Computer Science Day:
Mathematical Computer Science

Thomas Dueholm Hansen

June 15, 2011

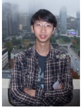
The Mathematical Computer Science group

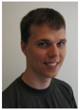
- Faculty:
  - Gudmund Skovbjerg Frandsen
  - Kristoffer Arnsfelt Hansen
  - Peter Bro Miltersen
- Postdocs:
  - Elias Tsigaridas
  - Elad Verbin (also at MADALGO)
  - Joshua Brody
  - Dominik Scheder
  - Wei Yu (also at MADALGO)
  - Jie Zhang
- PhD students:
  - Søren Stiil Frederiksen
  - Thomas Dueholm Hansen
  - Rasmus Ibsen-Jensen
- Administration:
  - Dorthe Haagen Nielsen







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Research projects

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Terminated within the last year:

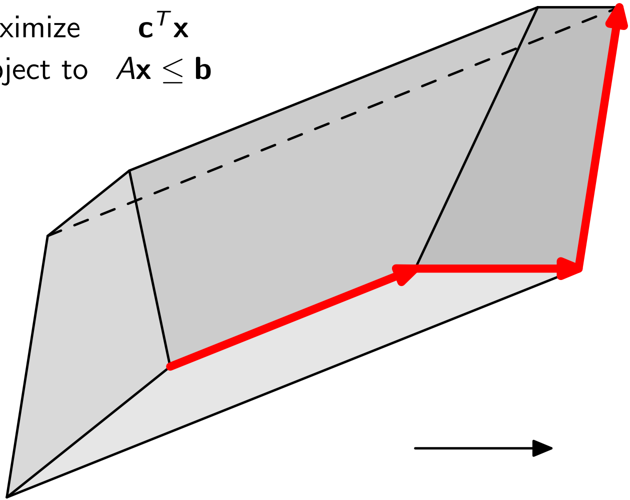
CAGT Center for Algorithmic Game Theory

COMA

Activities

- Computational Complexity Theory
- Algorithmic and Computational Game Theory
- Mathematical Programming
- Combinatorial Optimization
- Approximation Algorithms
- Computational Algebra
- Symbolic-Numerical Computations
- Real Algebraic Geometry
- Theoretical aspects of multiagent systems

maximize $\mathbf{c}^T\mathbf{x}$
subject to $A\mathbf{x} \leq \mathbf{b}$



Example: The diet problem

Buy food items in order to satisfy daily intake of energy (2000 kcal), protein (55 g) and calcium (800 mg), minimizing the cost.

Food	Serving size	Energy	Protein	Calcium	Price
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

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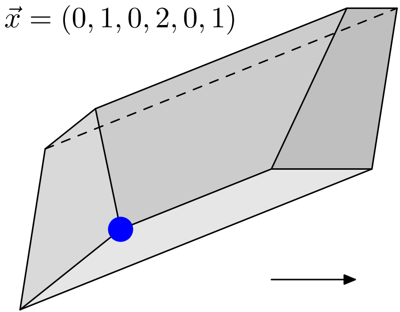
$$\begin{array}{llr} \textbf{Minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 & \\ \textbf{subject to} & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq & 2000 \\ & 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq & 55 \\ & 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq & 800 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq & 0 \end{array}$$

Basic feasible solutions and pivoting

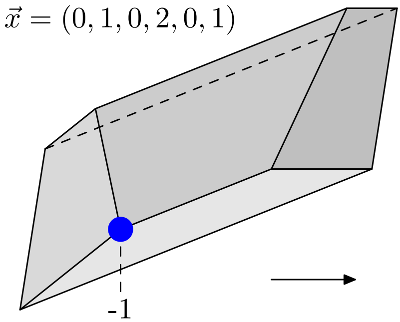
$$\begin{array}{lrcl} \max & 2x_1 - 2x_3 - 2x_5 - x_6 & & \\ \text{s.t.} & \frac{1}{3}x_1 + x_2 - \frac{2}{3}x_3 - \frac{2}{3}x_5 & = & 1 \\ & x_3 + x_4 - x_6 & = & 1 \\ & x_5 + x_6 & = & 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$



- The corners of the polytope correspond to **basic feasible solutions**: At most n, the number of equality constraints, variables are non-zero. A *basis* consists of n *basic* variables such that all non-basic variables are zero.

Basic feasible solutions and pivoting

$$\begin{array}{lrcl} \max & \multicolumn{3}{l}{-1 + 2x_1 - 2x_3 - x_5} \\ \text{s.t.} & x_2 & = & 1 - \frac{1}{3}x_1 + \frac{2}{3}x_3 + \frac{2}{3}x_5 \\ & x_4 & = & 2 - x_3 - x_5 \\ & x_6 & = & 1 - x_5 \\ & \multicolumn{3}{l}{x_1, x_2, x_3, x_4, x_5, x_6 \geq 0} \end{array}$$



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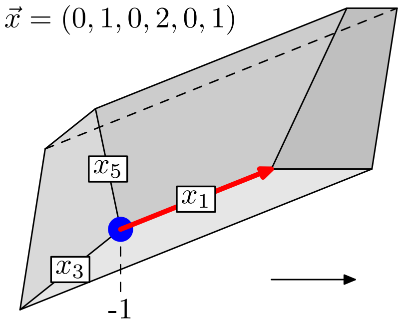

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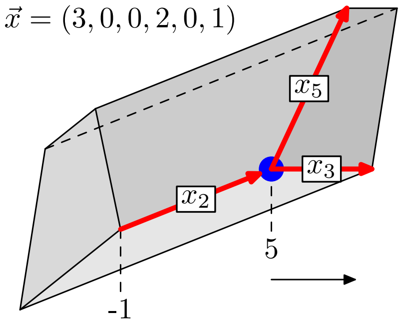
- Moving along an edge corresponds to **pivoting**: Exchange a variable in the basis with a non-basic variable.

