Kripke Models over Recursive Worlds
(The Joy of Ultrametric Spaces)

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Plan

- Intro and overview of recent work on using Kripke models over recursively defined worlds
  - to model type systems and logics for dynamically allocated, often higher order, recursive structures (e.g., higher-order store / storable locks)
  - where recursive worlds defined in category of ultrametric spaces
- Talk: partly technical intro to such uses via an example, partly an overview with pointers to literature (more details in other talks by collaborators this week).
- Will focus on the core issue of recursive worlds, using simple notions of worlds, both in denotational and operational settings — for useful applications necessary to
  1. use more sophisticated worlds (e.g., for reasoning about local state)
  2. also use recursively defined operations on worlds (e.g., for higher-order frame rules)
- Will skip 1 almost entirely (intro to such in Derek’s talk), will only touch briefly upon 2 (more about this in Jan’s talk)

Papers can be found at [www.itu.dk/people/birkedal/papers](http://www.itu.dk/people/birkedal/papers).
Case Study — Model of $F_{\mu,\text{ref}}$

[BST - FOSSACS’09, MSCS’10]
Slogan: one domain equation for each of $\forall$, ref, $\mu$.

$\forall$ impredicative polymorphism: choose to model types as relations $UARel(V)$ over a recursively defined predomain $V$.

ref general references with dynamic allocation: use Kripke model with recursively defined worlds, approximately of the form:

$$\begin{align*}
\mathcal{T} &= \forall \mathcal{W} \rightarrow UARel(V) \\
\mathcal{W} &= \mathbb{N} \rightarrow \mathcal{T}
\end{align*}$$

Solve in CBUlt.

$\mu$ recursive types: relations interpreting types also recursively defined,

- non-trivial for reference types, leads to novel modeling of locations involving some approximation information.
Proposition. There exists a uniform cpo \((V, (\pi_n)_{n \in \omega})\) satisfying:

In pCpo:

\[
V \cong \mathbb{Z} + Loc + 1 + (V \times V) + (V + V) + V + TV + (V \rightarrow TV) \tag{1}
\]

where

\[
TV = (V \rightarrow S \rightarrow Ans) \rightarrow S \rightarrow Ans
\]

\[
S = \mathbb{N} \rightarrow_{\text{fin}} V
\]

\[
Ans = (\mathbb{Z} + Err) \perp
\]

and

\[
Loc = \mathbb{N} \times \bar{\omega}
\]

\[
Err = 1.
\]
The functions $\pi_n : V \to V_\bot$ satisfy (and are determined by)

$$\pi_0 = \lambda v. \bot$$

$$\pi_{n+1}(in_{\mathbb{Z}}(k)) = \lfloor in_{\mathbb{Z}}(k) \rfloor$$

$$\pi_{n+1}(in_{\times}(v_1, v_2)) = \begin{cases} \lfloor in_{\times}(v'_1, v'_2) \rfloor & \text{if } \pi_n v_1 = \lfloor v'_1 \rfloor \text{ and } \pi_n v_2 = \lfloor v'_2 \rfloor \\ \bot & \text{otherwise} \end{cases}$$

... etc. as you’d expect, except:

$$\pi_{n+1}(in_{\text{Loc}}(l, m)) = \lfloor in_{\text{Loc}}(l, \min(n + 1, m)) \rfloor$$
Untyped Semantics of Terms, I

\([t]_X : V^X \rightarrow TV\) by induction on \(t\):

Mostly standard, e.g.,

\[\llbracket \lambda x. t \rrbracket_X \rho = \eta(in \rightarrow (\lambda v. \llbracket t \rrbracket_{X,x}(\rho[x \mapsto v])))\]

\[\llbracket t_1 t_2 \rrbracket_X \rho = \llbracket t_1 \rrbracket_X \rho \ast \lambda v_1. \llbracket t_2 \rrbracket_X \rho \ast \lambda v_2. \begin{cases} f \; v_2 & \text{if } v_1 = in \rightarrow f \\ \text{error} & \text{otherwise} \end{cases}\]
For lookup and assignment we need to consider semantic locations:

\[ \llbracket ! t \rrbracket_X \rho = \llbracket t \rrbracket_X \rho \star \lambda \nu \cdot \text{lookup } \nu \]

where \( \text{lookup } \nu = \) 

\[
\lambda k \lambda s. \begin{cases} 
  k s(l) s & \text{if } \nu = \lambda l \text{ and } l \in \text{dom}(s) \\
  k v' s & \text{if } \nu = \lambda_{l+1}^n, l \in \text{dom}(s), \text{ and } \pi_n(s(l)) = \lfloor v' \rfloor \\
  \perp_{\text{Ans}} & \text{if } \nu = \lambda_{l+1}^n, l \in \text{dom}(s), \text{ and } \pi_n(s(l)) = \perp \\
  \text{error}_{\text{Ans}} & \text{otherwise}
\end{cases}
\]
Adequacy wrt. standard operational semantics can be shown using recursively defined logical relation.

