W-Types and M-Types in Equilogical Spaces

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Abstract

We show that **Equ** has all W-types and all M-types. From this we conclude that induction and coinduction principles for polynomial functors are valid in the logic of equilogical spaces.

1 W-Types and M-types

Let **C** be a a finitely complete, locally cartesian closed category. For a morphism $f : B \to A$ define the "polynomial functor" $P_f : \mathbf{C} \to \mathbf{C}$ by

$$P_f(X) = \sum_{a \in A} X^{f^{-1}(a)}$$

where $\sum$ is a dependent sum and $f^{-1}(a)$ is the fiber in B over a. More precisely, $P_f(X)$ is the total space of the exponential

$$(X \times A \xrightarrow{\pi_2} A)^{(f : B \to A)}$$

in the slice category $\mathbf{C}/A$. For a morphism $[g] : X \to Y$, $P_f[g] = [id_{|A|} \times g^{id_{|B|}}]$. The *W-type* $W(f)$, if it exists, is an initial algebra for the functor P_f. The *M-type* $M(f)$, if it exists, is a final coalgebra for the functor P_f.

2 W-Types in PEqu

A morphism in **PEqu** is an equivalence class $[f] : B \to A$, but we often write f instead of $[f]$ where no confusion can arise. Wherever it makes sense, f should be interpreted as a continuous map $|B| \to |A|$, and otherwise it should be interpreted as the morphism represented by f.

Let $[f] : B \to A$ be a morphism in **PEqu**. For an object $X = (|X|, \approx_X)$ in **PEqu**, $P_f(X)$ is concretely defined as $P_f(X) = (|A| \times |X|^{|B|}, \approx_{P_f(X)})$, where

$$(a, u) \approx_{P_f(X)} (a', u')$$
$$\text{if and only if}$$
$$a \approx_A a' \text{ and } \forall\, b, b' \in |B| \,.\, (b \approx_B b' \wedge f(b) \approx_A a \Longrightarrow u(b) \approx_X u'(b'))\,.$$

For a morphism $[g] : X \to Y$, $P_f[g] = [id_{|A|} \times g^{id_{|B|}}]$. The rest of this section consists of a proof that P_f has an initial algebra $W = W(f)$.

(1) The underlying lattice $|W|$: The initial P_f-algebra W, if it exists, is isomorphic to $P_f(W)$. Thus, it makes sense to choose the underlying lattice $|W|$ so that $|W| \cong |A| \times |W|^{|B|}$. We know that such a lattice exists because domain equations can be solved in the category of algebraic lattices. In particular, we choose the lattice

$$|W| = \prod_{k=0}^{\infty} |A|^{|B|^k},$$

with an isomorphism $\langle \Box, \Box \rangle \colon |A| \times |W|^{|B|} \to |W|$, defined component-wise by:

$$\begin{aligned}
\pi_0 \langle a, u \rangle &= a \\
\pi_{i+1} \langle a, u \rangle &= \lambda(b, \vec{b}) \in |B|^{i+1} \,.\, \Big((\pi_i(u(b)))(\vec{b}) \Big) \,.
\end{aligned}$$

(2) The partial equivalence relation $\approx_W$: Let $\mathbf{PER}(|W|)$ be the complete lattice of partial equivalence relations on $|W|$, ordered by inclusion $\subseteq$. Define an operator $\Phi \colon \mathbf{PER}(|W|) \to \mathbf{PER}(|W|)$ by

$$\langle a, u \rangle \; \Phi(\approx) \; \langle a', u' \rangle$$

if and only if

$$a \approx_A a' \text{ and } \forall\, b, b' \in |B| \,.\, (b \approx_B b' \wedge f(b) \approx_A a \Longrightarrow u(b) \approx u'(b')) \,.$$

The operator Φ is a monotone operator on a complete lattice. Let $\approx_W$ be the least fixed point of Φ.

