madalgo
CENTER FOR MASSIVE DATA ALGORITHMICS

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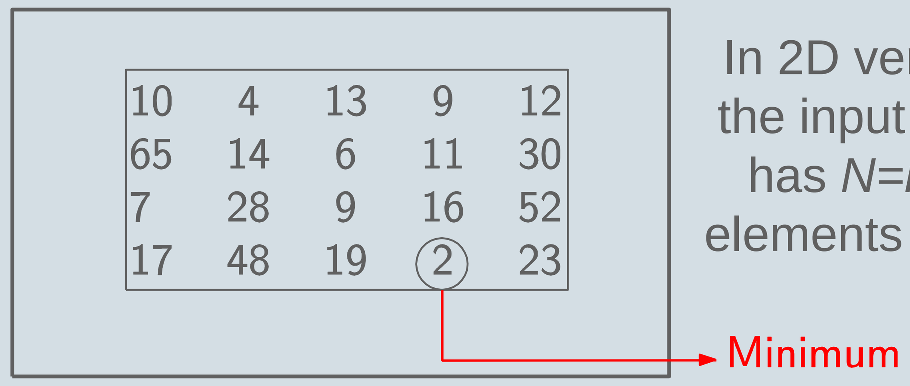
Space Efficient Range Minimum Queries

Introduction

Problem
Preprocessing an input array such that the given query asking for the index of the minimum element in a rectangular range within the array is solved efficiently.

Succinct Models
- Indexing: Probes into the input.
- Encoding: No access to the input.



In 2D version, the input array has $N=m\times n$ elements ($m\leq n$)

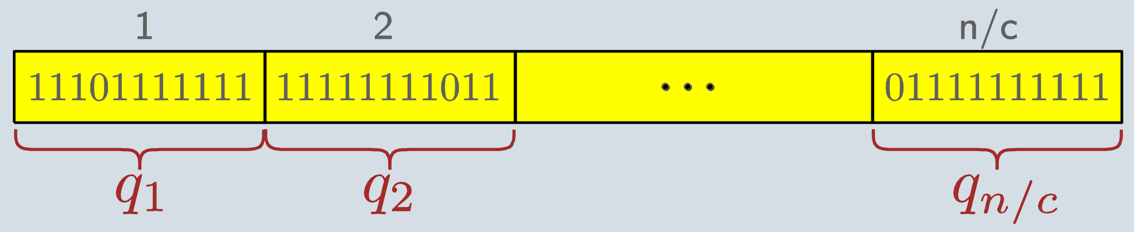
Applications
Databases, geographic information systems, graphics, computing lowest common ancestors in trees, pattern matching, document retrieval queries, maximum segment queries and more.

Indexing Lower Bound (1D and 2D)[1]

Theorem

Any RMQ algorithm using n / c bits additional space, requires $\Omega(c)$ probes into the input.

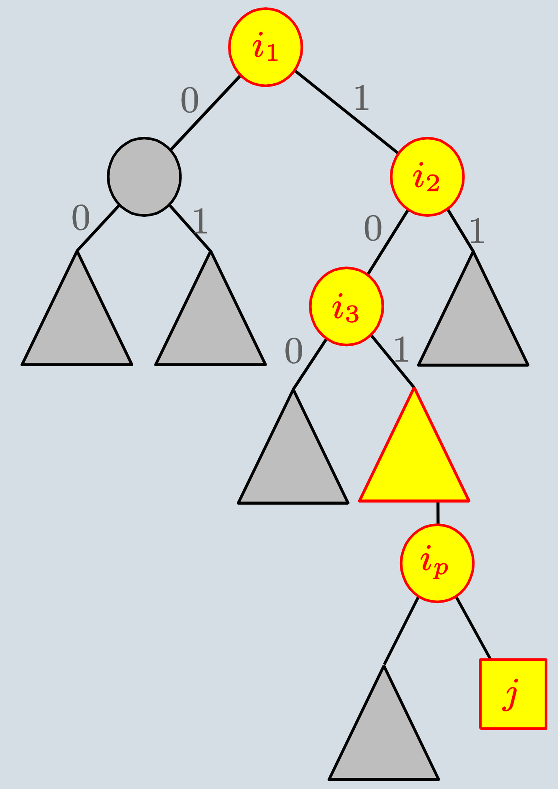
Proof
- Consider n/c queries for $c^{n/c}$ different {0,1} inputs with exactly one zero in each block.