Non-trivial, but not too hard using Pitts’ technique (with a function-space lattice to deal with nested recursive types), since suffices to consider only closed types for adequacy [BST, TLDI’09].

Now on to typed semantics, i.e., definition of logical relations over the untyped semantics. First define space of types using ultrametric spaces.
Recall:

- An *ultrametric space* is a metric space \((D, d)\) that instead of triangle inequality satisfies the stronger *ultrametric inequality*:

  \[ d(x, z) \leq \max(d(x, y), d(y, z)). \]

- A function \(f : D_1 \rightarrow D_2\) from a metric space \((D_1, d_1)\) to a metric space \((D_2, d_2)\) is *non-expansive* if \(d_2(f(x), f(y)) \leq d_1(x, y)\) for all \(x\) and \(y\) in \(D_1\).

- A function \(f : D_1 \rightarrow D_2\) from a metric space \((D_1, d_1)\) to a metric space \((D_2, d_2)\) is *contractive* if there exists \(\delta < 1\) such that \(d_2(f(x), f(y)) \leq \delta \cdot d_1(x, y)\) for all \(x\) and \(y\) in \(D_1\).

- CBUlt is the category with complete 1-bounded ultrametric spaces and non-expansive functions.
CBUlt, II

- CBUlt is cartesian closed; the exponential \((D_1, d_1) \rightarrow (D_2, d_2)\) is the set of non-expansive maps with the “sup”-metric \(d_{D_1 \rightarrow D_2}\) as distance function:

\[
d_{D_1 \rightarrow D_2}(f, g) = \sup \{d_2(f(x), g(x)) \mid x \in D_1\}.
\]

- Thm [America-Rutten]: Solutions to recursive domain equations for locally contractive functors exist.

- A functor \(F : \text{CBUlt}^{\text{op}} \times \text{CBUlt} \rightarrow \text{CBUlt}\) is \textit{locally contractive} if there exists \(\delta < 1\) such that

\[
d(F(f, g), F(f', g')) \leq \delta \cdot \max(d(f, f'), d(g, g'))
\]

for all non-expansive functions \(f, f', g,\) and \(g'\).
Recall [Amadio, Abadi-Plotkin]:

- \textit{UARel}(V) is the set of admissible relations that are \textit{uniform}: 
  \( \varpi_n \in R \rightarrow R_{\perp}, \) for all \( n. \)

- Such relations are determined by its elements of the form 
  \( (\varpi_n e, \varpi_n e') \).

- \( \textit{UARel}(V) \in \text{CBUlt}, \) distance function:

\[
d(R, S) = \begin{cases} 
2 - \max \{ n \in \omega \mid \varpi_n \in R \rightarrow S \land \varpi_n \in S \rightarrow R \} & \text{if } R \neq S \\
0 & \text{if } R = S.
\end{cases}
\]
Proposition. Let \((D, d) \in \text{CBUlt}\). The set \(\mathbb{N} \rightarrow_{\text{fin}} D\) with distance function:

\[
d'(\Delta, \Delta') = \begin{cases} 
\max \{ d(\Delta(l), \Delta'(l)) \mid l \in \text{dom}(\Delta) \} & \text{if } \text{dom}(\Delta) = \text{dom}(\Delta') \\
1 & \text{otherwise.}
\end{cases}
\]

is in \(\text{CBUlt}\).

Extension ordering: \(\Delta \leq \Delta'\) iff

\[
\text{dom}(\Delta) \subseteq \text{dom}(\Delta') \land \forall l \in \text{dom}(\Delta). \Delta(l) = \Delta'(l).
\]
Space of types

- **Proposition.**

\[ F(D) = (\mathbb{N} \rightarrow_{\text{fin}} D) \rightarrow_{\text{mon}} \text{UARel}(V) \]

(monotone, non-expansive maps) defines a functor
\( F : \text{CBUlt}^{\text{op}} \rightarrow \text{CBUlt} \).

