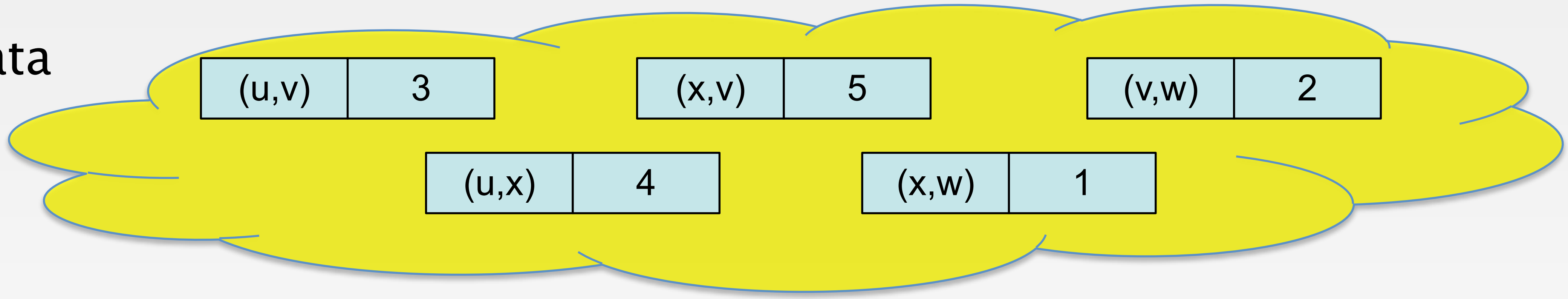
MapReduce Graph Algorithms

Sergei Vassilvitskii

Reminder: MapReduce

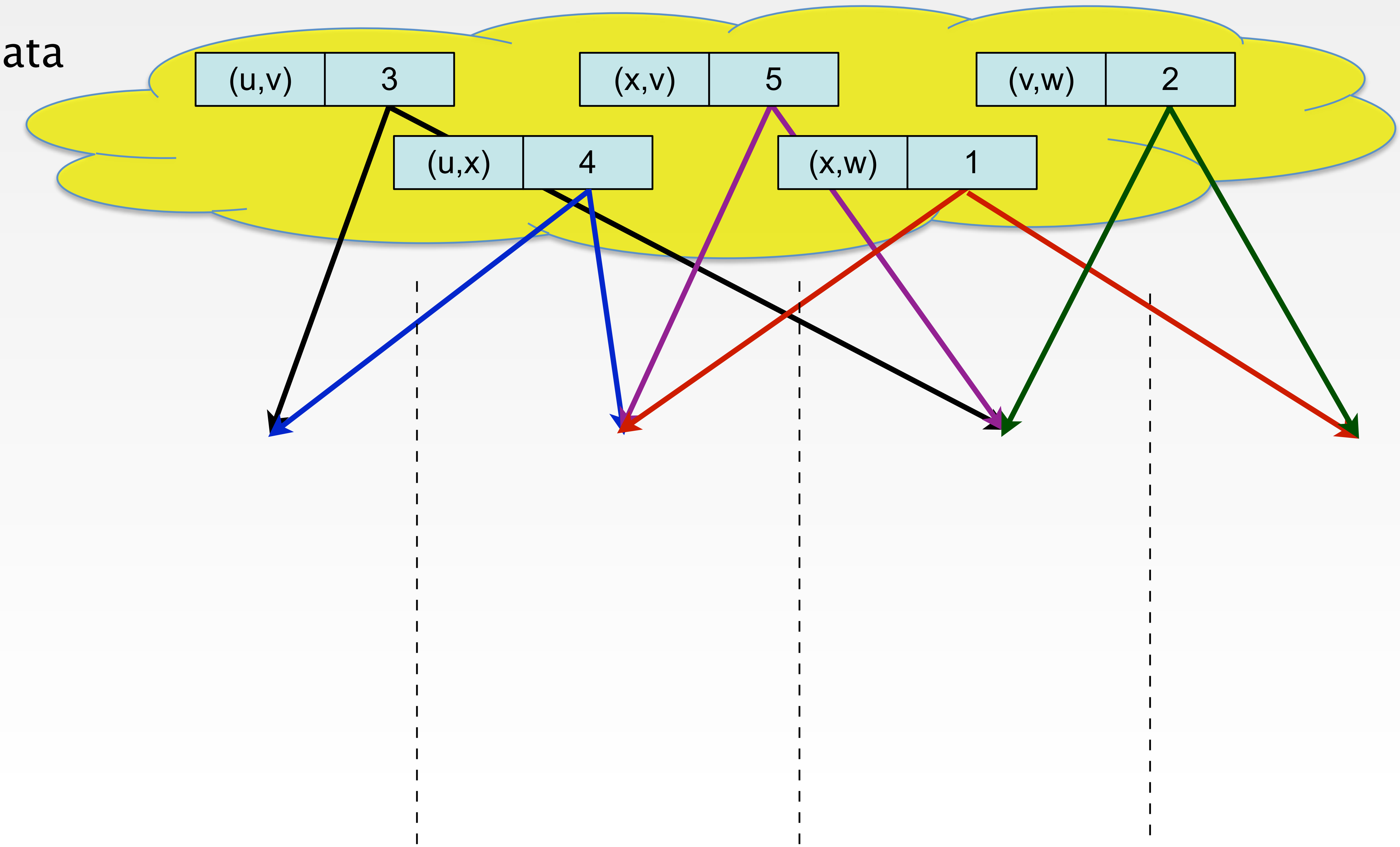
Unordered Data



MapReduce (Data View)

Unordered Data

Map



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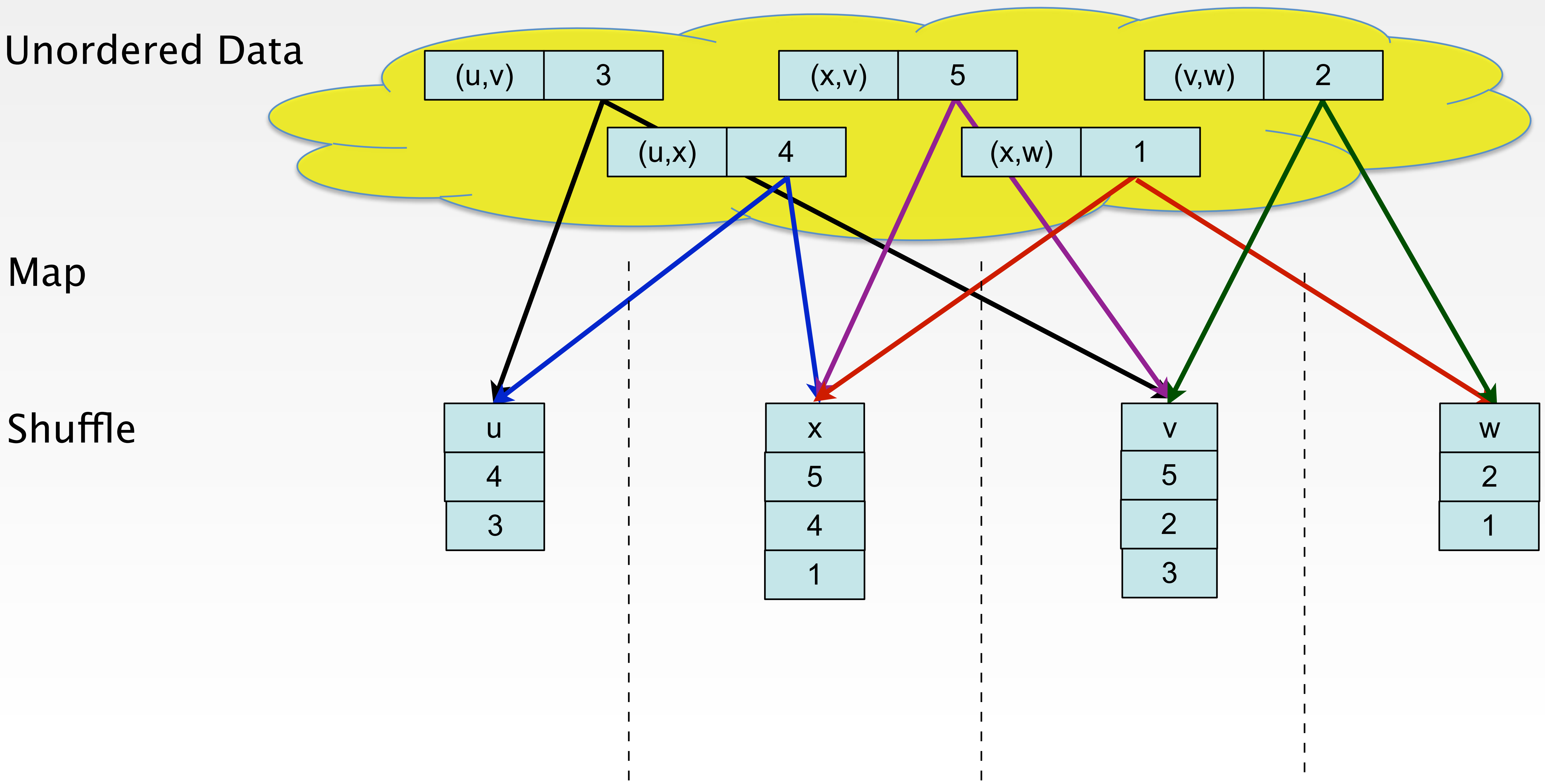

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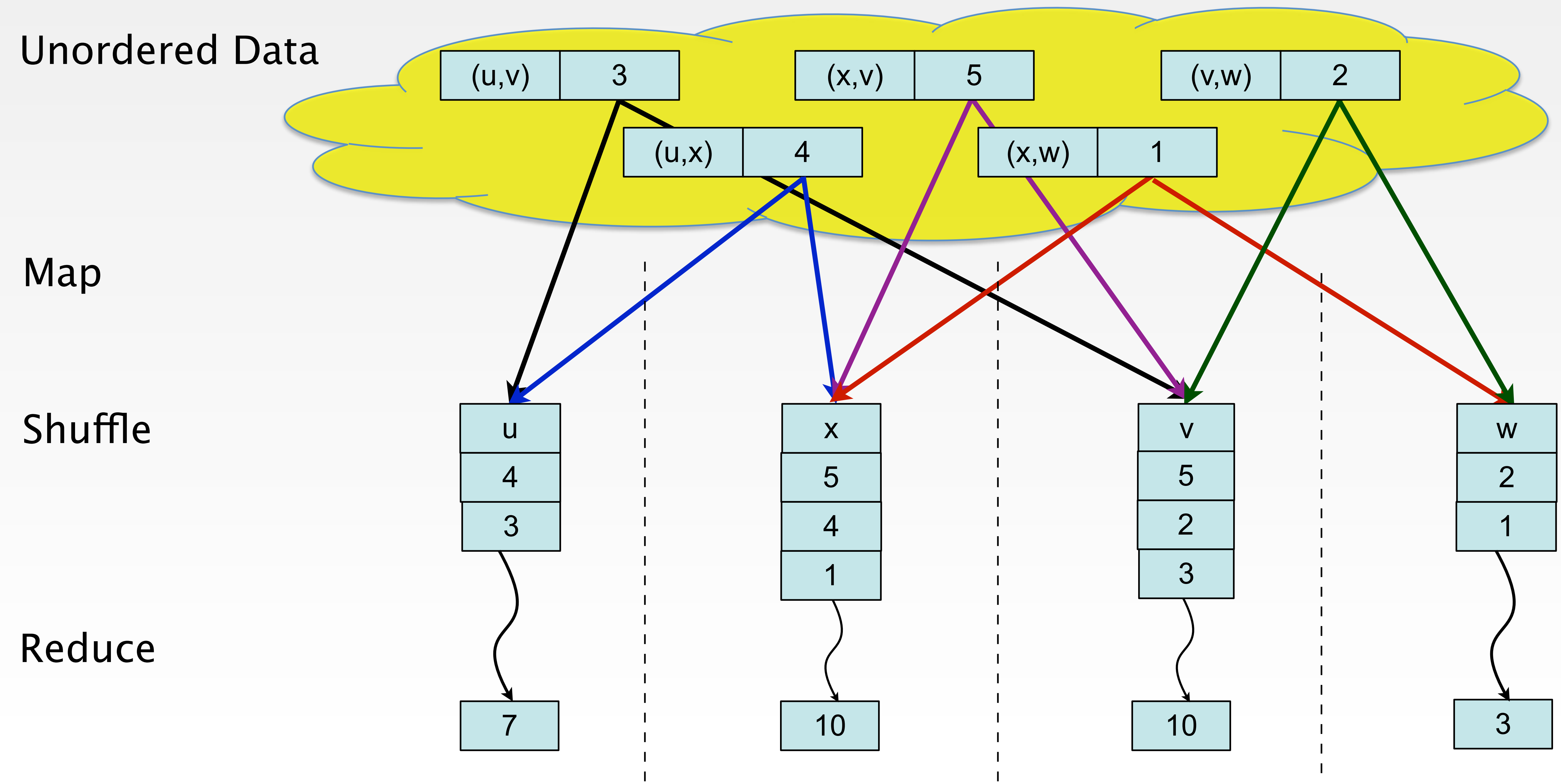

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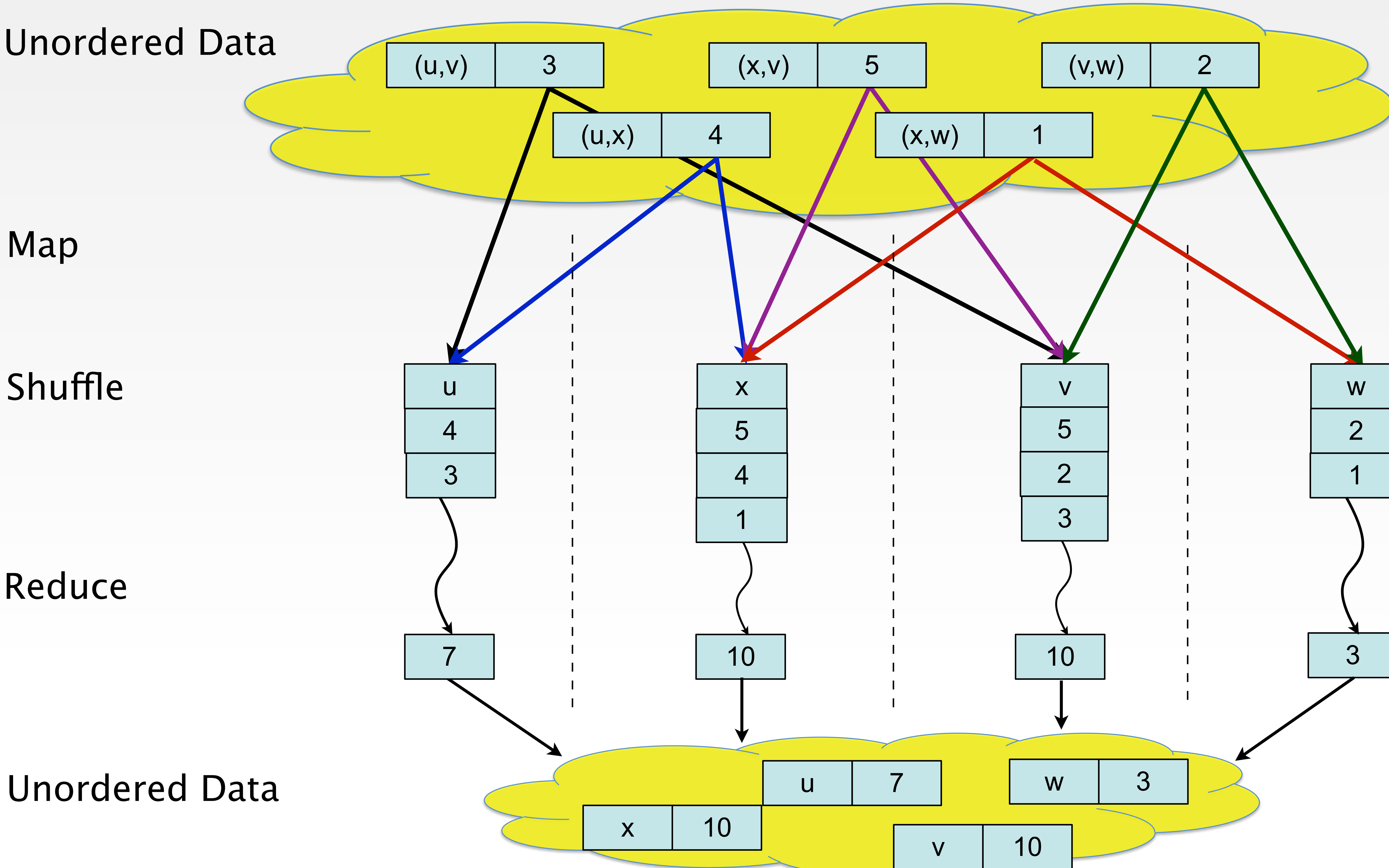