- $c^{n/c} / 2^{n/c}$ inputs share some data structure.
- Fix the data structure.
- Every query is a decision tree of height ≤ d.

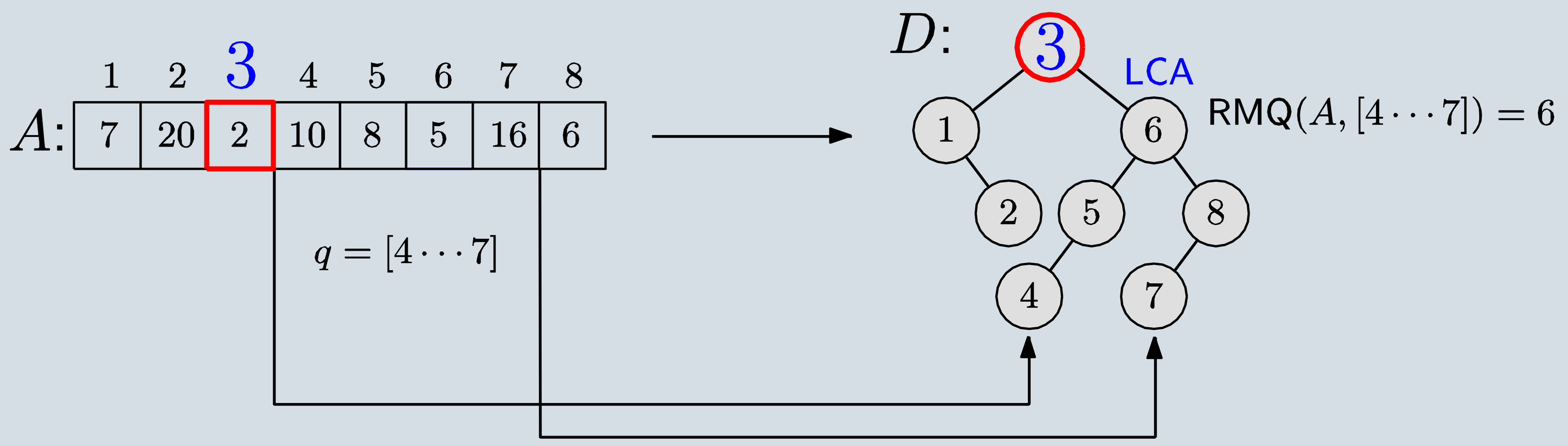
The path (1,0,1,...,1) ending at the leaf j shows that the answer of the query over the input matching with the probes of the path is j.

- Combine queries to a decision tree.
- Prune non-reachable branches for the inputs sharing the data structure.
- # zeroes on any path ≤ n/c.
- $\frac{c^{n/c}}{2^{n/c}} \leq$ #inputs = #leaves $\leq \binom{d+n/c}{n/c}$.
- Query time $d = \Omega(c)$.