- **Theorem.** There exists \( \hat{T} \in \text{CBUlt} \) such that the isomorphism

\[ \hat{T} \cong \frac{1}{2}((\mathbb{N} \rightarrow_{\text{fin}} \hat{T}) \rightarrow_{\text{mon}} \text{UARel}(V)) \quad (2) \]

holds in CBUlt.
Define:

- Worlds: $\mathcal{W} = \mathbb{N} \rightarrow_{\text{fin}} \mathcal{T}$
- Types: $\mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} \text{UARel}(V)$
- Computations: $\mathcal{T}_T = \mathcal{W} \rightarrow_{\text{mon}} \text{UARel}(TV)$
- Continuations: $\mathcal{T}_K = \mathcal{W} \rightarrow_{\text{mon}} \text{UARel}(K)$
- States: $\mathcal{T}_S = \mathcal{W} \rightarrow \text{UARel}(S)$ (note: not monotone)
Semantics of Types

For every $\Xi \models \tau$, define the non-expansive $[\tau]_\Xi : T^\Xi \to T$ by induction on $\tau$:

$[\alpha]_\Xi \varphi = \varphi(\alpha)$

$[\text{int}]_\Xi \varphi = \lambda \Delta. \{ (\text{in}_\mathbb{Z} k, \text{in}_\mathbb{Z} k) \mid k \in \mathbb{Z} \}$

$[1]_\Xi \varphi = \lambda \Delta. \{ (\text{in}_1 * , \text{in}_1 *) \}$

$[\tau_1 \times \tau_2]_\Xi \varphi = [\tau_1]_\Xi \varphi \times [\tau_2]_\Xi \varphi$

$[0]_\Xi \varphi = \lambda \Delta. \emptyset$

$[\tau_1 + \tau_2]_\Xi \varphi = [\tau_1]_\Xi \varphi + [\tau_2]_\Xi \varphi$

$[\text{ref} \, \tau]_\Xi \varphi = \text{ref}( [\tau]_\Xi \varphi)$

$[\forall \alpha. \tau]_\Xi \varphi = \lambda \Delta. \{ (\text{in}_\forall c, \text{in}_\forall c') \mid \forall \nu \in T. (c, c') \in\, \text{comp}( [\tau]_\Xi, \alpha \varphi[\alpha \mapsto \nu])(\Delta) \}$

$[\mu \alpha. \tau]_\Xi \varphi = \text{fix} \left( \lambda \nu. \lambda \Delta. \{ (\text{in}_\mu v, \text{in}_\mu v') \mid (v, v') \in\, [\tau]_\Xi, \alpha \varphi[\alpha \mapsto \nu] \Delta \} \right)$

$[\tau_1 \to \tau_2]_\Xi \varphi = ([\tau_1]_\Xi \varphi) \to (\text{comp}( [\tau_2]_\Xi \varphi))$
Semantic Type Constructors

\[(\nu_1 \times \nu_2)(\Delta) = \{ (in_x(\nu_1, \nu_2), in_x(\nu'_1, \nu'_2)) | (\nu_1, \nu'_1) \in \nu_1(\Delta) \land (\nu_2, \nu'_2) \in \nu_2(\Delta) \}\]

\[ref(\nu)(\Delta) = \{ (\lambda_l, \lambda_{l^1}) | l \in \text{dom}(\Delta) \land \forall \Delta_1 \geq \Delta. \text{App}(\Delta(l)) \Delta_1 = \nu(\Delta_1) \}\]

\[\cup \{ (\lambda_l^{n+1}, \lambda_{l^1}^{n+1}) | l \in \text{dom}(\Delta) \land \forall \Delta_1 \geq \Delta. \text{App}(\Delta(l)) \Delta_1 \equiv^n \nu(\Delta_1) \}\]