Outline: Graph Algorithms

Dense Graphs

- Connectivity
- Matching

Sparse Graphs

- Pregel/Giraph Model
- Connectivity
- Matchings

Application

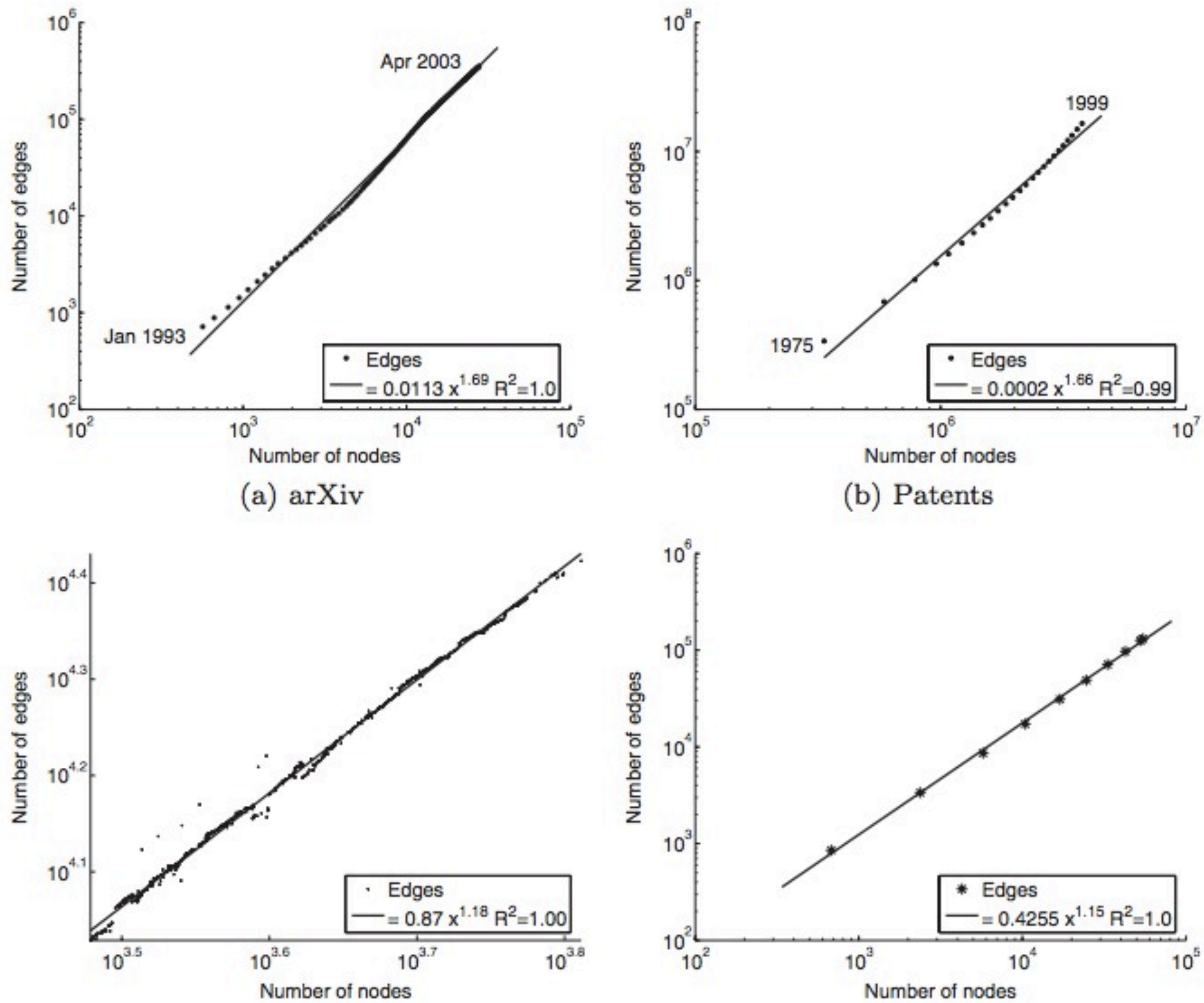
- Densest Subgraph

Dense Graphs

Are real world graphs:

- sparse: $m = \tilde{O}(n)$
- dense: $m = n^{1+c}$, for some $c > 0$?

Graphs over time



(a) arXiv

(b) Patents

(c) Autonomous Systems

(d) Affiliation network

Algorithmics

Find the core of the problem:

- Reduce the problem size in parallel
- Solve the smaller instance sequentially

Roadmap:

- Identify redundant information
- Filter out redundancy to reduce input size
- Solve the smaller problem

Connectivity

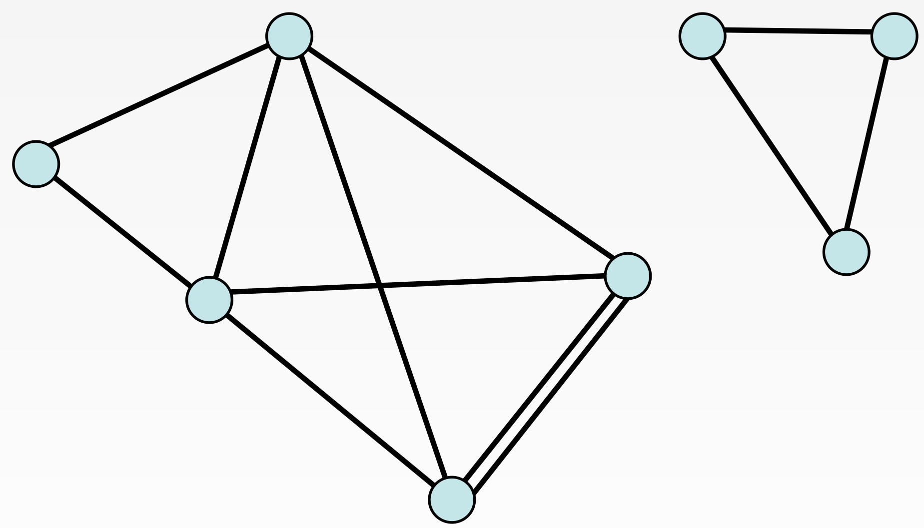
Given an undirected graph, find the number of connected components.

Sequential:

- Consider edges one at a time
- Maintain connected components (in a Union Find tree)

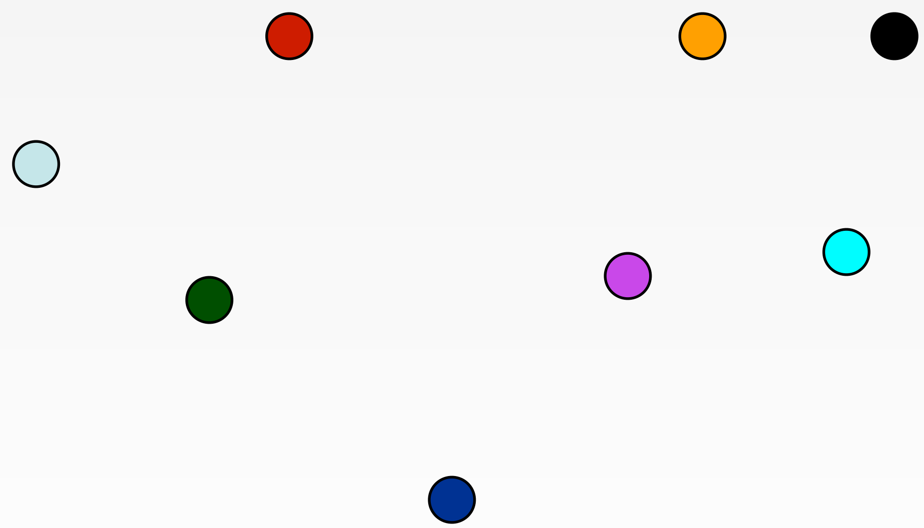
Connected Components

Given a graph:



Connected Components

Begin: Each node is a separate component



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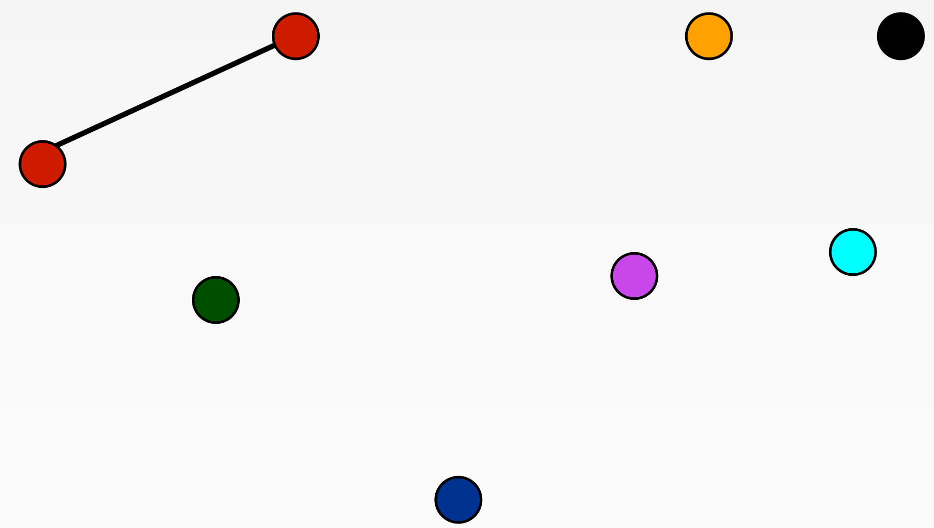

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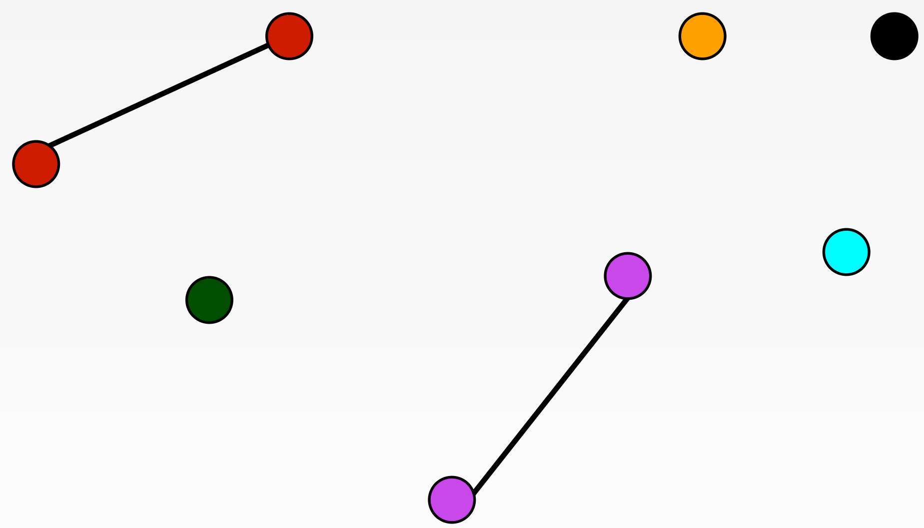

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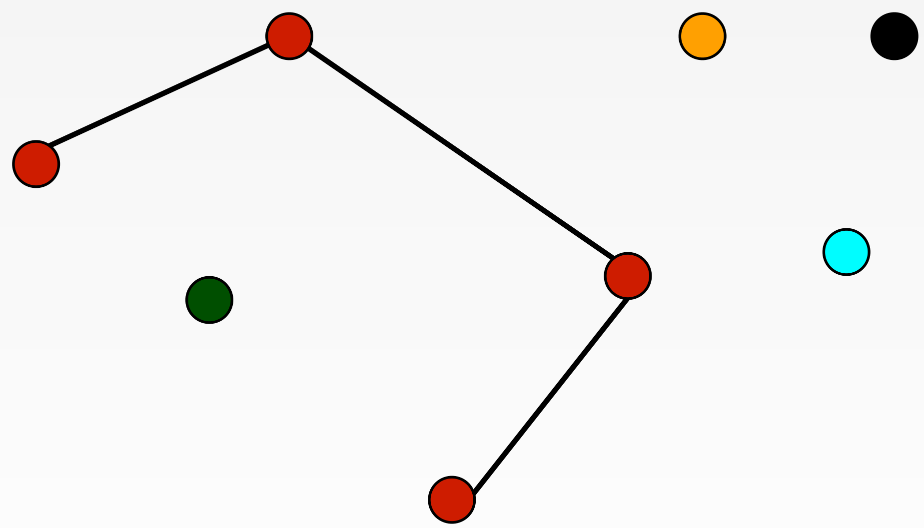
With every edge, select one of the colors

Update all of the colors in a component



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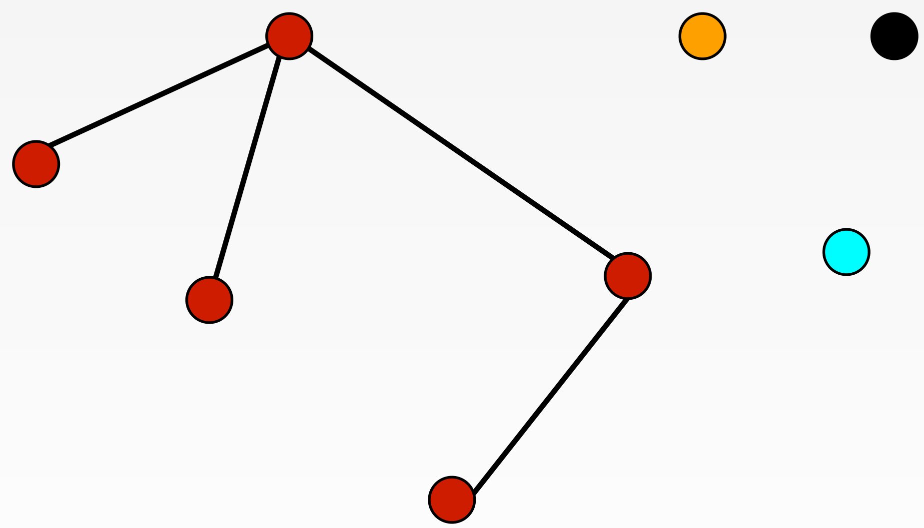

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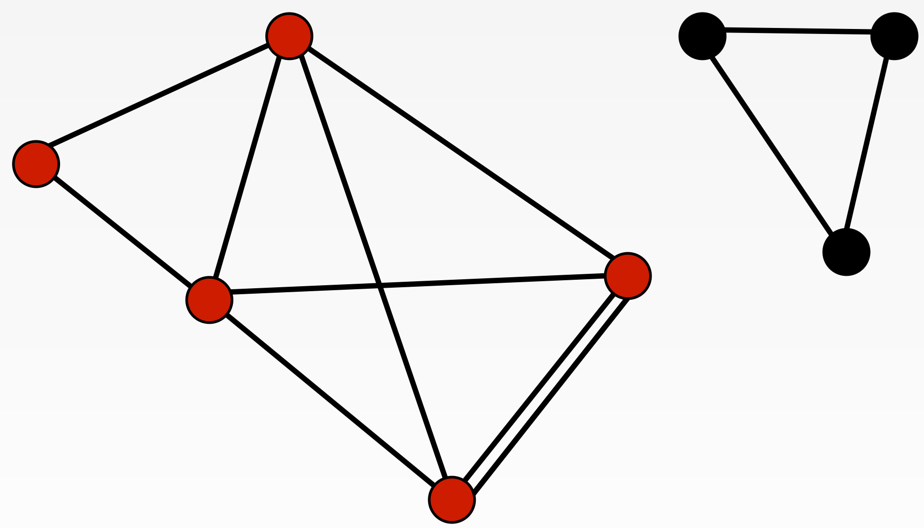

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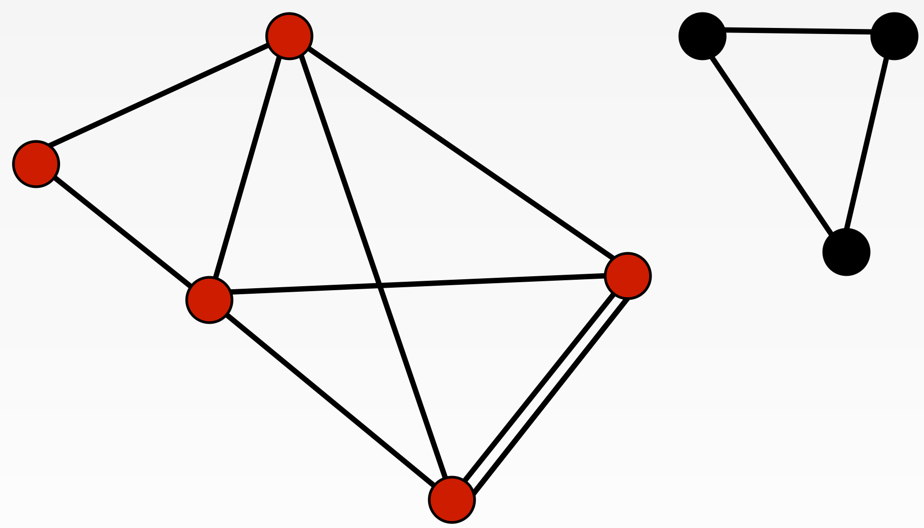

Count the number of colors: 2

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Sequential:

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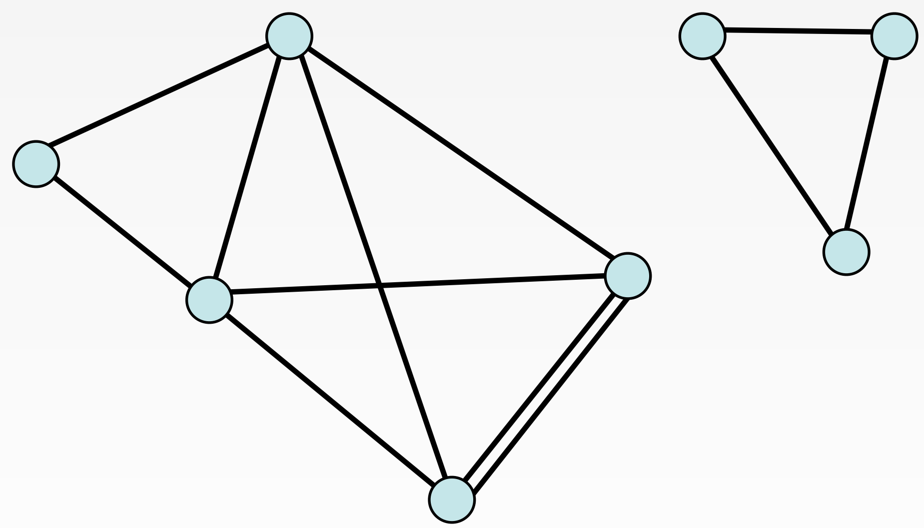
Filtering:

- What makes an edge redundant?
- If we already know the endpoints are connected

Connected Components

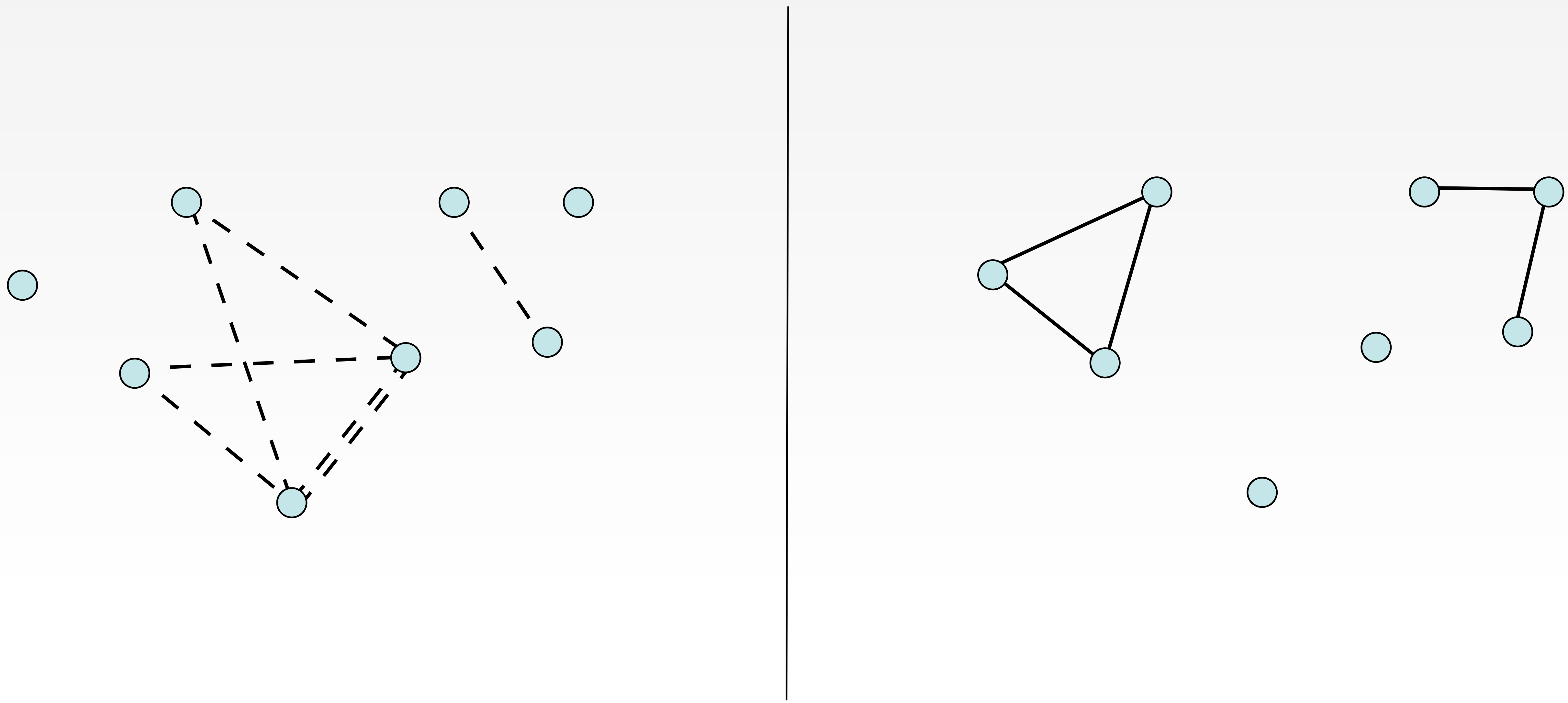
Given a graph:



Connected Components

Given a graph:

1. Partition edges (randomly)

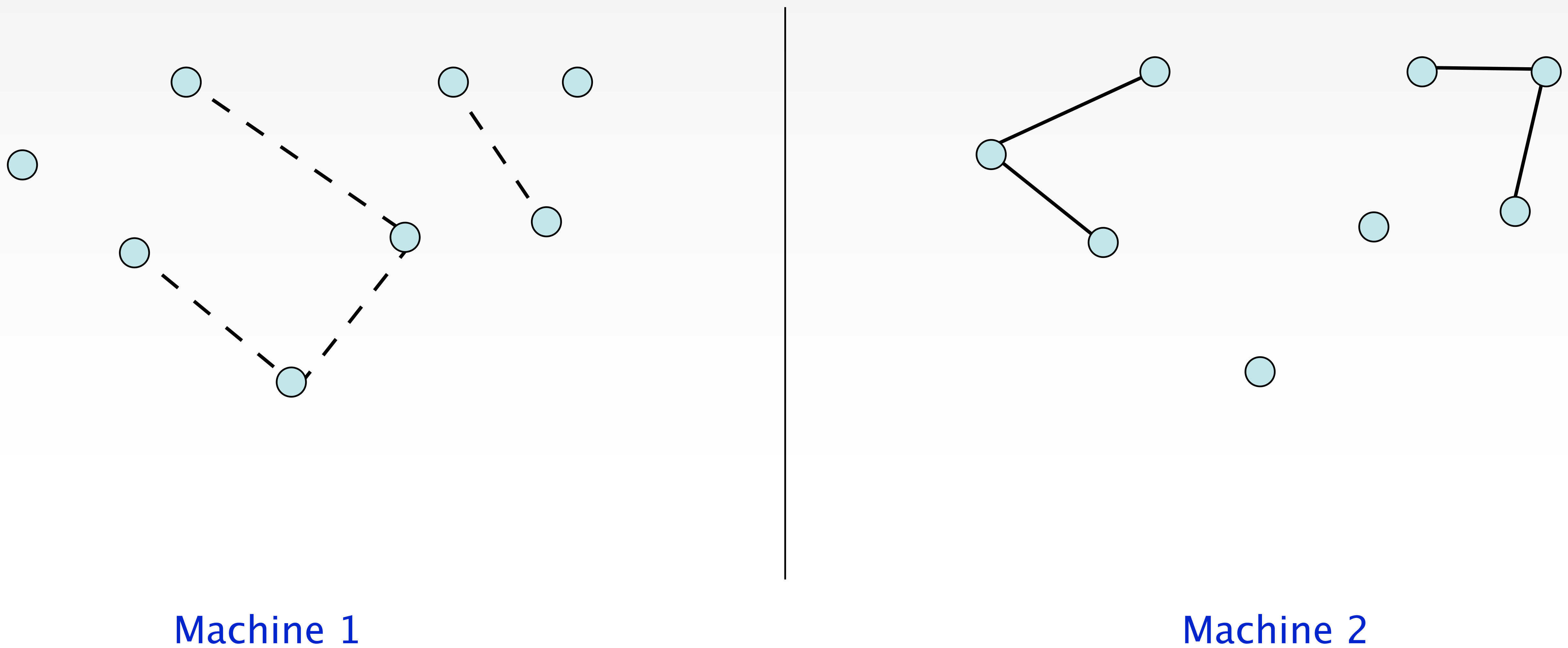


Machine 1 | Machine 2

Connected Components

Given a graph:

1. Partition edges (randomly)
2. Summarize (keep $\leq n-1$ edges per partition)



Connected Components

Given a graph:

1. Partition edges (randomly)
2. Summarize (keep $\leq n-1$ edges per partition)
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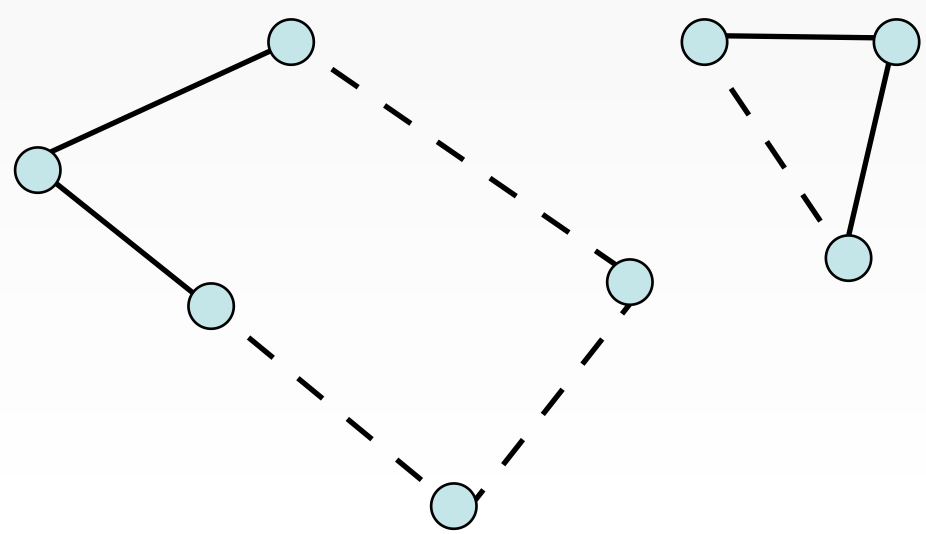

Connected Components

Given a graph:

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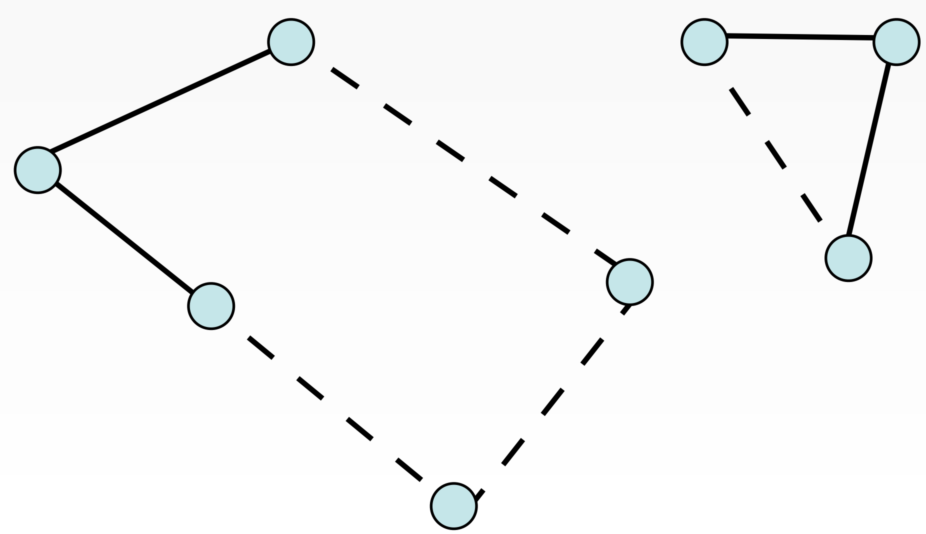
2. Summarize (keep $\leq n-1$ edges per partition)
3. Recombine
4. Compute CC's



Analysis

Given: k machines:

- Total Runtime: $T_{cc}(^m/_k) + T_{cc}(nk)$
- Memory per machine: $O(^m/_k + nk)$
  - Actually, can stream through edges so $O(n)$ suffices
- 2 Rounds total

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Notes:

- Semi-streaming model: vertices must fit in memory
- Instead of two passes can achieve a trade-off between memory and number of passes

Matchings

Finding matchings

- Given an undirected graph $G = (V, E)$
- Find a maximum matching

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Try random partitions:

- Find a matching on each partition
- Compute a matching on the matchings
- Does not work: may make very limited progress

Looking for redundancy

Matching:

- Could drop the edge if an endpoint already matched

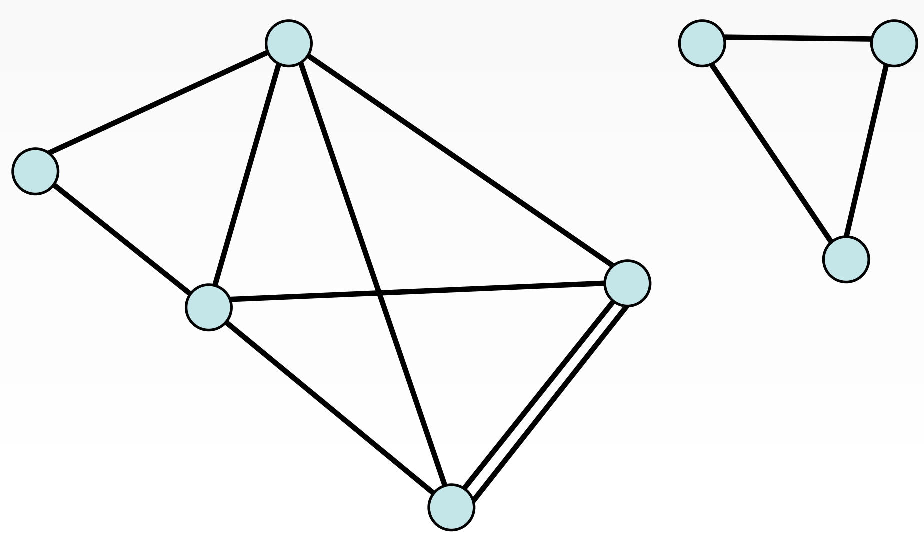
Idea:

- Find a seed matching (on a sample)
- Remove all 'dead' edges
- Recurse on remaining edges

Algorithm

Given a graph:

1. Take a random sample



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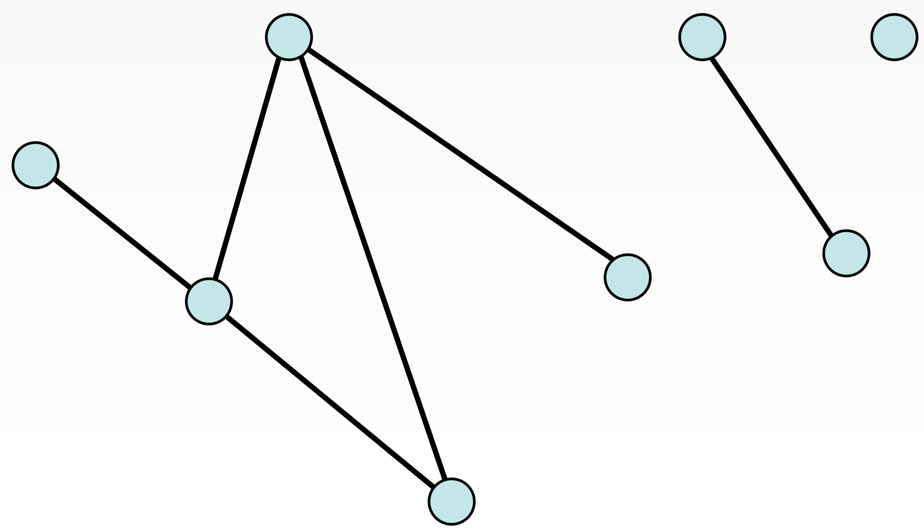
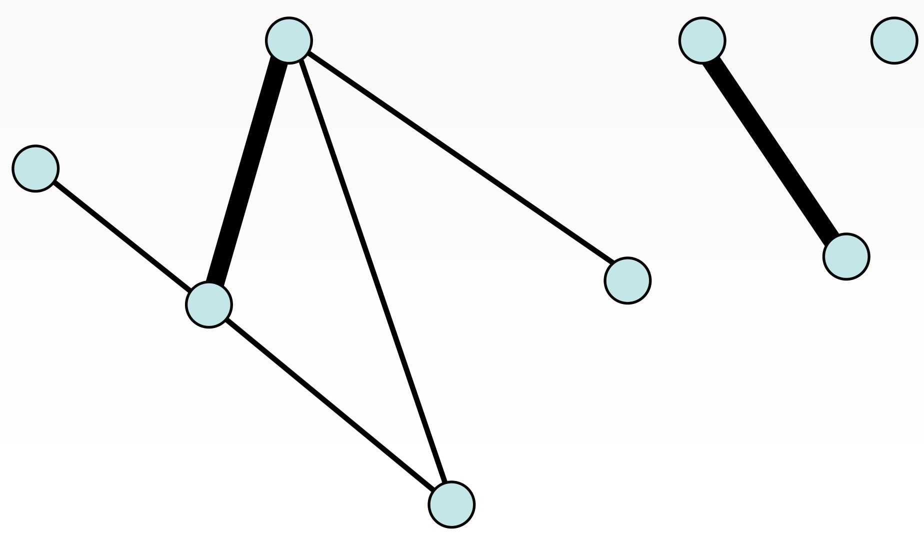


Algorithm

Given a graph:

1. Take a random sample

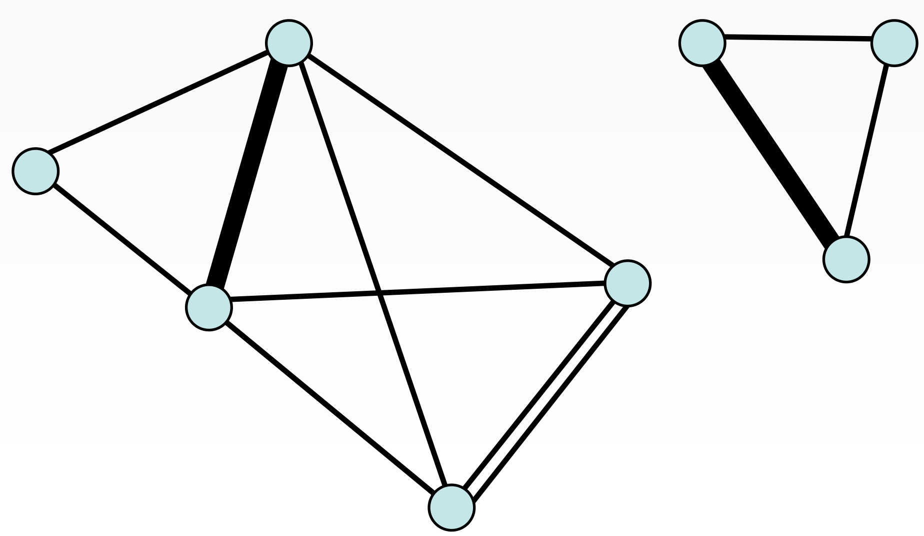
2. Find a maximal matching on sample



Algorithm

Given a graph:

1. Take a random sample
2. Find a maximal matching on sample
3. Look at original graph



Algorithm

Given a graph:

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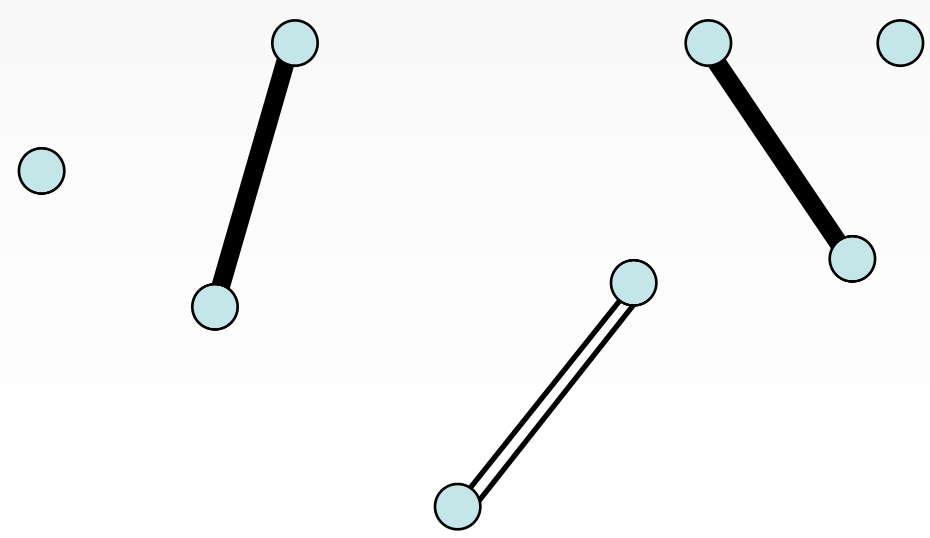
2. Find a maximal matching on sample
3. Look at original graph, drop dead edges



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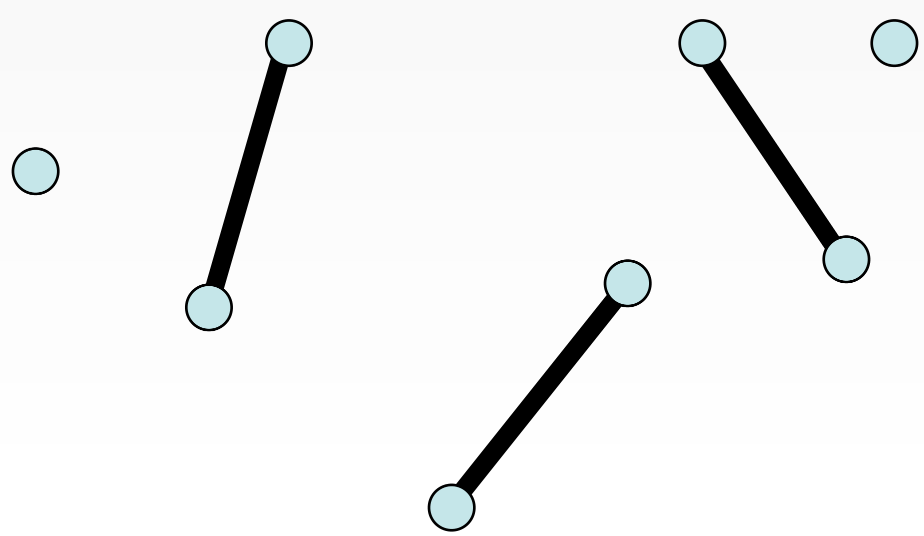

Analysis

Key Lemma:

- Suppose the sampling rate is $p = \frac{n^{1+c}}{m}$ for some $c > 0$.
- Then with high probability the number of edges remaining after the prune step is at most:

$$\frac{2n}{p} = \frac{2m}{n^c}$$

Analysis

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If $|E[I]| > O(n/p)$

i.e. we have a lot of edges left over

Then the probability that none of the edges were picked is at most

$$(1-p)^{n/p} \leq e^{-n}$$

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The total possible number of such sets I is 2^n

Thus the total probability of a bad event (too many edges left over) is:

$$2^n \cdot e^{-n} \leq 0.75^n$$

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Corollaries:

- Given n^{1+c} memory, algorithm requires $O(1)$ rounds
- Given $O(n \log n)$ memory, algorithm requires $O(\frac{\log n}{\log \log n})$ rounds.
- PRAM simulations: $\Theta(\log n)$ rounds

Outline: Graph Algorithms

Dense Graphs

- Connectivity
- Matching

Sparse Graphs

- Pregel Model
- Connectivity
- Matchings

Application

- Densest Subgraph

Optimizing Graphs

Computation:

- Most often computation is along the edges
- Djikstra's shortest path algorithm

Data:

- Graph itself usually does not change
- Pass values around vertices

Pregel & Giraph

Optimizations:

- Partition the graph once across the machines
- Keep the graph structure local (don't shuffle it!)

Pregel & Giraph

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Implications:

- Vertex central view of the data
- Each round:
  - Each vertex collects all messages sent to it
  - Send messages to its neighbors
  - Can also modify the graph

Pregel & Giraph

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Implications:

- Vertex central view of the data
- Each round:
  - Each vertex collects all messages sent to it
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  - Can also modify the graph
- Under the covers:
  - Vertices act as a key
  - Edges are stored locally with each vertex, reducing shuffle time