One Dimensional Data Structure

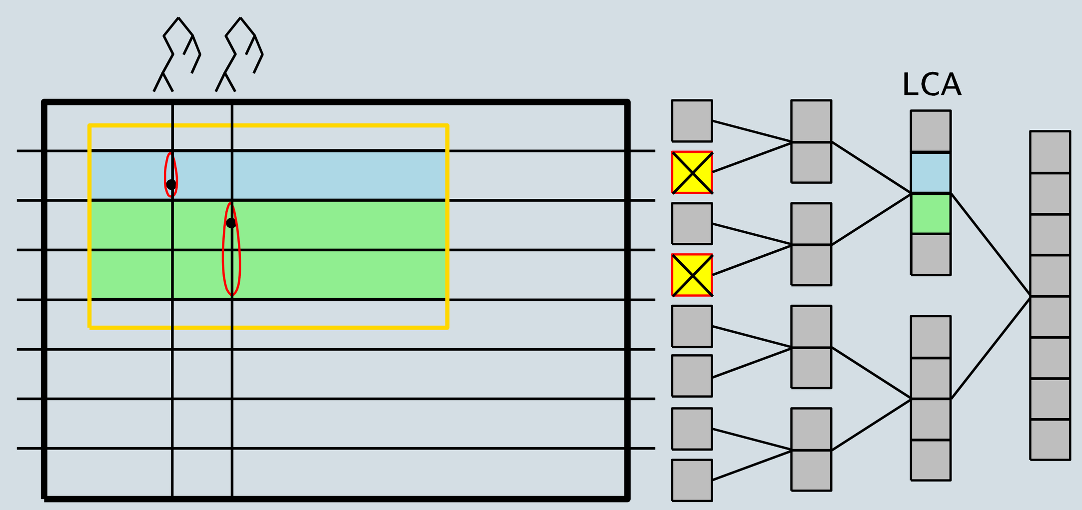
- A Cartesian Tree encodes the input 1D array of size n elements in $4n$+o(n) bits s.t. the query can be solved without accessing the input in O(1) time (Sadakane'07).



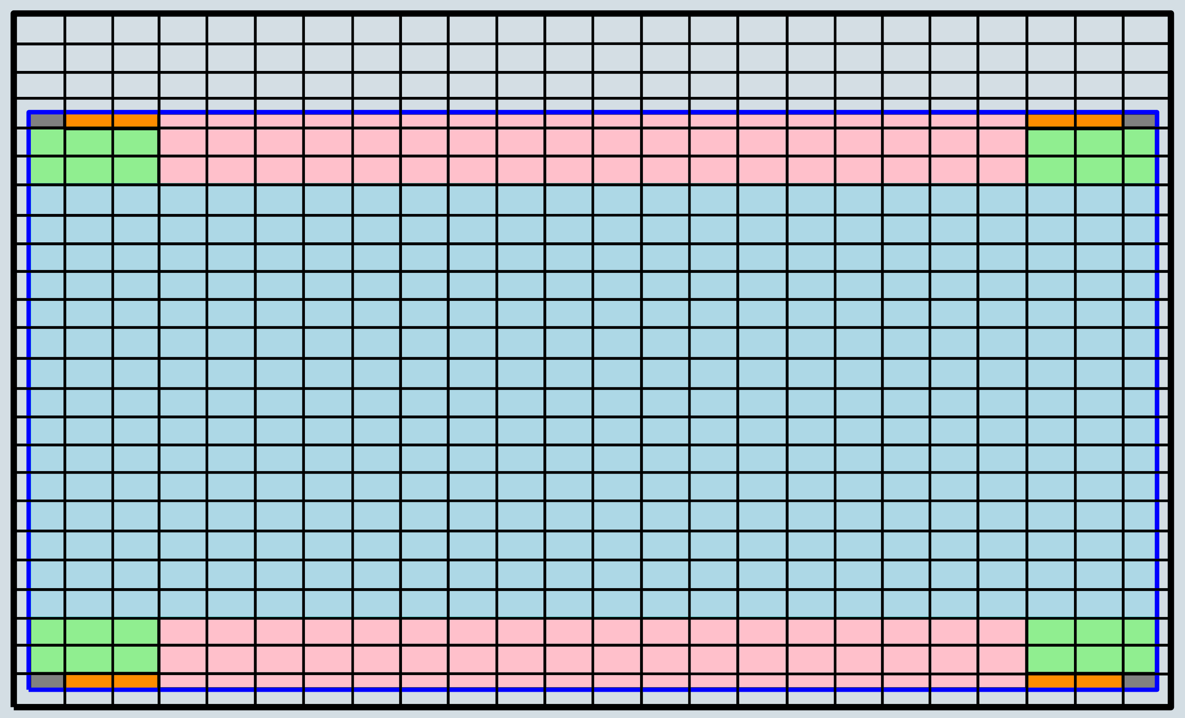
- The best indexing and encoding data structures have size $2n$+o(n) bits matching the lower bound of $2n$-Θ(log n) bits (Fischer and Heun'03, Fischer'07).