- Note the use of semantic locations to ensure non-expansiveness in \textit{ref} case.
- Necessary: see Kristian's talk.
- Because of relational parametricity, we need to model \textit{open} types; hence need to compare semantic types above, cannot simply use syntactic worlds and compare types syntactically.
\[(\nu \to \xi)(\Delta) = \{ (\text{in} \to f, \text{in} \to f') \mid \forall \Delta_1 \geq \Delta. \forall (v, v') \in \nu(\Delta_1). (f v, f' v') \in \xi(\Delta_1) \}\]

\[\text{cont}(\nu)(\Delta) = \{ (k, k') \mid \forall \Delta_1 \geq \Delta. \forall (v, v') \in \nu(\Delta_1). \forall (s, s') \in \text{states}(\Delta_1). (k v s, k' v' s') \in R_{\text{Ans}} \}\]

\[\text{comp}(\nu)(\Delta) = \{ (c, c') \mid \forall \Delta_1 \geq \Delta. \forall (k, k') \in \text{cont}(\nu)(\Delta_1). \forall (s, s') \in \text{states}(\Delta_1). (c k s, c' k' s') \in R_{\text{Ans}} \}\]

\[\text{states}(\Delta) = \{ (s, s') \mid \text{dom}(s) = \text{dom}(s') = \text{dom}(\Delta) \\land \forall l \in \text{dom}(\Delta). (s(l), s'(l)) \in \text{App}(\Delta(l)) (\Delta) \}\]

\[R_{\text{Ans}} = \{ (\bot, \bot) \} \cup \{ ([\nu_1 k], [\nu_1 k]) \mid k \in \mathbb{Z} \}\]
Typed Semantics of Terms

- For $\Xi \vdash \Gamma$ and $\varphi \in T^\Xi$, let $\llbracket \Gamma \rrbracket_\Xi \varphi$ be the binary relation on $V^{\text{dom}(\Gamma)}$ defined by
  \[
  \llbracket \Gamma \rrbracket_\Xi \varphi = \{ (\rho, \rho') \mid \forall x \in \text{dom}(\Gamma). (\rho(x), \rho'(x)) \in \llbracket \Gamma(x) \rrbracket_\Xi \varphi \} .
  \]

- Two typed terms $\Xi \mid \Gamma \vdash t : \tau$ and $\Xi \mid \Gamma \vdash t' : \tau$ of the same type are **semantically related**, written $\Xi \mid \Gamma \models t \sim t' : \tau$, if for all $\varphi \in T^\Xi$, all $(\rho, \rho') \in \llbracket \Gamma \rrbracket_\Xi \varphi$, and all $\Delta \in \mathcal{W}$,
  \[
  \left( \llbracket t \rrbracket_{\text{dom}(\Gamma)} \rho, \llbracket t' \rrbracket_{\text{dom}(\Gamma)} \rho' \right) \in \text{comp}(\llbracket \tau \rrbracket_\Xi \varphi)(\Delta) .
  \]
Theorem. Semantic relatedness is a congruence.

Corollary. (FTLR) If $\Xi | \Gamma \vdash t : \tau$, then $\Xi | \Gamma \models t \sim t : \tau$.

Corollary. If $\emptyset | \emptyset \vdash t : \tau$ is a closed term of type $\tau$, then $\llbracket t \rrbracket_\emptyset \emptyset \neq \text{error}$.

Corollary. If $\Xi | \Gamma \models t \sim t' : \tau$ then $t = \text{ctx} t'$.
Overview of Applications and Extensions

- Four strands of work plus mention couple of other applications
Strand I: Nested Triples and (Anti)Frame Rules

Separation Logic with Nested Hoare Triples for reasoning about stored code (higher-order store) with higher-order frame rules [SBRY-CSL’09]

- Interpretation indexed over Kripke world describing “hidden invariants”.

- But invariants are simply predicates (think of frame rule where any predicate can be used as an invariant), so get equation in CBUlt:

\[
\text{Pred} = \frac{1}{2}(W \rightarrow UAdm(H))
\]

\[
W \cong \text{Pred}
\]

Iso \( \iota : \text{Pred} \rightarrow W \).
For higher-order frame rules: define non-expansive map $$\circ : W \times W \rightarrow W$$, s.t., for all $$p, r, w \in W$$,

$$\iota^{-1}(p \circ r)(w) = \iota^{-1}(p)(r \circ w) \ast \iota^{-1}(r)(w).$$

Intuition:
- $$p$$ and $$r$$ world-dependent invariants
- world-dependency via application
- $$p \circ r$$ is the extension of $$p$$ with $$r$$: first extend $$r$$ with $$w$$, and then apply $$p$$ to that, in addition to “starring on” $$r(w)$$.

Well-defined by Banach: intuitively because the $$\circ$$ on the right is as an argument, below an unfolding via $$\iota^{-1}$$.