The operator Φ is defined on partial equivalence relations on $|M|$. Nevertheless, it can be applied to an arbitrary binary relation R on $|M|$. If R is a relation on $|M|$, let $\sigma(R)$ be its symmetric closure, and let $\tau(R)$ be its transitive closure. It is not hard to check that Φ satisfies

$$\begin{aligned}
\sigma(\Phi(R)) &\subseteq \Phi(\sigma(R)) \\
\tau(\Phi(R)) &\subseteq \Phi(\tau(R)),
\end{aligned}$$

and thus also

$$\tau(\sigma(\Phi(R))) \subseteq \Phi(\tau(\sigma(R))) \,.$$

(3) W is a P_f-algebra: To show that $[\langle \Box, \Box \rangle] \colon P_f(W) \to W$ is a P_f-algebra all that has to be checked is that $\langle \Box, \Box \rangle$ preserves the partial equivalence relation. Suppose $(a, u) \approx_{P_f(W)} (a', u')$. This means that

$$a \approx_A a' \text{ and } \forall\, b, b' \in |B| \,.\, (b \approx_B b' \wedge f(b) \approx_A a \Longrightarrow u(b) \approx_W u'(b'))$$

which is equivalent to $\langle a, u \rangle \; \Phi(\approx_W) \; \langle a', u' \rangle$, and since $\approx_W$ is a fixed point of Φ, this is just $\langle a, u \rangle \approx_W \langle a', u' \rangle$.

(4) Uniqueness of homomorphisms: Let $[v]\colon P_f(V) \to V$ be a P_f-algebra, and suppose $[s],[t]\colon W \to V$ are P_f-homomorphisms from W to V. Let $\sim$ be the partial equivalence relation on $|W|$, such that $\langle a,u\rangle \sim \langle a',u'\rangle$ if, and only if, all of the following hold:

- $(a, s\circ u) \approx_{P_f(V)} (a', s\circ u')$,
- $(a, t\circ u) \approx_{P_f(V)} (a', t\circ u')$,
- $(a, s\circ u) \approx_{P_f(V)} (a', t\circ u')$,
- $(a', s\circ u') \approx_{P_f(V)} (a, t\circ u)$.

If $\langle a,u\rangle \sim \langle a',u'\rangle$, then it follows from the first and the second conditions, that

$$\begin{aligned} s\langle a,u\rangle &\approx_V v(a,s\circ u) \approx_V v(a',s\circ u') \approx_V s\langle a',u'\rangle \\ t\langle a,u\rangle &\approx_V v(a,t\circ u) \approx_V v(a',t\circ u') \approx_V t\langle a',u'\rangle, \end{aligned}$$

which means that t and s preserve $\sim$. Similarly, using the third condition, it follows from $\langle a,u\rangle \sim \langle a',u'\rangle$ that

$$s\langle a,u\rangle \approx_V v(a,s\circ u) \approx_V v(a',t\circ u') \approx_V t\langle a,u\rangle.$$

To show that $[t]=[s]$, we demonstrate that $\approx_W \subseteq \sim$ by proving that $\sim$ is a prefixed point of Φ. Suppose $\langle a,u\rangle\ \Phi(\sim)\ \langle a',u'\rangle$. Then $a \approx_A a'$, and for all $b,b' \in |B|$ such that $b \approx_B b'$ and $f(b) \approx_A a$ it is the case that $u(b) \sim u'(b')$. Because s and t preserve $\sim$ and they coincide on it up to equivalence in V it follows that:

- $s(u(b)) \approx_V s(u'(b'))$,
- $t(u(b)) \approx_V t(u'(b'))$,
- $s(u(b)) \approx_V t(u'(b'))$,
- $s(u'(b')) \approx_V t(u(b))$.

It is now clear that $\langle a,u\rangle \sim \langle a',u'\rangle$.