Example: BFS

```
for each vertex v:
   for every message received: m
      if (value > m)
          value = m;
if (value changed)
    SendMessageToAllNeighbors(value + 1);
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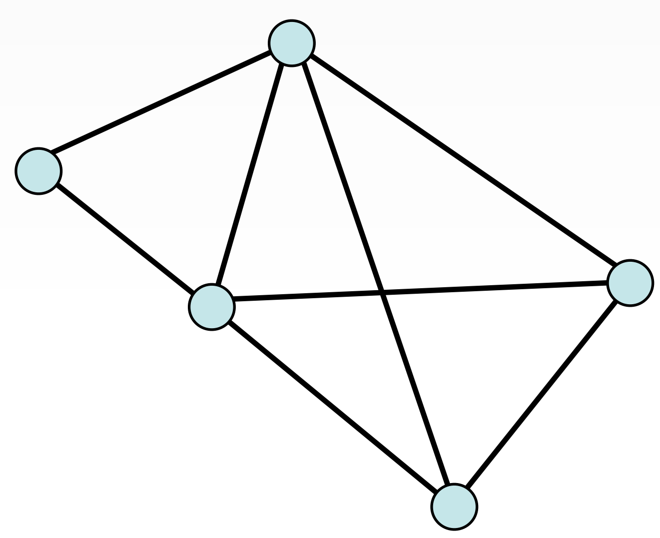

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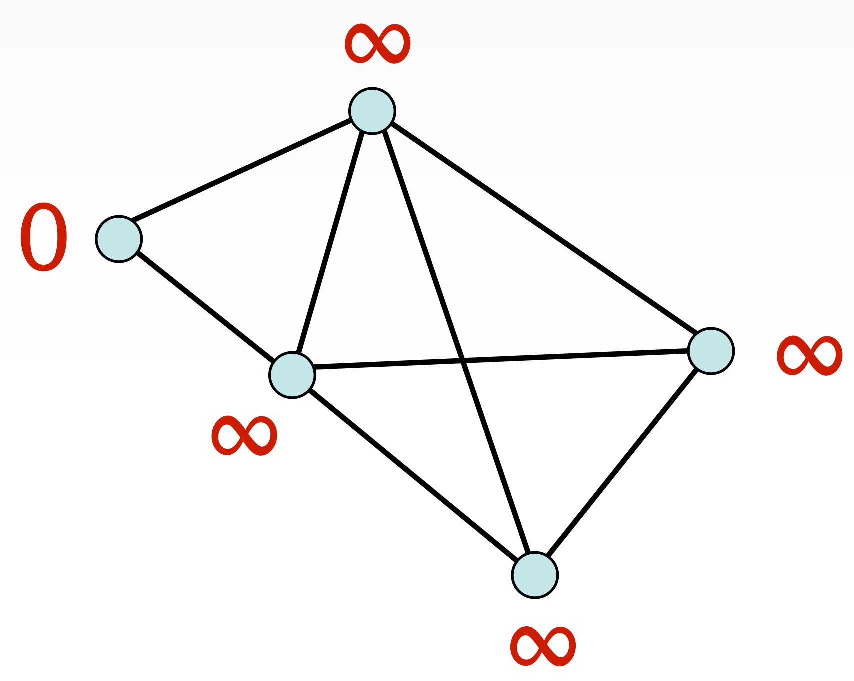

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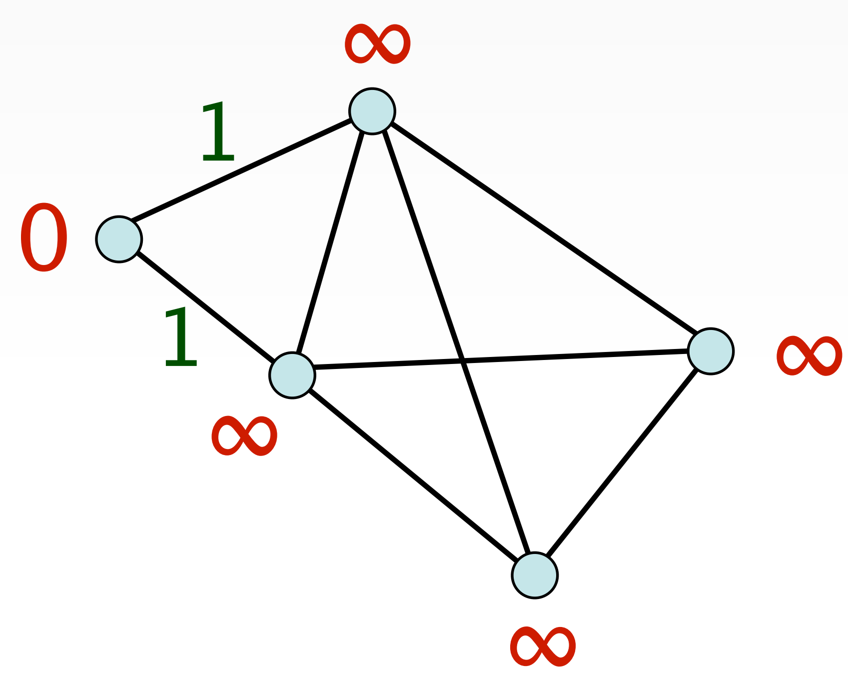

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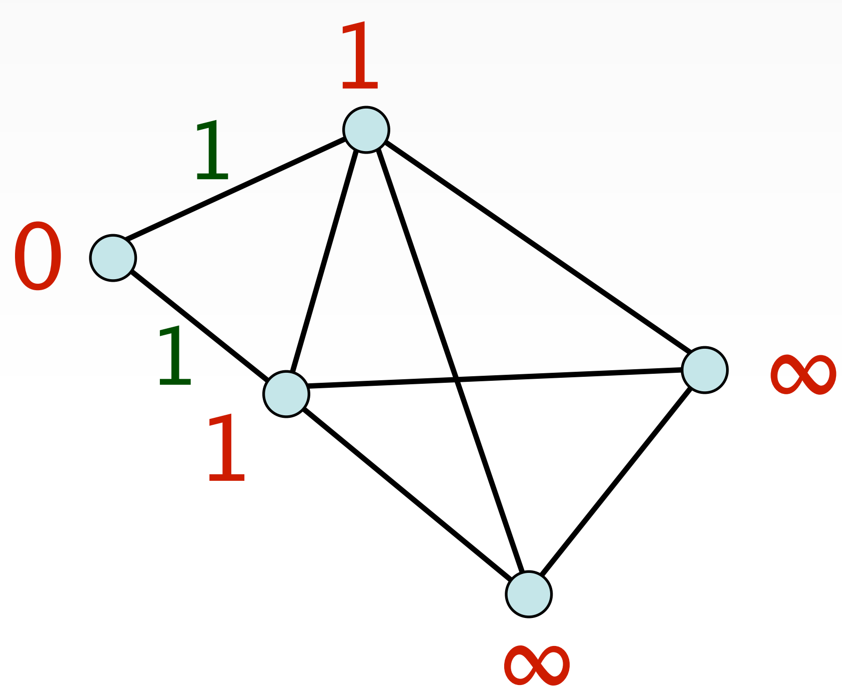

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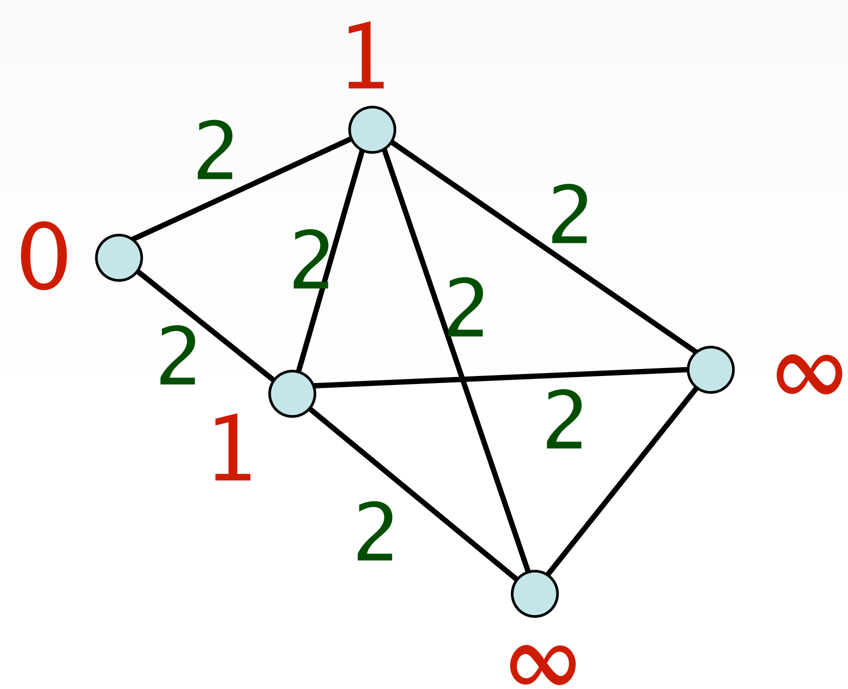

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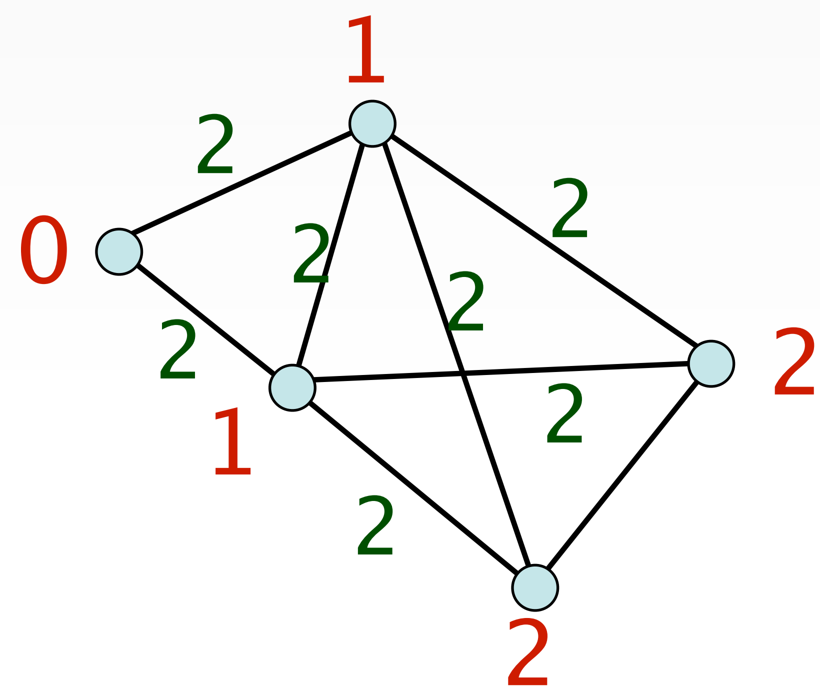

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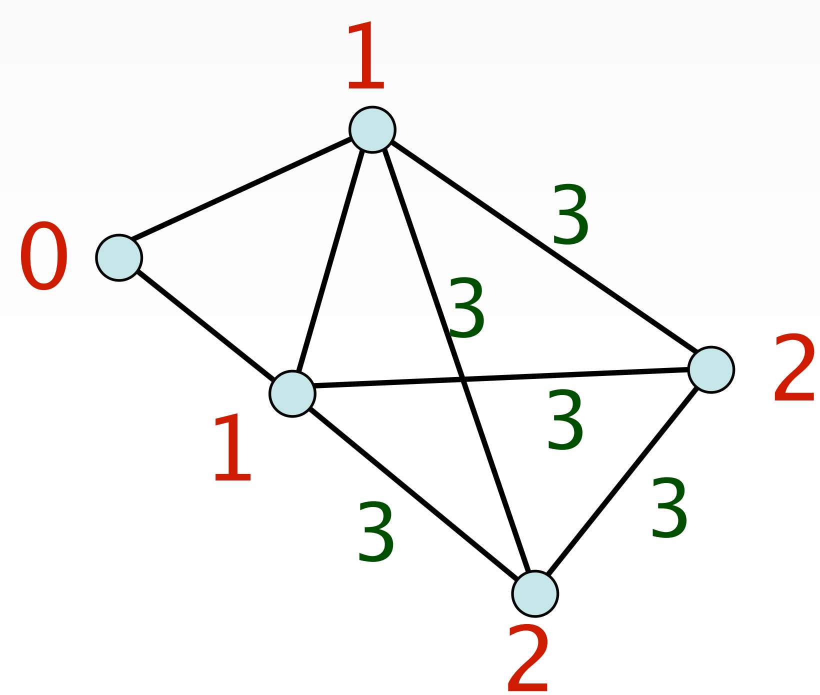

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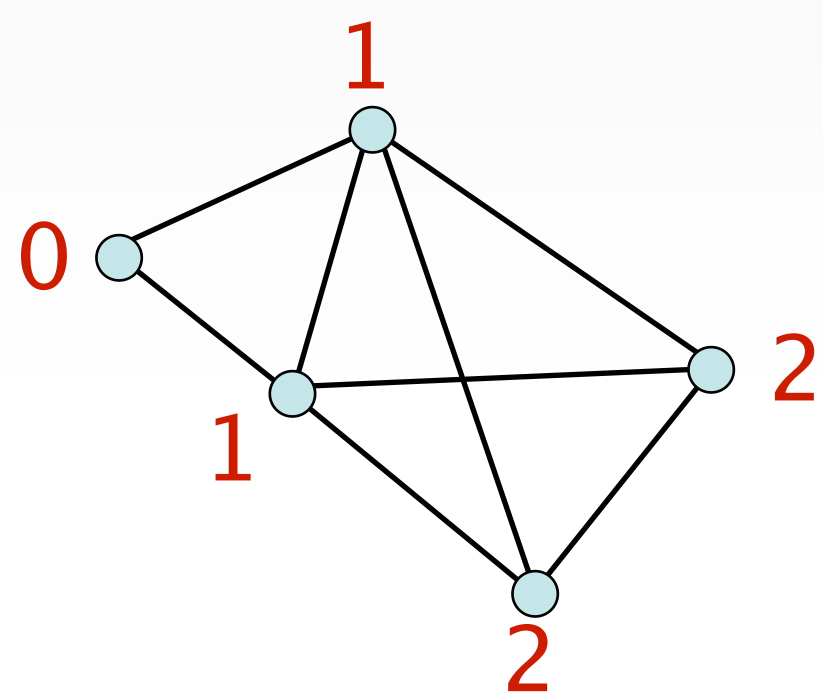

Beyond BFS

Simple Algorithms:

- BFS,
- shortest paths (single source & all pairs)
- ..

What about:

- connectivity?
- matchings?
- ...

Connectivity

Connectivity, Try 1:

- Begin with a unique id at every node
- In each super-round:
  - Every node identifies the minimum in its 2-neighborhood
  - Adds edges from all neighbors at least as big to the minimum
  - Sets own id to the minimum

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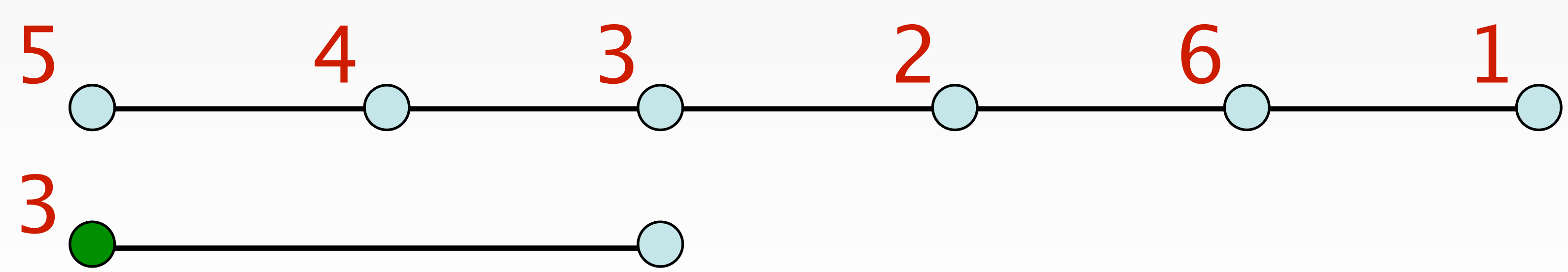

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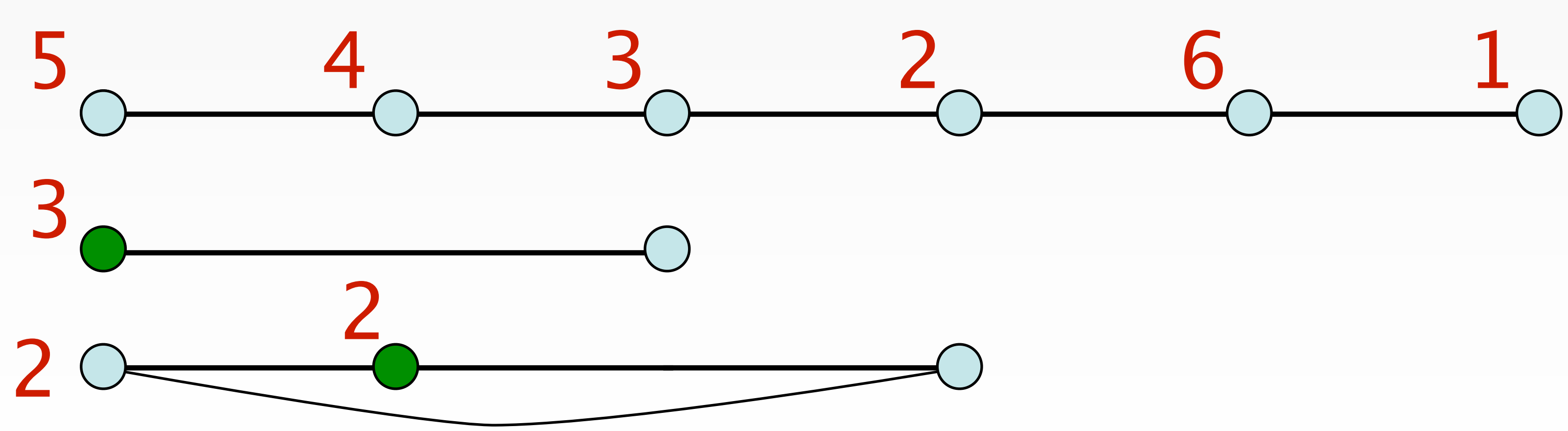

Saturday, August 25, 12

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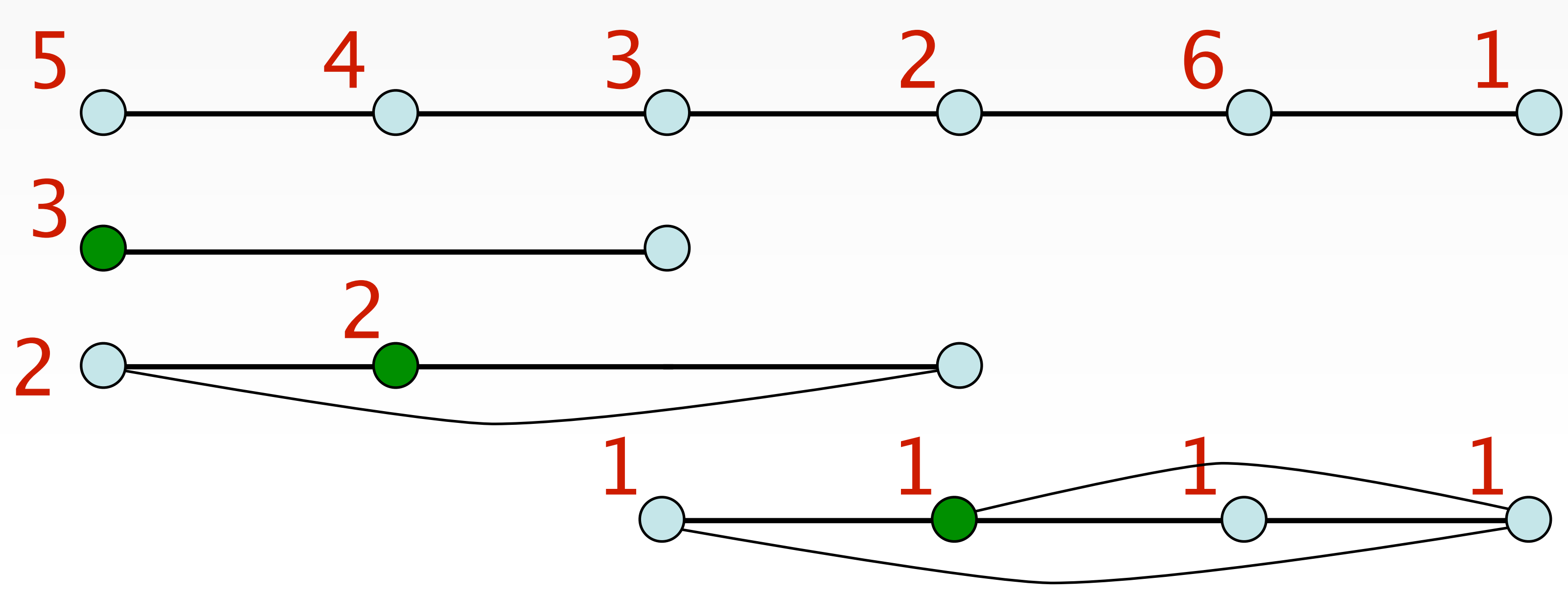

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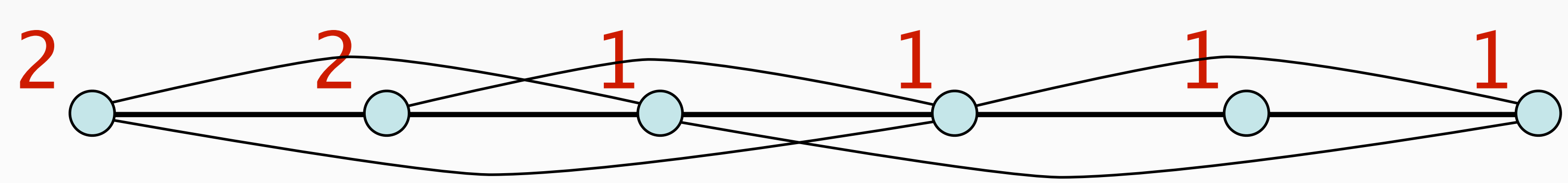

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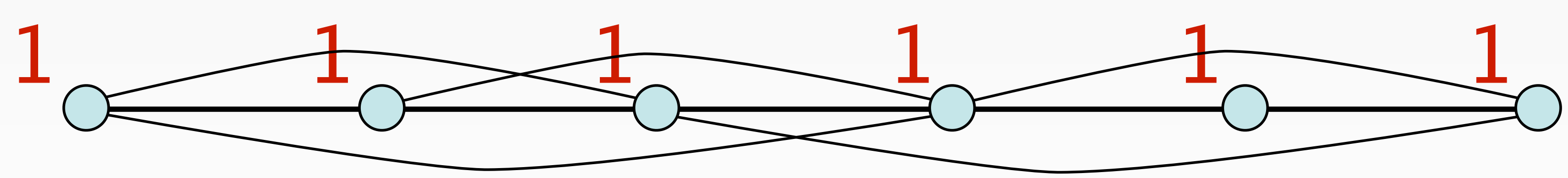

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- Begin with a unique id at every node
- In each super-round:
  - Every node identifies the minimum in its 2-neighborhood
  - Adds edges from all neighbors at least as big to the minimum
  - Sets own id to the minimum
- Analysis:
  - Takes $O(\log n)$ rounds to complete
  - Add $O(n)$ edges per round

Connectivity

Connectivity, Try 2:

- Begin with a unique id at every node
- In each super-round:
  - Every node identifies the minimum in its 2-neighborhood
  - Adds edge from itself to the minimum
  - Sets own id to the minimum

- Conjecture
  - This takes also $O(\log n)$ rounds to complete

Sparse Matchings

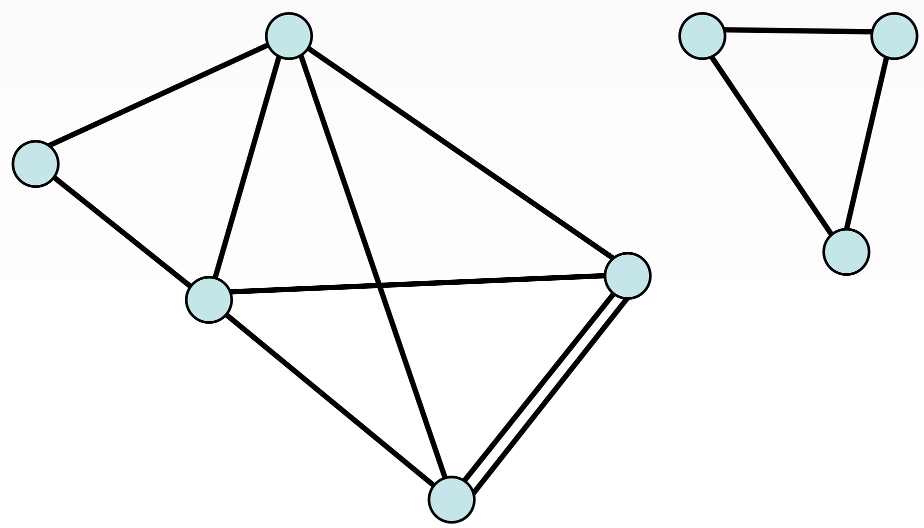