Semantics allowed to investigate soundness of various higher-order frame rules (tricky, some formulations are not sound, others are, see paper for examples)

**Take-home**: Use worlds form metric space to ensure well-definedness (via Banach) of recursive operation on worlds.
Models of Pottier’s Anti-frame rule [SYBRS - FOSSACS’10]

- Separation Logic for higher-order store with nested triples and formulation of Pottier’s anti-frame rule for hiding invariants in direct style.
- Standard existence theorems for the world equation used did not apply directly, constructed solution by hand.

(Jan’s talk)
Strand II: Step-Indexed Models

(cf. Derek’s talk)

- Step-indexed model of $F_{\mu, \mathrm{ref}}$ for reasoning about ctx. equiv., with more refined worlds, allows to prove more example programs equivalent. [ADR-POPL’09]

- Logics (LSLR, LADR) for step-indexed models, to avoid reasoning about steps when reasoning about examples (and a bit of the meta-theory) [DAB-LICS’09, DNRB-POPL’10]

- Worlds as transition systems describing how local state can evolve, studying the influence of different language features (first vs. higher-order state, with or without call/cc). Handles all known examples. [DNB-ICFP’10]

**Take-home:** (1) logics for steps for more high-level reasoning, (2) expressive worlds for more useful models.
Strand III: expressive ultrametric worlds

- Scaling up the denotational approach to recursively defined worlds a la those in LADR. [BST-TR] [Thamsborg dissertation]
  - involves using new form of relations that we call Bohr relations (chain-complete and downwards-closed in left-hand side), capturing ctx. approximation (instead of equiv.) [as in step-indexed models]
  - involves solving world equation in category of preordered ultrametric spaces

  - The Category-Theoretic Solution to Recursive Metric-Space Equations [BST-TCS’10]. Supporting theory. M-categories. (Jacob’s talk.)

**Take-home** (1) Techniques scale well, (2) Resulting model allows for proofs of examples in the model at same level of abstraction as the LADR logic for step-indexed model.
Recent work on showing how the approach applies to operational semantics via step-indexing [BRSSTY]

- arguably simpler than denotational approach, scales well to concurrency
- high-level understanding of step-indexing
  - essence of step-indexing
  - generalizes Hobor et. al.’s Indirection Theory [POPL’10], which is aimed at giving general description of step-indexed models
- has been formalized in Coq

To explain idea, let’s consider a simple unary model $F_{\mu,\text{ref}}$. 
Uniform Predicates

Idea: replace domain $V$ by the set of $Val$ of operational values

- Uniform predicates:

$$UPred(Val) = \{ p \subseteq \mathbb{N} \times Val \mid \forall (k, v) \in p. \forall j \leq k. (j, v) \in p\}$$

- For $p \in UPred(Val)$ and $k \in \mathbb{N}$, let

$$p^k = \{(m, v) \in p \mid m < k\}$$

- Distance:

$$d(p, q) = \begin{cases} 2^{-\max\{k \mid p^k = q^k\}} & \text{if } p \neq q \\ 0 & \text{otherwise.} \end{cases}$$

- Lemma ($UPred(Val), d$) is a well-defined object in CBUlt.
Theorem

There exists $\hat{T} \in \text{CBUlt}$ such that

$$\hat{T} \cong \frac{1}{2} \cdot ((\mathbb{N} \rightarrow_{\text{fin}} \hat{T}) \rightarrow_{\text{mon}} \text{UPred}(\text{Val}))$$

is an iso in CBUlt.
Interpretation of Types, I

Define non-expansive map

$$[[\Xi \vdash \tau]] : \mathcal{T}^{[\Xi]} \rightarrow \mathcal{T}$$

by induction on $\tau$ (only some cases):

$$[[\Xi \vdash \tau]]_{\eta} : \mathcal{W} \rightarrow_{mon} UPred(Val)$$

$$[[\Xi \vdash 1]]_{\eta} w = \{(k,()) \mid k \in \mathbb{N}\}$$

$$[[\Xi \vdash \text{ref } \tau]]_{\eta} w = \{(k, l) \mid l \in \text{dom}(w) \land \forall w' \sqsupseteq w. i(w(l))(w') = k[[\Xi \vdash \tau]]_{\eta} w'\}$$
Interpretation of Types, II

$$\left[\Gamma \vdash \tau \rightarrow \tau'\right]_\eta w = \{(k, v) \mid \forall v' \in \text{Val}. \forall w' \supseteq w. \forall i \leq k. (i, v') \in \left[\Gamma \vdash \tau\right]_\eta w' \implies (i, v, v') \in E \left[\Gamma \vdash \tau'\right]_\eta w'\}$$