(5) Existence of homomorphisms: Let $[v]\colon P_f(V) \to V$ be a P_f-algebra. We show that there exists a P_f-homomorphism $[w]\colon W \to V$. Let $\Psi\colon |V|^{|W|} \to |V|^{|W|}$ be the operator defined by

$$(\Psi g)\langle a,u\rangle = v(a, g\circ u).$$

Let $w \in |W| \to |V|$ be the least fixed point of Ψ, so that

$$w\langle a,u\rangle = v(a, w\circ u).$$

We need to show that $[w]$ is a P_f-homomorphism. Let $\sim$ be a partial equivalence relation on $|W|$ defined by

$$\langle a, u\rangle \sim \langle a', u'\rangle \iff (a, w \circ u) \approx_{P_f(V)} (a', w \circ u').$$

First, observe that w preserves $\sim$: if $\langle a, u\rangle \sim \langle a', u'\rangle$, then $(a, w \circ u) \approx_{P_f(V)} (a', w \circ u')$, hence

$$w\langle a, u\rangle = v(a, w \circ u) \approx_V v(a', w \circ u') = w\langle a', u'\rangle.$$

To see that w preserves $\approx_W$, we show that $\approx_W \subseteq \sim$. This is the case because $\sim$ is a prefixed point of the operator Φ. Indeed, suppose $\langle a, u\rangle\ \Phi(\sim)\ \langle a', u'\rangle$. Then $a \approx_A a'$, and for every $b, b' \in |B|$ such that $b \approx_B b'$ and $f(b) \approx_A a$ we have $u(b) \sim u'(b')$. Since w preserves $\sim$, it follows that $w(u(b)) \approx_V w(u'(b'))$, therefore $(a, w \circ u) \approx_{P_f(V)} (a', w \circ u')$, which is just $\langle a, u\rangle \sim \langle a', u'\rangle$.

3 M-Types in PEqu

In this section we prove that every polynomial functor P_f in **PEqu** has a final coalgebra $M = M(f)$.

(1) The underlying lattice $|M|$: Let $|M|$ be the algebraic lattice $|W|$ defined in Section 2. We consistently switch the notation from W's to M's to indicate the duality between W-types and M-types.

(2) The partial equivalence relation $\approx_M$: Recall that in Section 2 we defined a monotone operator Φ on the complete lattice $\mathbf{PER}(|M|)$ and considered its least fixed point. Now let $\approx_M$ be the *greatest* fixed point of the operator Φ, and let $M = (|M|, \approx_M)$.

(3) M is a P_f-coalgebra: To show that $[\langle \Box, \Box\rangle^{-1}]\colon M \to P_f(M)$ is a P_f-coalgebra all that needs to be checked is that $\langle \Box, \Box\rangle^{-1}$ preserves $\approx_M$. The proof is analogous to the case of W-types and $\approx_W$, since $\approx_M$ is a fixed point of Φ.

(4) Uniqueness of homomorphisms: Let $[n]\colon N \to P_f(N)$ be a P_f-coalgebra and suppose that $[s], [t]\colon N \to M$ are P_f-coalgebra morphisms. We show that $[s] = [t]$. Let $\sim_0$ be the relation on $|M|$ defined by

$$\langle a, u\rangle \sim_0 \langle a', u'\rangle$$
$$\text{if and only if}$$
$$\exists\, x, x' \in |N|\,.\,(x \approx_N x' \wedge \langle a, u\rangle \approx_M s(x) \wedge \langle a', u'\rangle \approx_M t(x))\,,$$

and let $\sim$ be the least partial equivalence relation that contains $\sim_0$. In other words, $\sim$ is the transitive closure of the symmetric closure of $\sim_0$.

We show that $\sim$ is a postfixed point of Φ, i.e., that $\sim \subseteq \Phi(\sim)$, from which it follows that $\sim \subseteq \approx_M$ because $\approx_M$ is the greatest postfixed point of Φ. Then $[s] = [t]$ holds because $\sim$ is defined so that $x \approx_N x'$ implies $s(x) \sim t(x')$.

