates if a strategy β is a PSNE or not.*

Proof. The strategy β is a PSNE if and only if $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for every $1 \leq i \leq k$ and every b. The algorithm looks at every i. It also looks at every b, except for $b < b_1$, $b > b_k$ and $b_j + 1 < b < b_{j'}$ (with no bid between b_j and $b_{j'}$).

- $b < b_1$

  We have $E_\Pi(v_i, b, \beta) = 0$ for every i. So $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta)$ for every i. So we do not have to look at such b.

Algorithm 3 Verification

```
1: Compute A with the Algorithm 2.
2: equi ← true
3: for j = 1 to k do
4:     for j' = 1 to k do
5:         if E_Π(v_j, b_j, β) < E_Π(v_j, b_{j'}, β) or E_Π(v_j, b_j, β) < E_Π(v_j, b_{j'} + 1, β) then
6:             equi ← false
7:         end if
8:     end for
9: end for
10: if equi then
11:     β is a symmetric PSNE.
12: else
13:     β is not a symmetric PSNE.
14: end if
```

- $b \geq b_k + 1$

  $$E_\Pi(v_i, b, \beta) = (\sum_{h=1}^{k} P_h)(v_i - b) = v_i - b \leq v_i - (b_k + 1) = E_\Pi(v_i, b_k + 1, \beta)$$

  So $E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b, \beta) \Rightarrow E_\Pi(v_i, b_i, \beta) < E_\Pi(v_i, b_k + 1, \beta)$. Moreover, the algorithm looks at $b_k + 1$. So we do not have to look at such b.

- $b_j + 1 < b < b_{j'}$

  We have

  $$E_\Pi(v_i, b, \beta) = (\sum_{h \backslash b_h < b} P_h)(v_i - b) < (\sum_{h \backslash b_h < b} P_h)(v_i - (b_j + 1)) = E_\Pi(v_i, b_j + 1, \beta)$$

  So $E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b, \beta) \Rightarrow E_\Pi(v_i, b_i, \beta) \geq E_\Pi(v_i, b_{j+1}, \beta)$. Moreover, the algorithm looks at b_{j+1}. So we do not have to look at such b.

So the algorithm looks at every comparison between $E_\Pi(v_i, b_i, \beta)$ and $E_\Pi(v_i, b, \beta)$ for every i, and also for every b we need to look at. So the algorithm is correct and indicates if the strategy β is a PSNE or not. □

Proposition 4. *Algorithm 3 runs in quadratic time.*

Proof. First, the algorithm computes A (line 1) in time $O(k)$. Then, the algorithm is composed of a loop L_1 (line 3). L_1 starts at 1 and ends at k. So there are k iterations in L_1. Each iteration is composed of a loop L_2 (line 4). L_2 starts at 1 and ends at k. So there are k iterations in L_2. Each iteration of L_2 is composed of two comparisons between two payoffs (line 5). Since we have computed A, we can compute the payoffs in a constant time, with the formulas we got previously. So each iteration of L_2 has a complexity of $O(1)$. So each iteration of L_1 has a complexity of $O(k)$. So, the complexity of L_1 is $O(k^2)$. So the complexity of the algorithm is $O(k) + O(k^2) = O(k^2)$, which is the statement of the proposition. □