$$E \left[\Gamma \vdash \tau\right]_\eta : \mathcal{W} \rightarrow_{\text{mon}} \text{UPred}(\text{Exp})$$

$$E \left[\tau\right]_\eta w = \{(k, t) \mid \forall i \leq k. \forall h, h'. \forall t'. (h :_k w \land (t \mid h) \mapsto^i (t' \mid h') \land (t \mid h') \text{ irreducible}) \implies (\exists w' \supseteq w. h' :_{k-i} w' \land (k - i, v) \in \left[\tau\right]_\eta w')\}$$

$$h :_k w \iff \forall i < k. \text{dom}(h) = \text{dom}(w) \land \forall l \in \text{dom}(w). (i, h(l)) \in w(l)(w)$$
Recursive Types:

$$
\left[ \Delta \vdash \mu \alpha. \tau \right]_\eta = \text{fix}(\lambda r. \lambda w. \{(k, \text{fold } t) \mid k > 0 \Rightarrow (k - 1, v) \in \left[ \Delta, \alpha \vdash \tau \right]_{\eta[\alpha \mapsto r]} w\})
$$

- Uses Banach’s fixed point theorem.
- Contractiveness ensured by use of $k - 1$. 
Well-definedness

- Metric setup tells you what you have to show:
  - non-expansiveness of $[\exists \vdash \tau]$  
  - non-expansiveness of $[\exists \vdash \tau]_\eta$  
  - contractiveness of map for recursive types.

- Simple calculations.
Lemma

If $s :_k w$ and $w \models w'$ and $k < n$, then also $s :_k w'$.

Proof.

TS: $\forall j < k. \text{dom}(s) = \text{dom}(w') \land \forall l \in \text{dom}(w'). (j, s(l)) \in w'(l)(w')$.

Sps. $k > 0$; then $n > 0$. Let $j < k$. By $w \models w'$, we get

$\text{dom}(w) = \text{dom}(w') \land \forall l \in \text{dom}(w). \forall w_0. w(l)(w_0)^{n-1} = w'(l)(w_0)$.

Since $\text{dom}(s) = \text{dom}(w)$ by the assumption that $s :_k w$ (using $k > 0$), we get $\text{dom}(s) = \text{dom}(w')$. Moreover,

$$w(l)(w) \overset{n}{=} w(l)(w') \overset{n-1}{=} w'(l)(w')$$

since $w(l)$ is non-expansive, and since $w \models w'$. Thus, as $(j, s(l)) \in w(l)(w)$ by assumption, and since $j < k \leq n - 1$, we also get $(j, s(l)) \in w'(l)(w')$, as desired.
Indirection Theory. Hobor et. al. POPL’10

- General formulation of step-indexed models. Also observe cannot solve world-equation in sets. Instead describe approximate solutions and show how they can be used in many step-indexed models.

- We prove that one can derive an approx. solution à la Indirection Theory from one of our metric equations (see paper for detailed formulation and formal theorems).

- Corollary: applies to all the models described by indirection theory.
Advantages of metric approach

(some propaganda :-)

- Useful guiding framework.
- Supporting theory (e.g., recursive equations when spaces equipped with structure).
- Supports recursively-defined operations on worlds.
- Connection between step-indexing and metric spaces known from start of step-indexing (Appel-McAllester); but useful not to forget the connection!
- Also formalized in Coq [BBKV-TR]
Applications of Operational Approach

- We have given an alternative model of nested Hoare triples based directly on operational semantics with higher-order frame rules.
- Defined a model of Pottier’s Capability Calculus, shown soundness of extension with higher-order frame and anti-frame rules.
- Capability Calculus setup: $\mathcal{W} \simeq \frac{1}{2} \mathcal{W} \rightarrow UPred^\dagger(Heap)$.
- Model expresses that capabilities can be understood as separation logic assertions and is used to show soundness of the type system (new result).
Other recent applications

- A Metric Model of Nakano’s calculus of Guarded Recursion [BSS - FICS’10]
  - Kripke logical relation for adequacy proof defined using family of natural-number indexed relations.
- Separation logic for storable locks [BBS - ongoing].
- Model of type-and-effect system for higher-order store, extending work of Benton, Beringer, Hofmann, Kennedy [TB - ongoing] (again seems to involve a recursively defined operation on the set of worlds, though a quite different one).
- Krishnaswami-Benton: model of reactive programming using ultrametric spaces, see Neel’s talk.
Thank You!