An example of adapting the spirit of a PRAM model

- Leads to an $O(\log n)$ algorithm
- With very simple computations per round
- Can be implemented either in MapReduce or in the Congest model
- Due to Israeli & Itai, 1986

Algorithm

Each Super-Round:

- Each Node picks one neighboring edge, directed away



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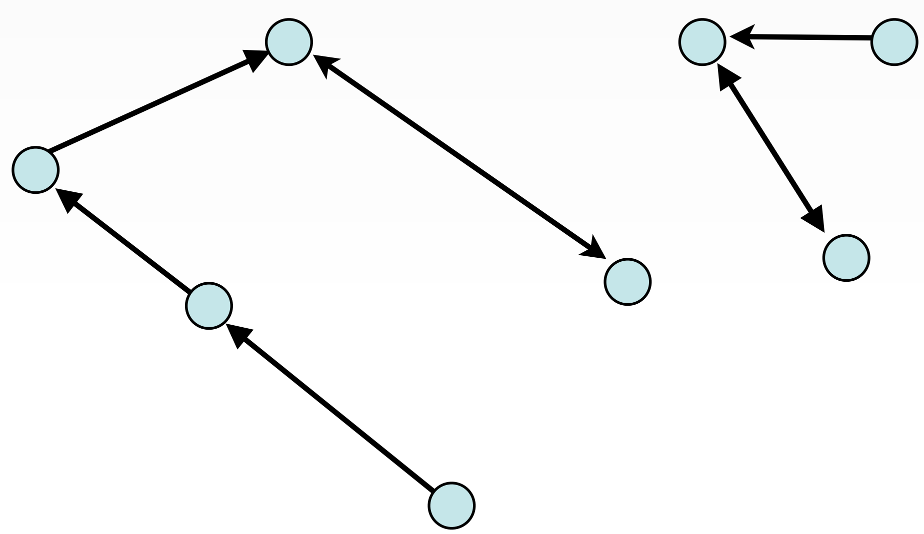

Algorithm

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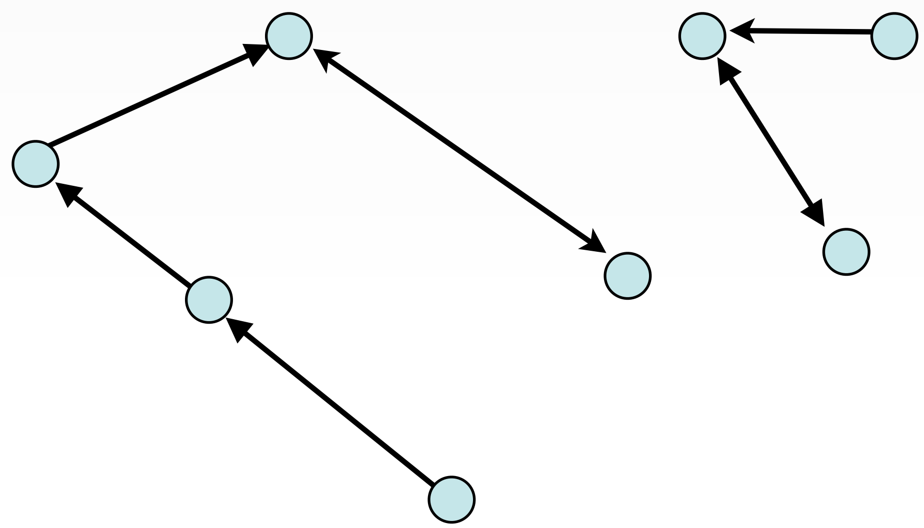
- Each Node picks one neighboring edge, directed away
- Nodes with in-degree > 1, pick one edge at random



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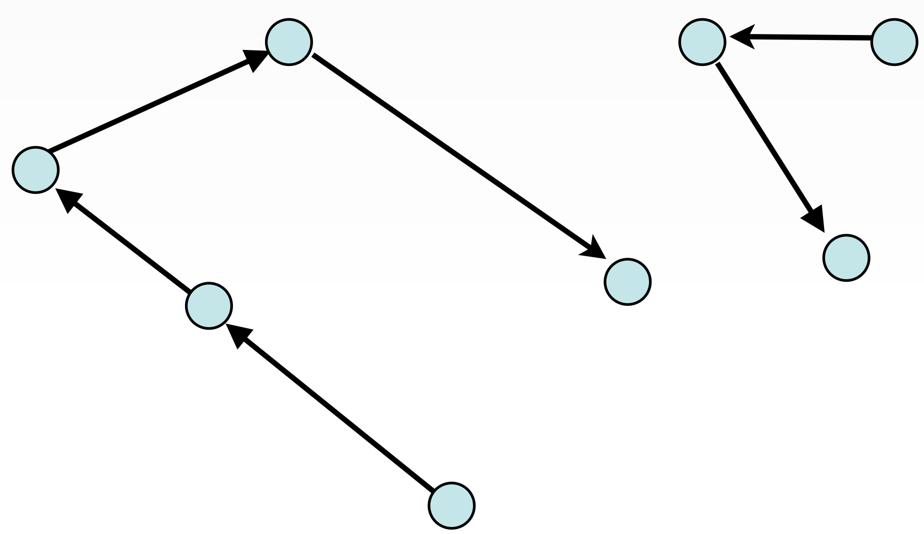

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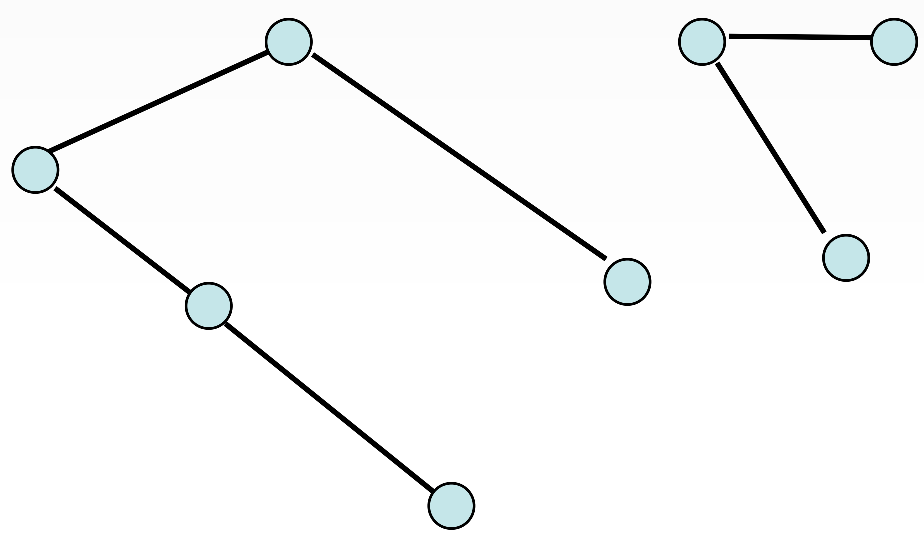

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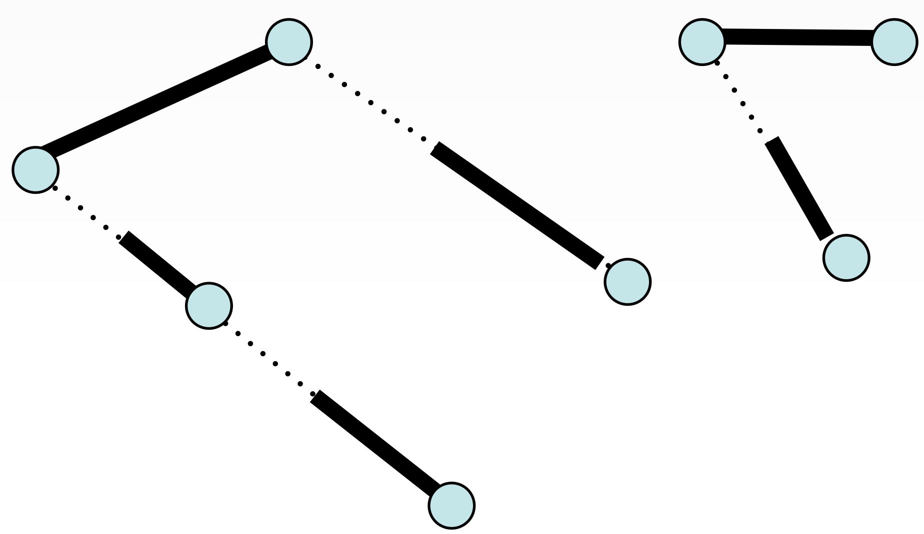

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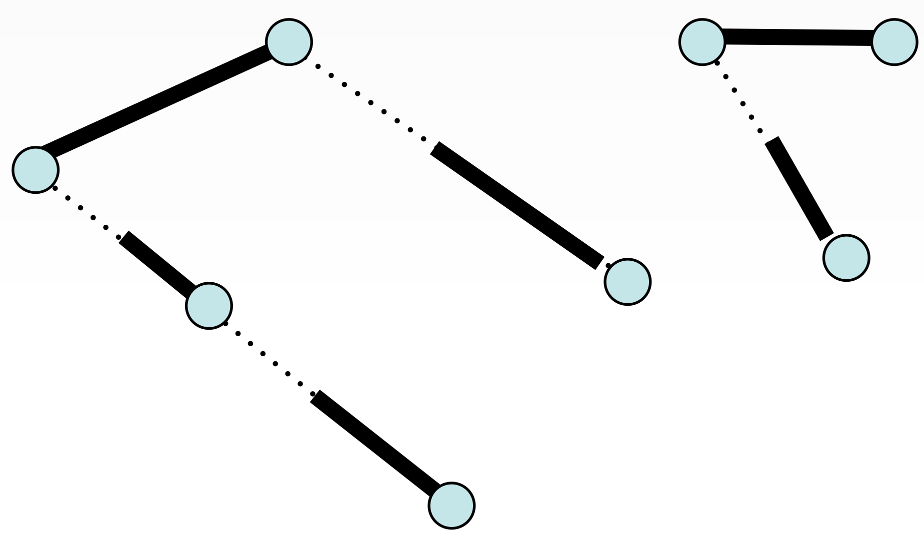

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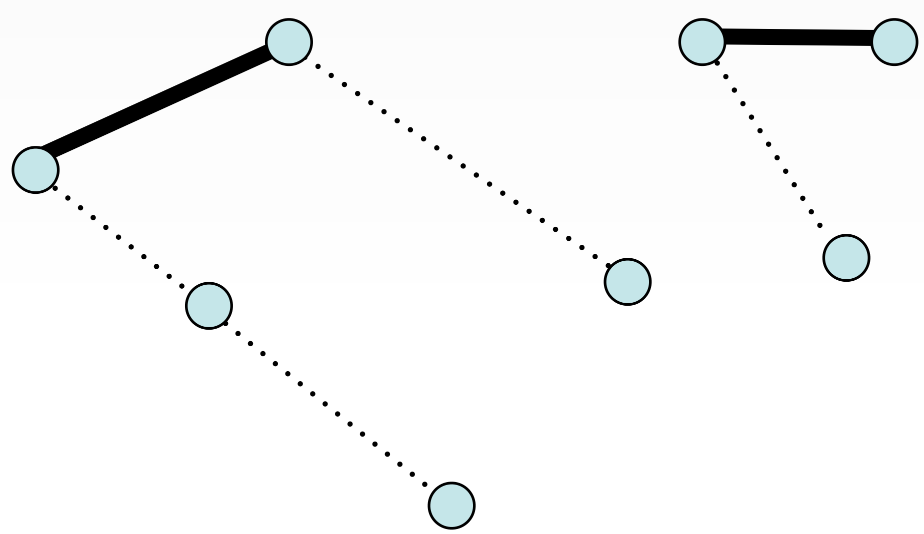

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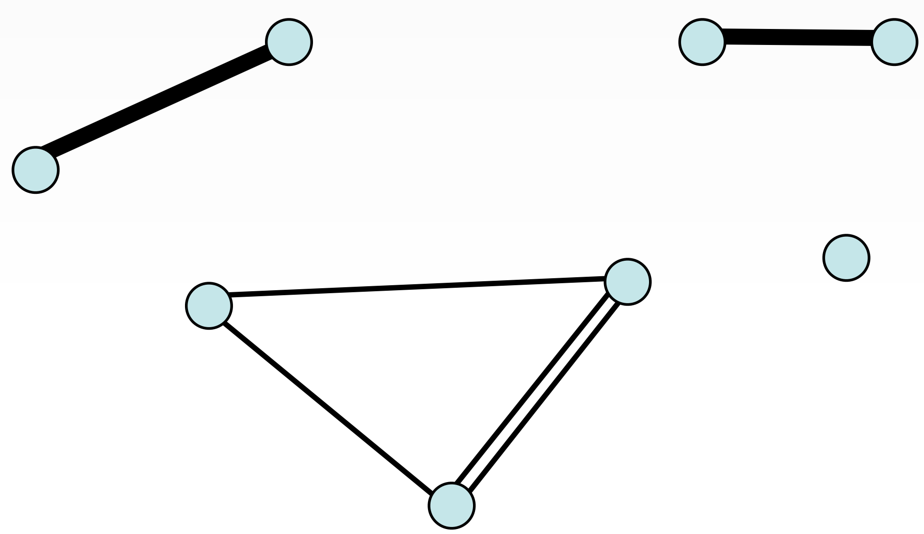
Recurse on unmatched nodes



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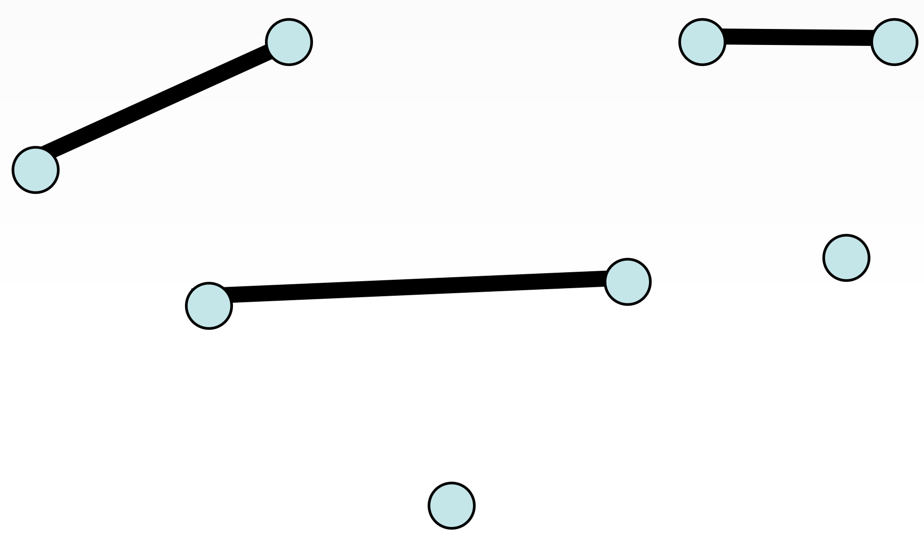


Sparse Graphs Conclusion

When nodes don't fit into memory

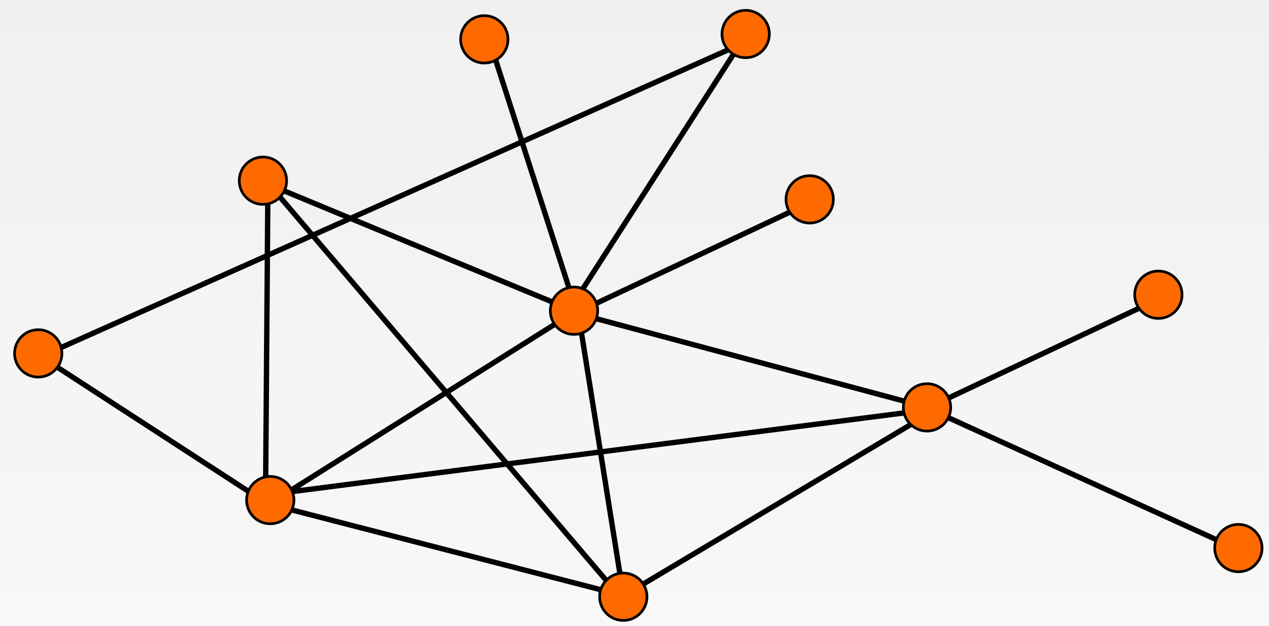
- Very different from Streaming algorithms
- Possible to adapt PRAM algorithms
- Many open questions!

Applications

Back to Social Graph Mining

- Yesterday: Finding tight knit communities
- Today: Finding large communities

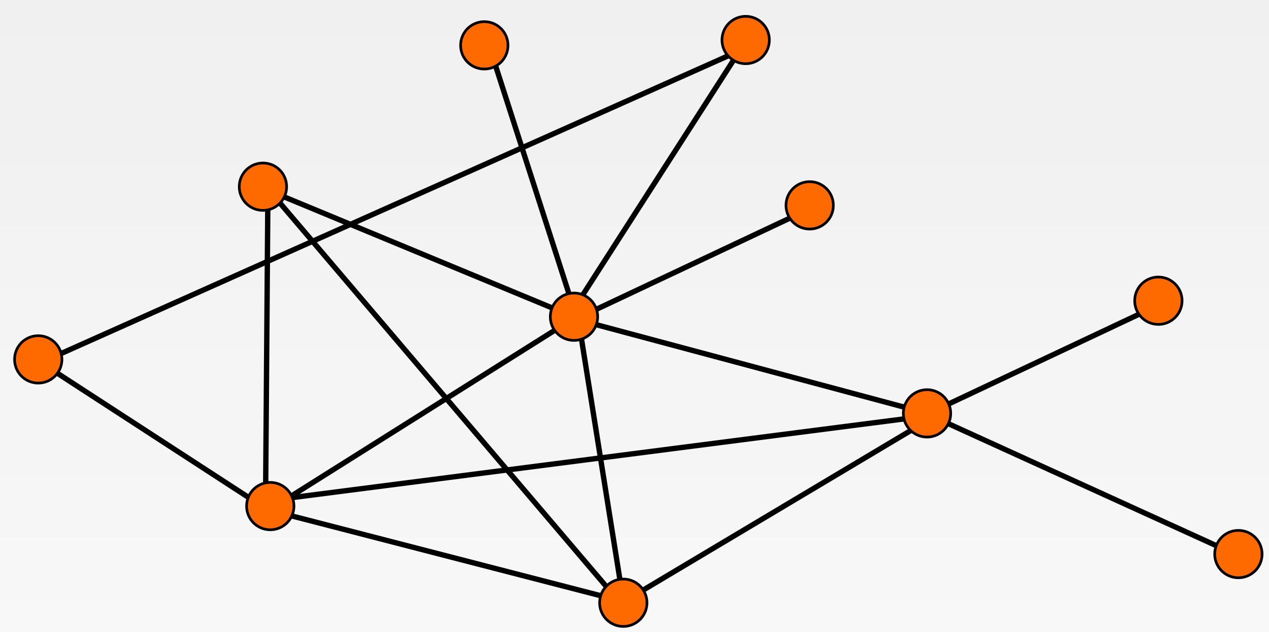
Finding Densest Subgraph



Problem: Given a graph $G = (V, E)$, find $V' \subseteq V$ that maximizes:

$$\rho = \frac{|E(V')|}{|V'|}$$

Finding Densest Subgraph



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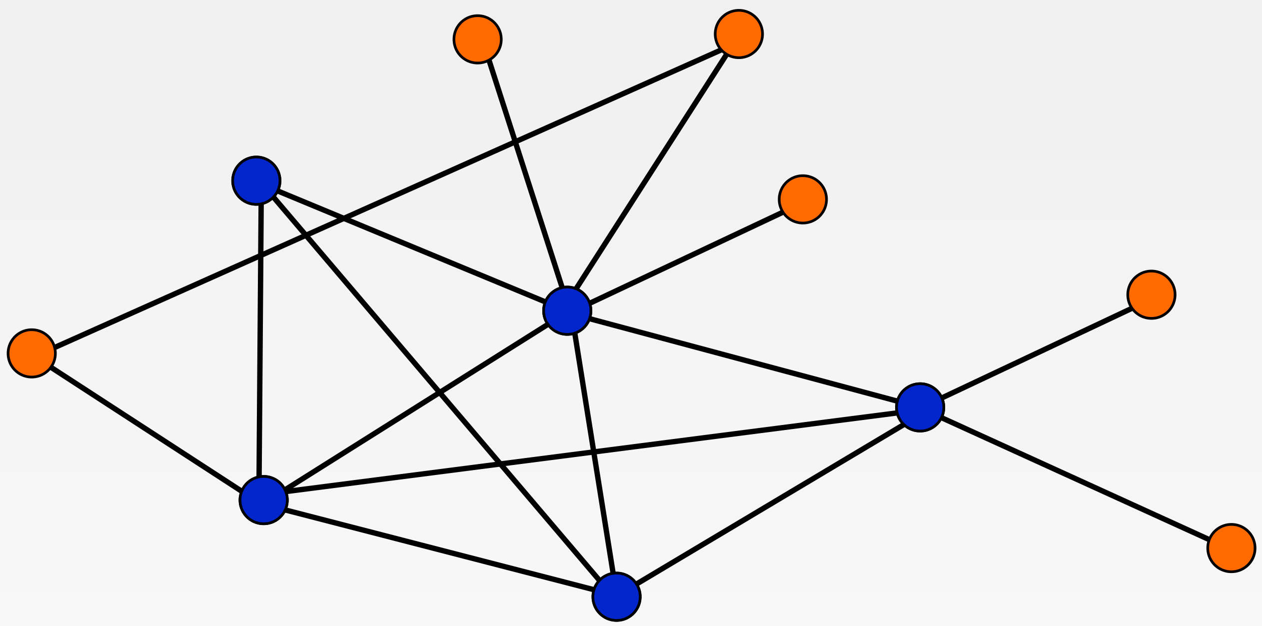
Useful Primitive in Graph Analysis:

- Community Detection
- Graph Compression
- Link SPAM Mining
- Many other applications

Finding Densest Subgraph



Problem: Given a graph $G = (V, E)$, find $V' \subseteq V$ that maximizes:

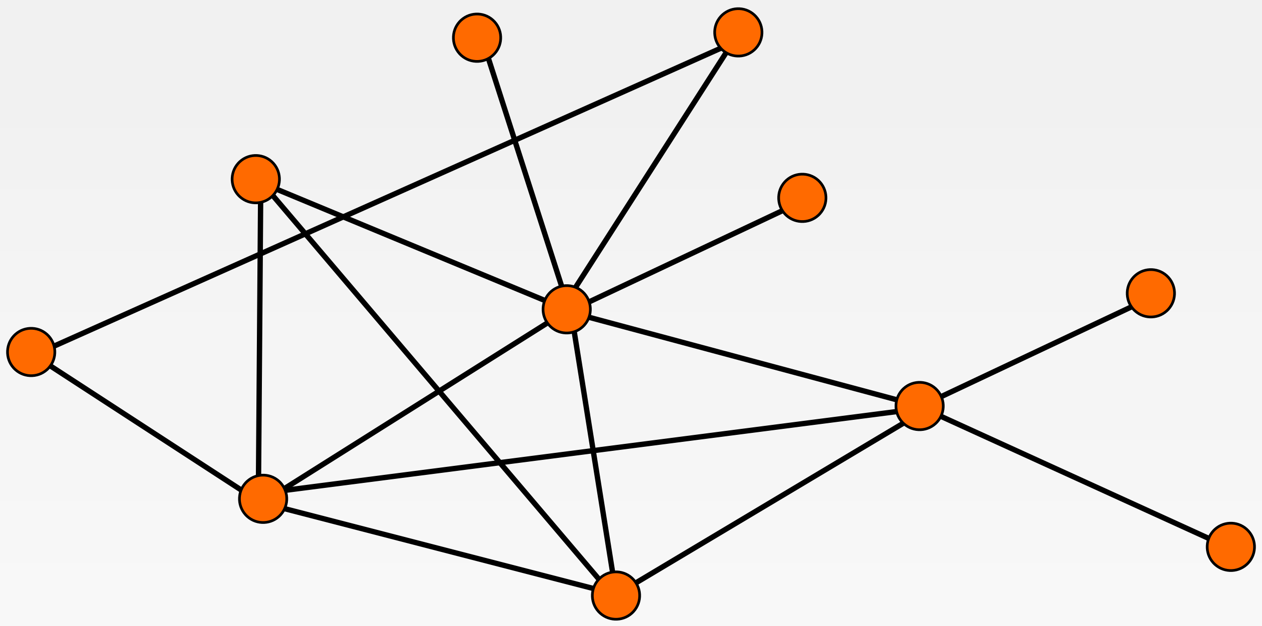
$$\rho = \frac{|E(V')|}{|V'|}$$

Useful Primitive in Graph Analysis

Can be solved exactly:

- LP Formulation
- Multiple Max flow computations

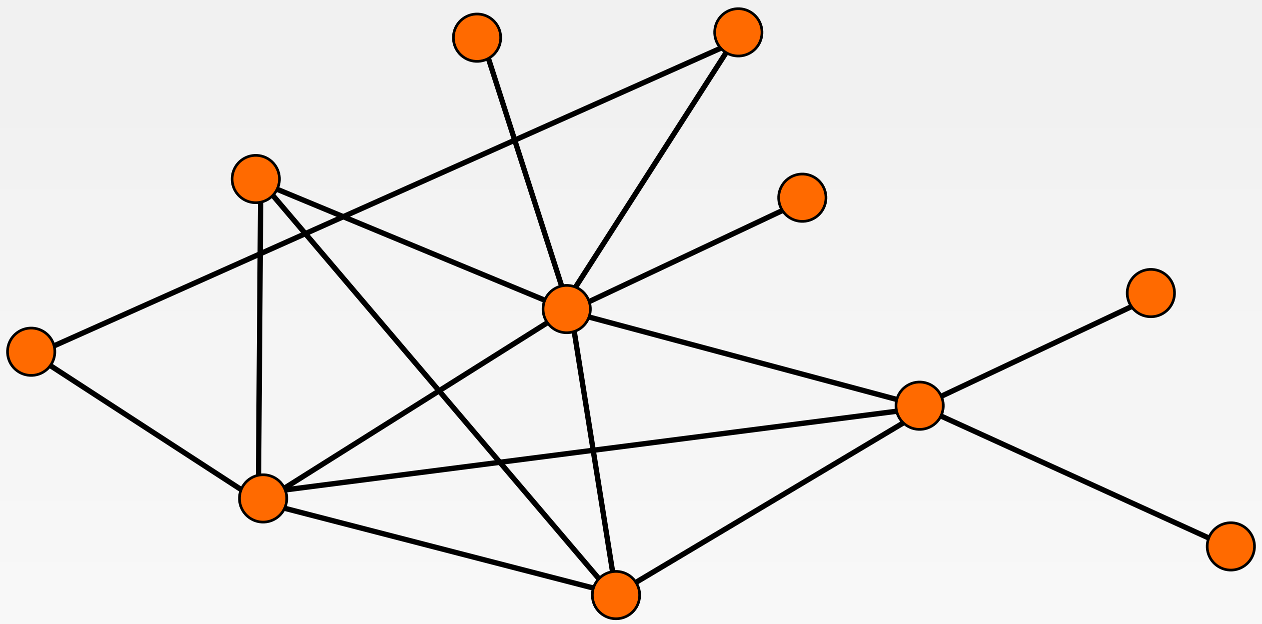
Finding Densest Subgraphs



Simple Algorithm [Charikar '00]:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs



```
Best Density: 16/11
Current Density: 16/11
```

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
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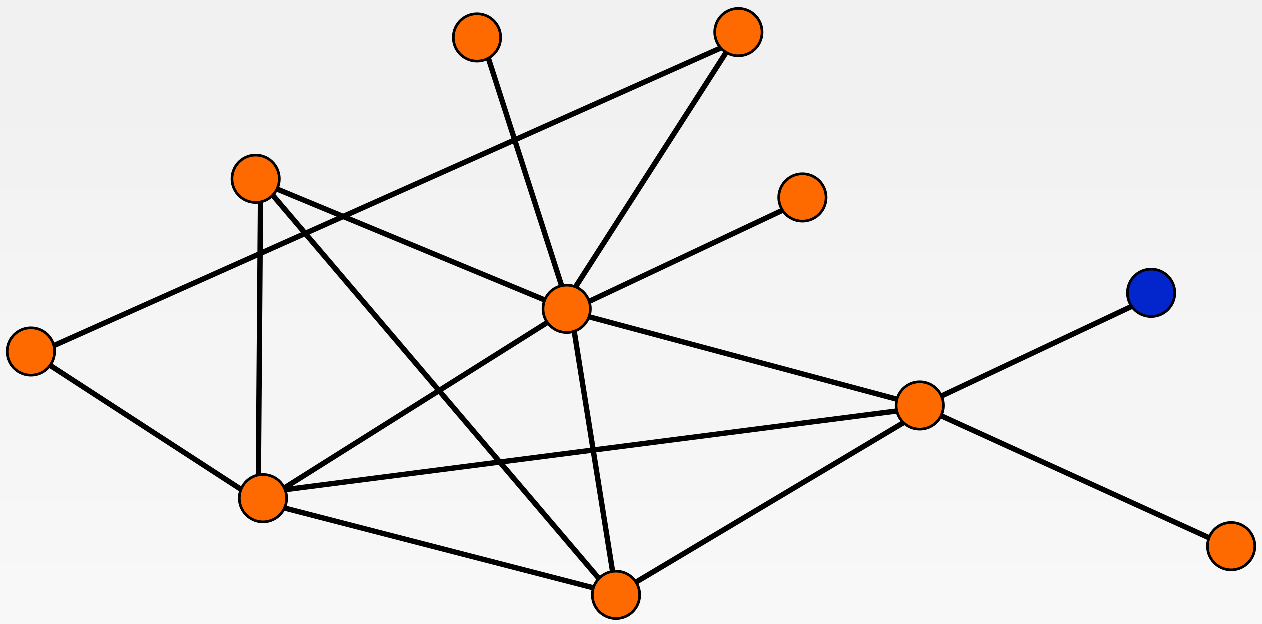
Finding Dense Subgraphs



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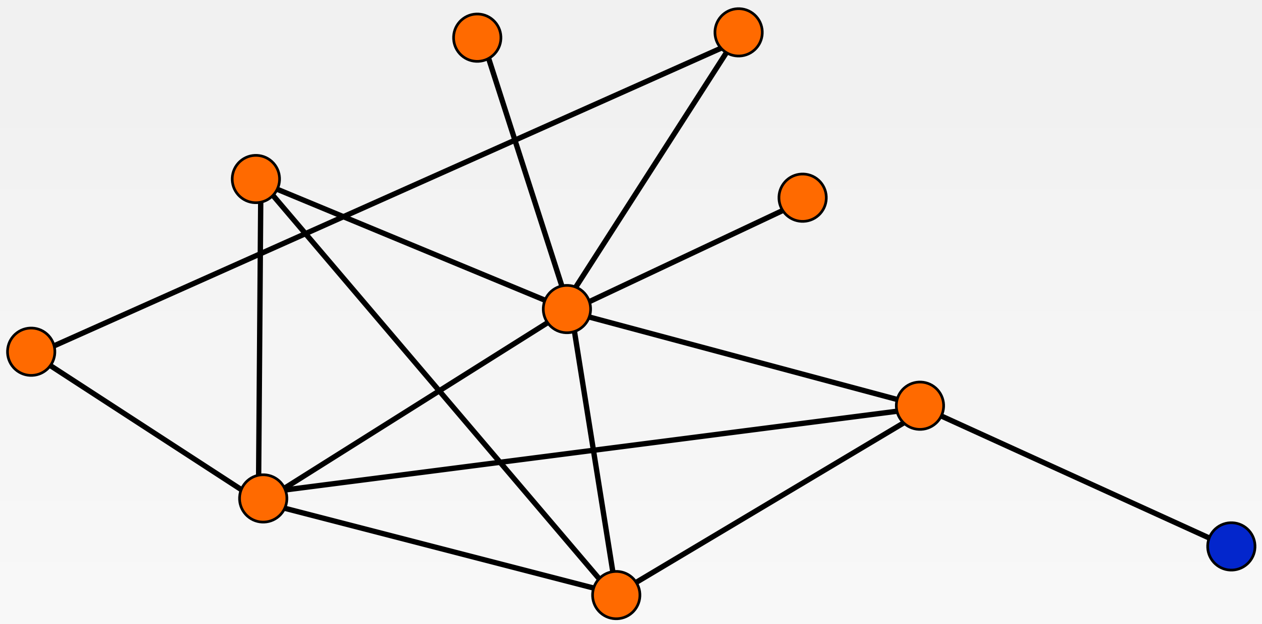
Finding Dense Subgraphs



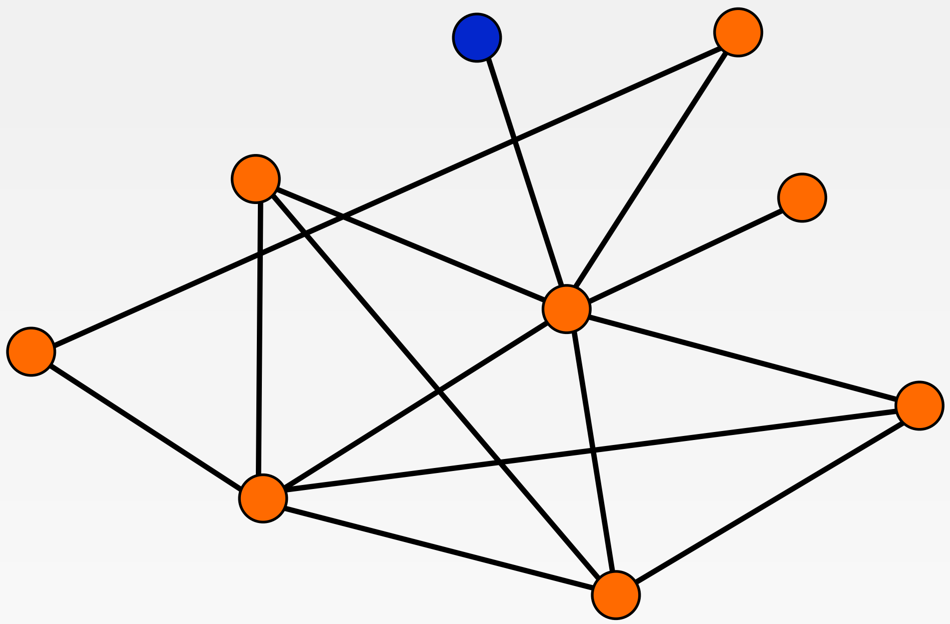
Best Density: 15/10

Current Density: 15/10

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs

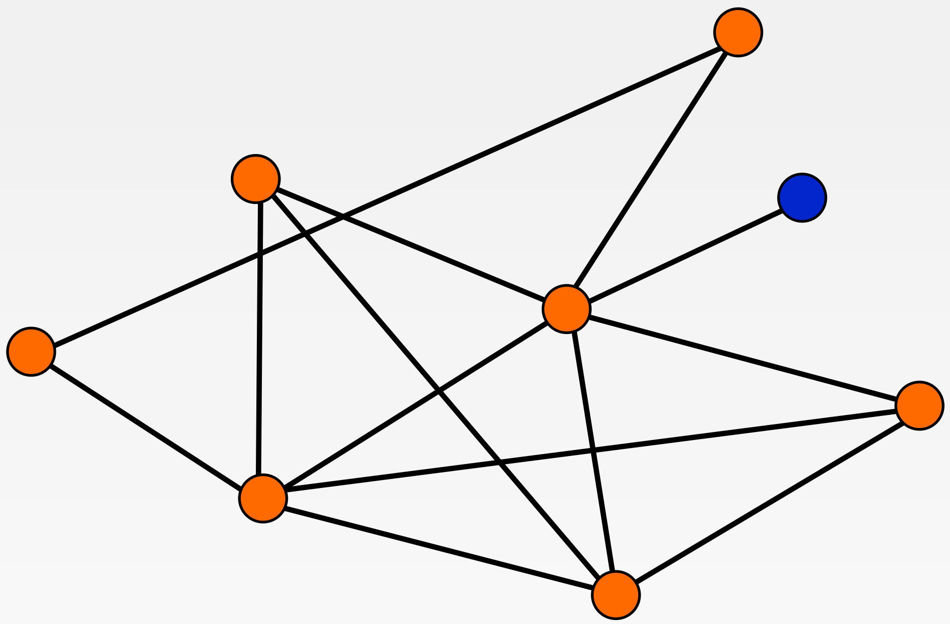