By the remarks at the end of the second paragraph in Section 2, in order to show that $\sim \subseteq \Phi(\sim)$, we only need to check that $\sim_0 \subseteq \Phi(\sim_0)$. Suppose that for some $x, x' \in |N|$ it is the case that $x \approx_N x'$, $\langle a, u\rangle \approx_M s(x)$ and $\langle a', u'\rangle \approx_M t(x)$. Taking into account that $[s]$ and $[t]$ are P_f-coalgebra morphisms, we see that

$$\langle a, u\rangle \approx_M s(x) \approx_M \langle n_1(x), s \circ n_2(x)\rangle$$
$$\langle a', u'\rangle \approx_M t(x') \approx_M \langle n_1(x'), t \circ n_2(x')\rangle,$$

where $n = (n_1, n_2)\colon |N| \to |A| \times |N|^{|B|}$. Since $\approx_M$ is a fixed point of Φ, it follows that

$$a \approx_A n_1(x) \approx_A n_1(x') \approx_A a'. \tag{1}$$

Also, if $b, b' \in |B|$, $b \approx_B b'$, and $f(b) \approx_A a$ then $n_2(x)(b) \approx_N n_2(x')(b')$ and

$$u(b) \approx_M s(n_2(x)(b)),$$
$$u'(b') \approx_M t(n_2(x')(b')).$$

By definition of $\sim_0$,

$$u(b) \sim u'(b'). \tag{2}$$

Putting (1) and (2) together, we get $\langle a, u\rangle\ \Phi(\sim_0)\ \langle a', u'\rangle$, as required.

(5) Existence of homomorphisms: Let $[n]\colon N \to P_f(N)$ be a P_f-coalgebra. We show that there is a P_f-coalgebra homomorphism $[m]\colon N \to M$. Define a continuous operator $\Psi\colon |M|^{|N|} \to |M|^{|N|}$ by

$$(\Psi g)x = \langle n_1(x), g \circ n_2(x)\rangle,$$

where $n = (n_1, n_2)\colon |N| \to |A| \times |N|^{|B|}$. Let m be the least fixed point of Ψ, so that for all $x \in |N|$,

$$m(x) = \langle n_1(x), m \circ n_2(x)\rangle. \tag{3}$$

We need to prove that m represents a morphism $[m]\colon N \to M$. Let $\sim_0$ be a relation on $|M|$ defined by

$$\langle a, u\rangle \sim_0 \langle a', u'\rangle$$
$$\text{if and only if}$$
$$\exists\, x, x' \in |N| \,.\, (x \approx_N x' \wedge \langle a, u'\rangle \approx_M m(x) \wedge \langle a', u'\rangle \approx_M m(x'))\,,$$

and let $\sim$ be the least partial equivalence relation that contains $\sim_0$. Just like in the previous paragraph, we can easily check that $\sim \subseteq \approx_M$ by verifying that $\sim \subseteq \Phi(\sim)$.

The map m represents a morphism $[m]\colon N \to M$ because $x \approx_N x'$ implies $m(x) \sim m(x')$, which in turn implies $m(x) \approx_M m(x')$. That $[m]$ is a P_f-coalgebra homomorphism is expressed exactly by the fixed point property (3).

4 Induction and Coinduction Principles

A *polynomial functor* is a functor built up from identity, constants, finite products, and finite coproducts, see [HJ96] for a more precise definition.

Since polynomial functors are special cases of the functors P_f, any polynomial functor T on the category of equilogical spaces has an initial algebra and a final coalgebra.

Consider the logic of subobjects $\begin{array}{c}\scriptstyle \mathrm{Sub}(\mathbf{PEqu})\\ \downarrow\\ \scriptstyle \mathbf{PEqu}\end{array}$ described in detail in [Bir99]. (Here we do not consider dependencies.) Define $\begin{array}{c}\scriptstyle \mathrm{Rel}(\mathbf{PEqu})\\ \downarrow\\ \scriptstyle \mathbf{PEqu}\end{array}$ by change-of-base:

$$
\begin{array}{ccc}
\mathrm{Rel}(\mathbf{PEqu}) & \longrightarrow & \mathrm{Sub}(\mathbf{PEqu}) \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{PEqu} & \xrightarrow[X \mapsto X \times X]{} & \mathbf{PEqu}.
\end{array}
$$

Thus the fibre category $\mathrm{Rel}(\mathbf{PEqu})_X$ over $X \in \mathbf{PEqu}$ is the subobjects on $X \times X$, *i.e.*, binary relations on X.