Indexing 2D Data Structure: Linear Space[2]

- Partition the input recursively.
- Construct a binary tree in each level of the recursion.
- Solve the queries spanning over the blocks.



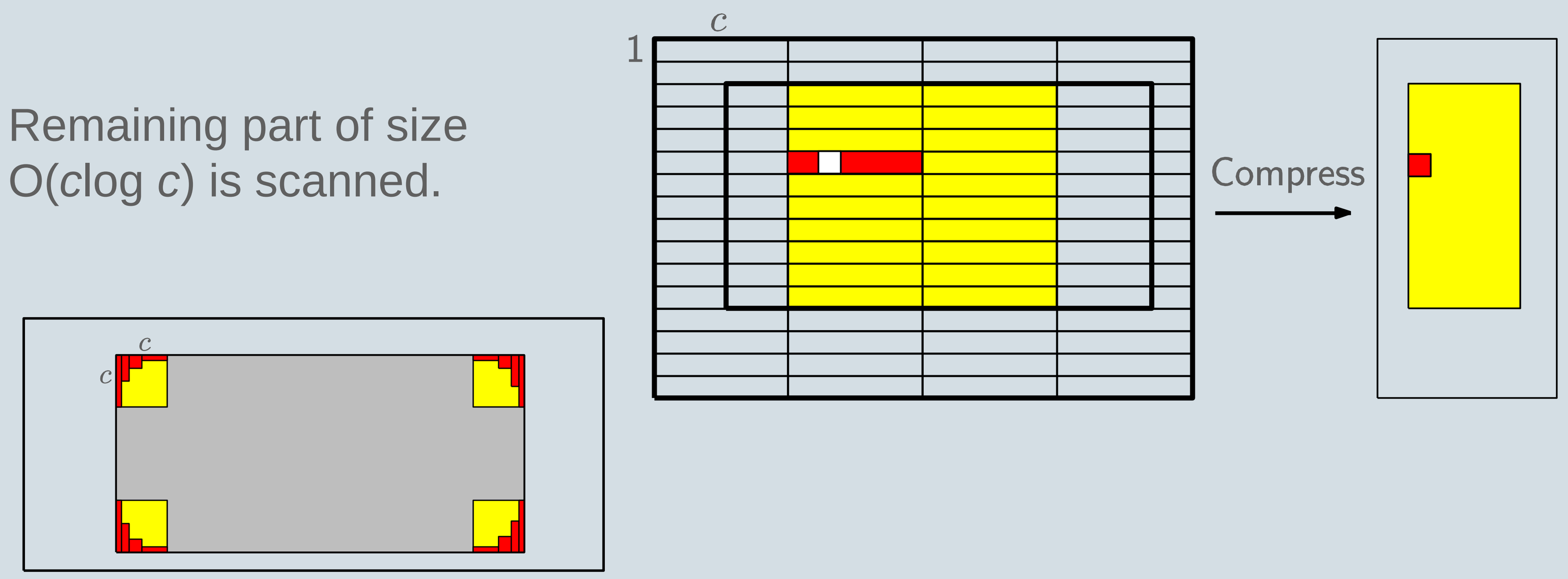
- Use Four-Russians trick in the last level where the block size is O(log N).



Indexing 2D Data Structure: Time-space Trade-off[3]

- Divide the input to blocks of size 2^i x $c/2^i$ in log c steps.
- In each step, construct an indexing data structure of size O(N/c) bits for the compressed matrix.

Remaining part of size O($c$log c) is scanned.



Results

No.	Query time	Space (bits)	Preprocessing time
1	Ω(c)	O(N/c)+|A|	-
2	O(1)	O(N)+|A|	O(N)
3	O($c\log^2 c$)	O(N/c)+|A|	O(N)
4	-	Ω(N log m)	-
5	O(1)	O(N log n)	O(N)

Encoding 2D Lower Bound[4]

- Define a set of $\left(\frac{m}{2}!\right)^{\frac{n}{2}-\frac{m}{4}}$ different matrices.

Bits required is at least:

$$\log\left(\frac{m}{2}!\right)^{\frac{n}{2}-\frac{m}{4}} = \Theta(N \log m)$$