Basic feasible solutions and pivoting

$$\begin{aligned} \max \quad & 5 - 6x_2 + 2x_3 + 3x_5 \\ \text{s.t.} \quad & x_1 = 3 - 3x_2 + 2x_3 + 2x_5 \\ & x_4 = 2 - x_3 - x_5 \\ & x_6 = 1 - x_5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$



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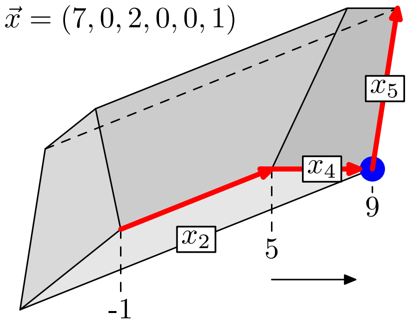
$$\begin{array}{lrcl} \max & 9 - 6x_2 - 2x_4 + x_5 & & \\ \text{s.t.} & x_1 & = & 7 - 3x_2 - 2x_4 \\ & x_3 & = & 2 - x_4 - x_5 \\ & x_6 & = & 1 - x_5 \\ & \multicolumn{3}{l}{x_1, x_2, x_3, x_4, x_5, x_6 \geq 0} \end{array}$$



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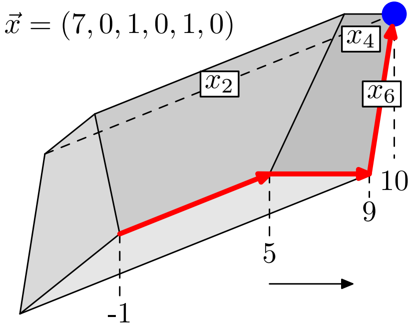
$$\begin{array}{lrcl} \max & 10 - 6x_2 - 2x_4 - x_6 & & \\ \text{s.t.} & x_1 & = & 7 - 3x_2 - 2x_4 \\ & x_3 & = & 1 - x_4 + x_6 \\ & x_5 & = & 1 - x_6 \\ & & & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$



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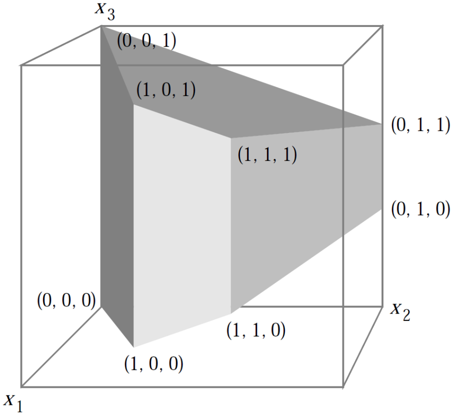
Deterministic pivoting rules

- Klee and Minty (1972): Dantzig's original LARGESTCOEFFICIENT pivoting rule may require exponentially many steps.



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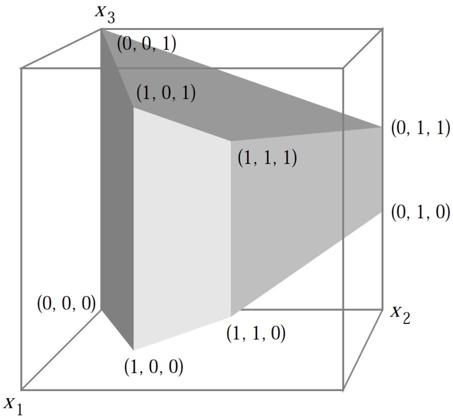

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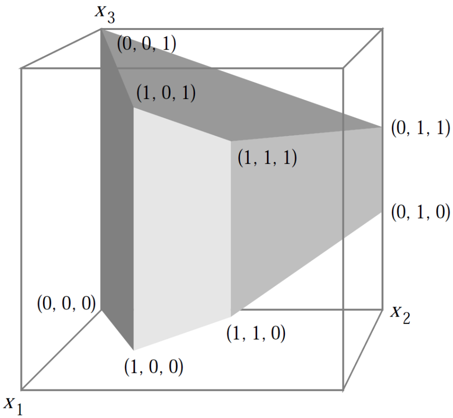

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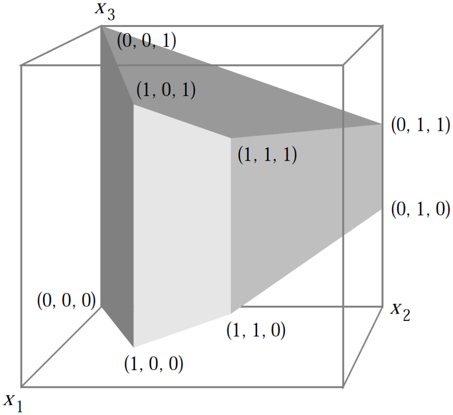
- What about randomized pivoting rules?
    - No good lower bounds were known for forty years.



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- RandomEdge: Make a uniformly random improving pivot in every step.
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Theorem (Friedmann, Hansen and Zwick (2011))

RandomEdge *and* RandomFacet *may require an expected number of pivoting steps that is almost exponential in the number of constraints and variables:* $2^{\Omega(n^{1/4})}$ *and* $2^{\Omega(\sqrt{n}/\log n)}$*, respectively.*

The end

Thank you for listening!