Every polynomial functor $T\colon \mathbf{PEqu} \to \mathbf{PEqu}$ can be lifted to a functor $\mathrm{Pred}(T)\colon \mathrm{Sub}(\mathbf{PEqu}) \to \mathrm{Sub}(\mathbf{PEqu})$, called the **logical predicate lifting** of T, by induction on the structucture of T as described in [HJ96]: Every constant $A \in \mathbf{PEqu}$ occurring in T is replaced by the true predicate $\top_A$ and the bicartesian structure of $\mathbf{PEqu}$ used in T is replaced by the bicartesian structure in $\mathrm{Sub}(\mathbf{PEqu})$ (*i.e.*, $\wedge$ and $\vee$).

Similarly, a polynomial functor T can be lifted to a functor $\mathrm{Rel}(T)\colon \mathrm{Rel}(\mathbf{PEqu}) \to \mathrm{Rel}(\mathbf{PEqu})$, called the **logical relation lifting** of T, by induction on the structure of T. Now we replace a constant $A \in \mathbf{PEqu}$ occurring in T by the equality predicate $\mathrm{Eq}(A) = \coprod_{\delta}(A)(\top_A) \in \mathrm{Sub}(A \times A) = \mathrm{Rel}(A)$, where $\delta(A)$ is the diagonal on A.

Because $\mathbf{PEqu}$ is bicartesian closed with disjoint and stable coproducts (hence distributive), and $\begin{array}{c}\scriptstyle \mathrm{Sub}(\mathbf{PEqu})\\ \downarrow\\ \scriptstyle \mathbf{PEqu}\end{array}$ is a first-order fibration, it admits comprehension (subset types) and has quotient types, see [Bir99], we can conclude from the general results in [HJ96], that the following induction and coinduction principles are valid.

Induction Principle Let T be a polynomial functor and let $c\colon TD \to D$ be the initial T-algebra. Let $s\colon TX \to X$ be any T-algebra and let $!\colon D \to X$ be the unique algebra map. The following inference rule is valid, for any prediate $\varphi \in \mathrm{Sub}(\Gamma, x\colon X)$.

$$
\frac{\Gamma, x\colon TX \mid \Theta, \mathrm{Pred}(T)(\varphi)(x) \vdash \varphi(s(x))}{\Gamma, d\colon D \mid \Theta \vdash \varphi(!d)}
$$

Example 4.1. Let T be the functor $X \mapsto 1 + N \times X$, with N the natural numbers object of **PEqu**. Write $s = [n, c] \colon TL \to L$ for the initial algebra of T. Let $TD \to D$ above be $TL \to L$ and let $TX \to X$ above also be $TL \to L$, so $! = id$. Let $\varphi \in \mathrm{Sub}(\Gamma, l \colon L)$ be any predicate. Then the inference rule specializes to the expected induction principle for lists

$$\frac{\Gamma \mid \Theta \vdash \varphi(n) \qquad \Gamma, m \colon N, l \colon L \mid \Theta, \varphi(l) \vdash \varphi(c(m, l))}{\Gamma, l \colon L \mid \Theta \vdash \varphi(l)}$$

Coinduction Principle Let T be a polynomial functor and let $c \colon D \to TD$ be the final T-coalgebra. Let $s \colon X \to TX$ be any T-coalgebra and let $! \colon X \to D$ be the unique coalgebra map. The following inference rule is valid, for any relation $R \in \mathrm{Rel}(X)$,

$$\frac{\Gamma, x, y \colon X \mid \Theta, R(x, y) \vdash \mathrm{Rel}(T)(R)(sx, sy)}{\Gamma, x, y \colon X \mid \Theta \vdash !x =_D !y}$$

5 Comments

If $|B|$ and $|A|$ are countably based algebraic lattices then also the lattice $|W|$ is countably based. This means that ω**PEqu**, the countably based version of **PEqu**, has W-types and M-types as well.

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