```
Best Density: 14/9
Current Density: 14/9
```

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs



```
Best Density: 13/8
Current Density: 13/8
```

Simple Algorithm:

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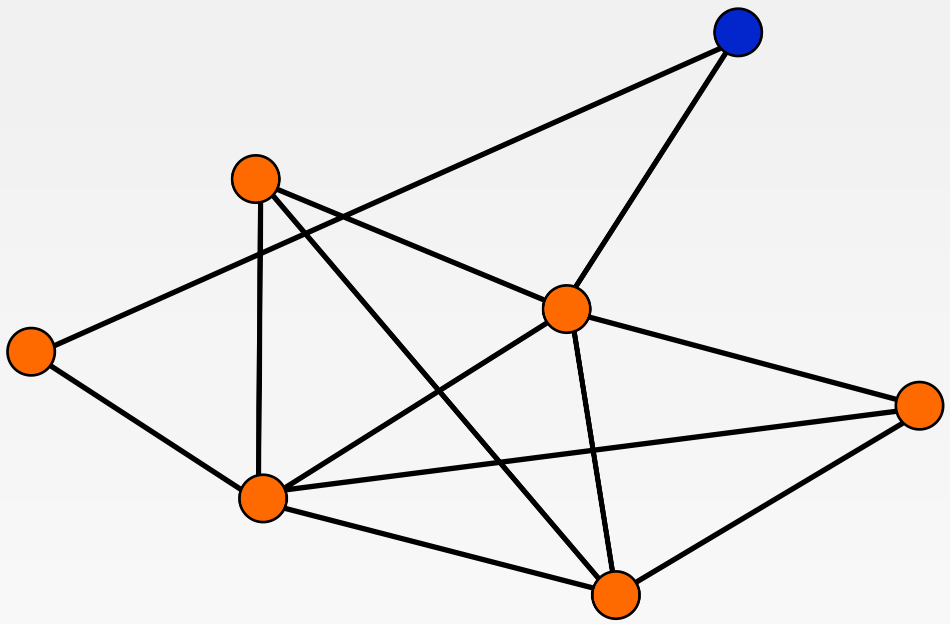
- Keep the densest intermediate subgraph

Finding Dense Subgraphs



```
Best Density: 12/7
Current Density: 12/7
```

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
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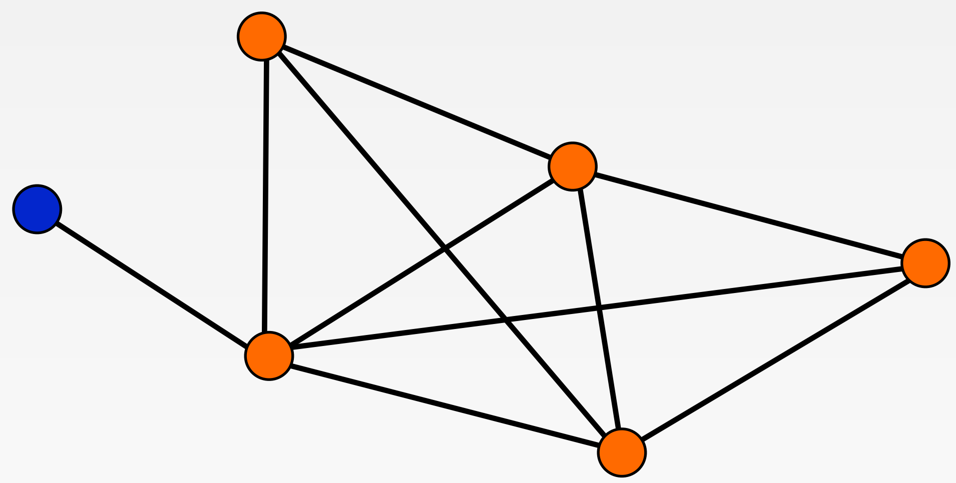
Finding Dense Subgraphs

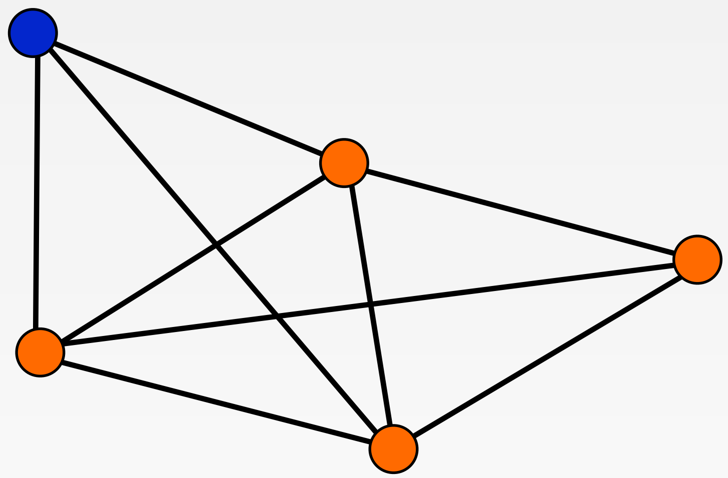


```
Best Density: 12/7
Current Density: 10/6
```

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs

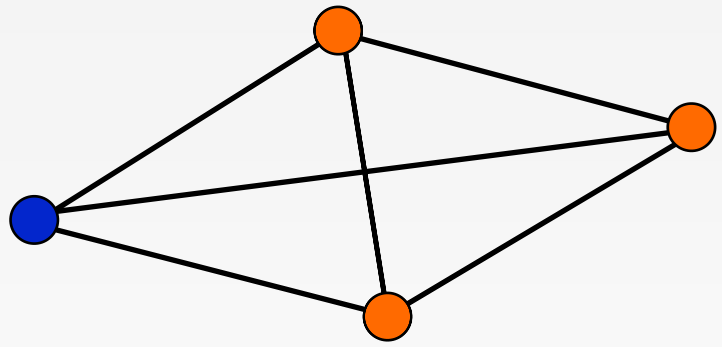


```
Best Density: 9/5
Current Density: 9/5
```

Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs



Best Density: 9/5

Current Density: 6/4

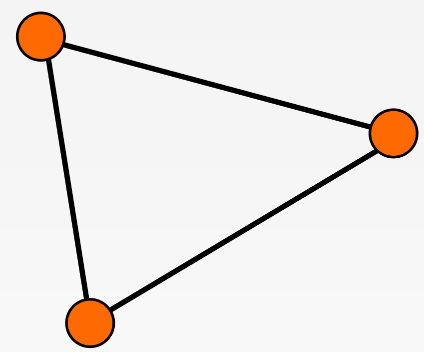
Simple Algorithm:

- Iteratively remove the lowest degree node and update vertex degrees
- Keep the densest intermediate subgraph

Finding Dense Subgraphs

Best Density: 9/5

Current Density: 3/3



Simple Algorithm:

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- Keep the densest intermediate subgraph

Finding Dense Subgraphs (Analysis)

Approximation Ratio:

- Guaranteed to return a 2-approximation

Proof:

- Let $V^* \subseteq V$ be the optimal solution, and $\lambda^* = \frac{|E[V^*]|}{|V^*|}$ the optimal density.
- Consider the first time a vertex from V^* is removed.
- Every vertex in V^* has degree at least λ^*.
  - Otherwise can improve optimum density
- Therefore the density of that subgraph is at least:

$$\frac{\lambda^* |V^*|}{2|V^*|} = \lambda^*/2$$

Finding Dense Subgraphs (Analysis)

Approximation Ratio:

- Guaranteed to return a 2-approximation

Running Time:

- RAM:
  - Maintain a heap on vertex degrees
  - Update keys upon removing every edge
  - Straightforward implementation in $O(m \log n)$
- Streaming:
  - Seemingly need one pass per vertex to adapt this algorithm
  - Can show that need $\Omega(n/\log n)$ memory if using $O(\log n)$ passes
- MapReduce?
  - Open question in Chierichetti, Kumar and Tompkins WWW '10.

Parallel Dense Subgraphs

Sequential Algorithm:

- Remove the node with the smallest degree

Parallel Dense Subgraphs

Sequential Algorithm:

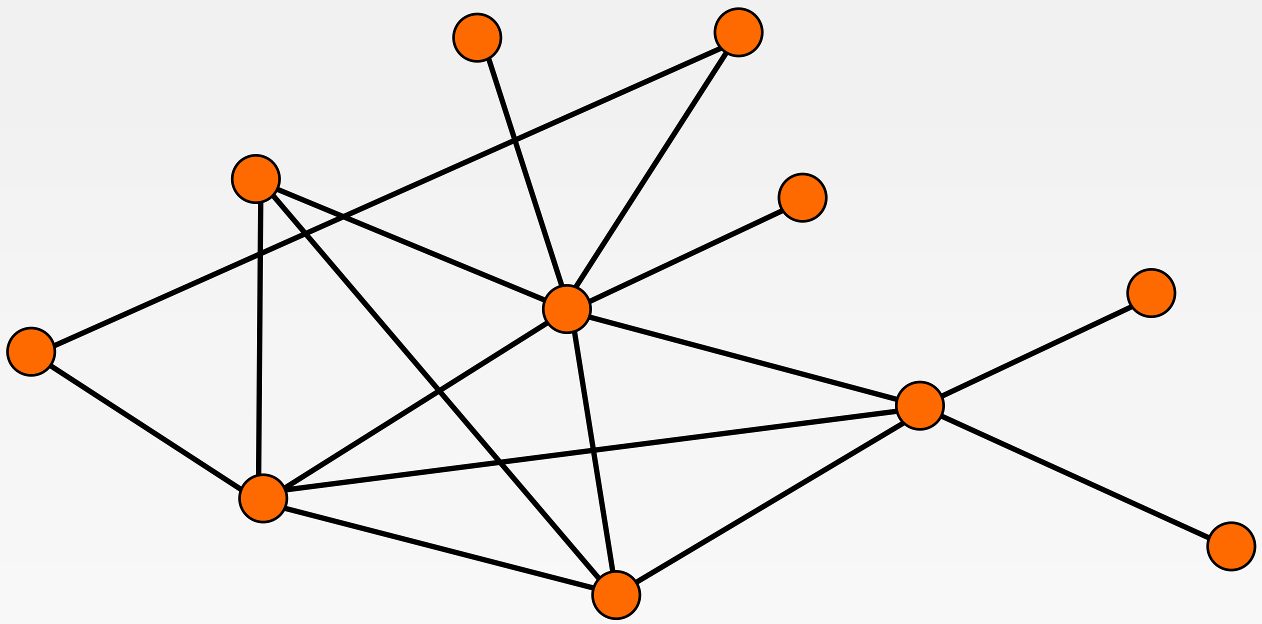
- Remove the node with the smallest degree

Parallel Version:

- Remove all nodes with less degree less than $(1+\epsilon)$* average degree
- Of course this also includes the smallest degree node

- Every Step:
  - Round 1: Count remaining edges, vertices, compute vertex degrees
    - Distributed counting
  - Round 2: Remove vertices with degree below threshold
    - Distributed checking

Parallel Dense Subgraphs



Best Density: 16/11

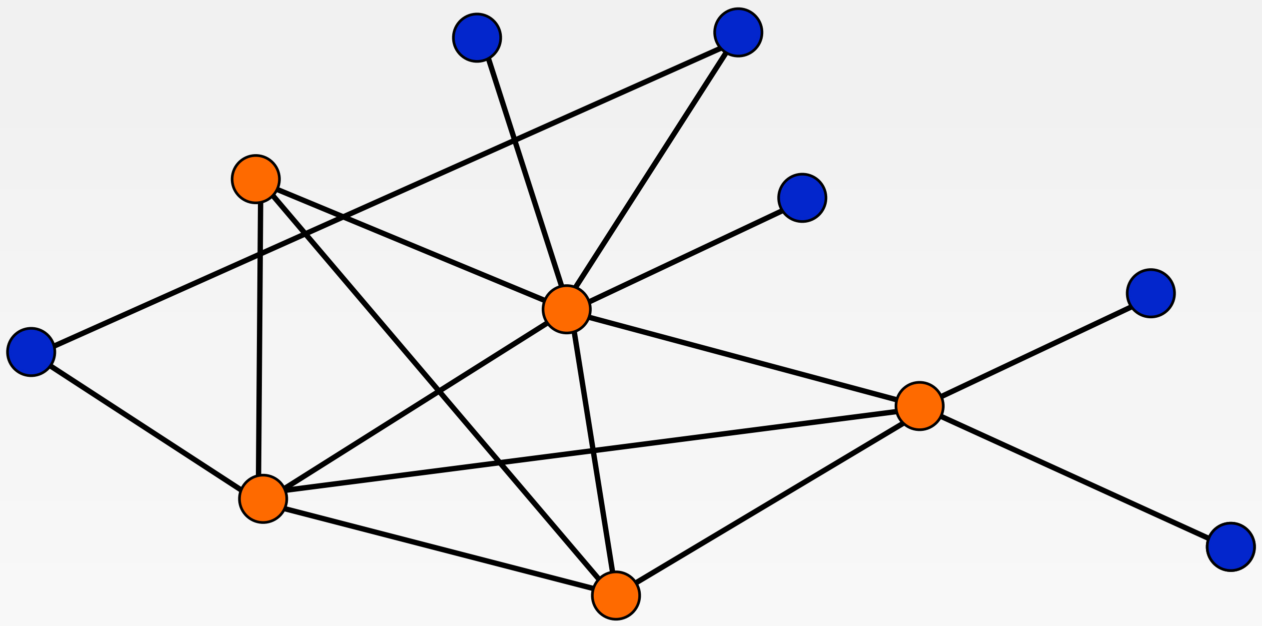
Current Density: 16/11

Average Degree: 32/11

Parallel Algorithm:

- Iteratively remove nodes with degree below average and update vertex degrees
- Keep the densest intermediate subgraph

Parallel Dense Subgraphs



Best Density: 16/11

Current Density: 16/11

Average Degree: 32/11

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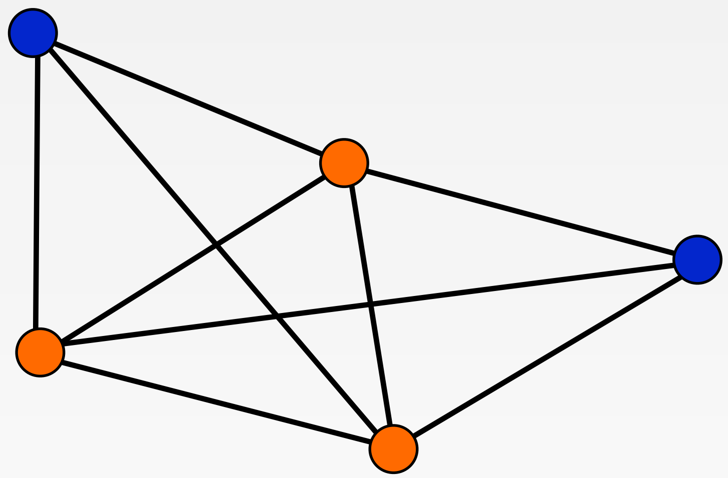
Parallel Dense Subgraphs



Best Density: 9/5

Current Density: 9/5

Average Degree: 18/5

Parallel Algorithm:

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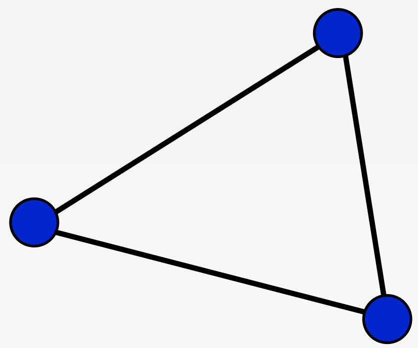
Parallel Dense Subgraphs



```
Best Density: 9/5
Current Density: 3/3
Average Degree: 6/3
```

Parallel Algorithm:

- Iteratively remove nodes with degree below average and update vertex degrees
- Keep the densest intermediate subgraph

Parallel Densest Subgraph (Analysis)

Algorithm:

- Each round remove all vertices with degree less than $(1+\epsilon)$ * average.

How many vertices do we remove?

- One cannot have too many vertices above average (This is not Lake Wobegon)
- Easy [Markov inequality] : at most a $\frac{1}{1+\epsilon}$ fraction of vertices remains in every round.
- Therefore algorithm terminates after $O\left(\frac{1}{\epsilon}\log n\right)$ rounds

Parallel Densest Subgraph (Analysis)

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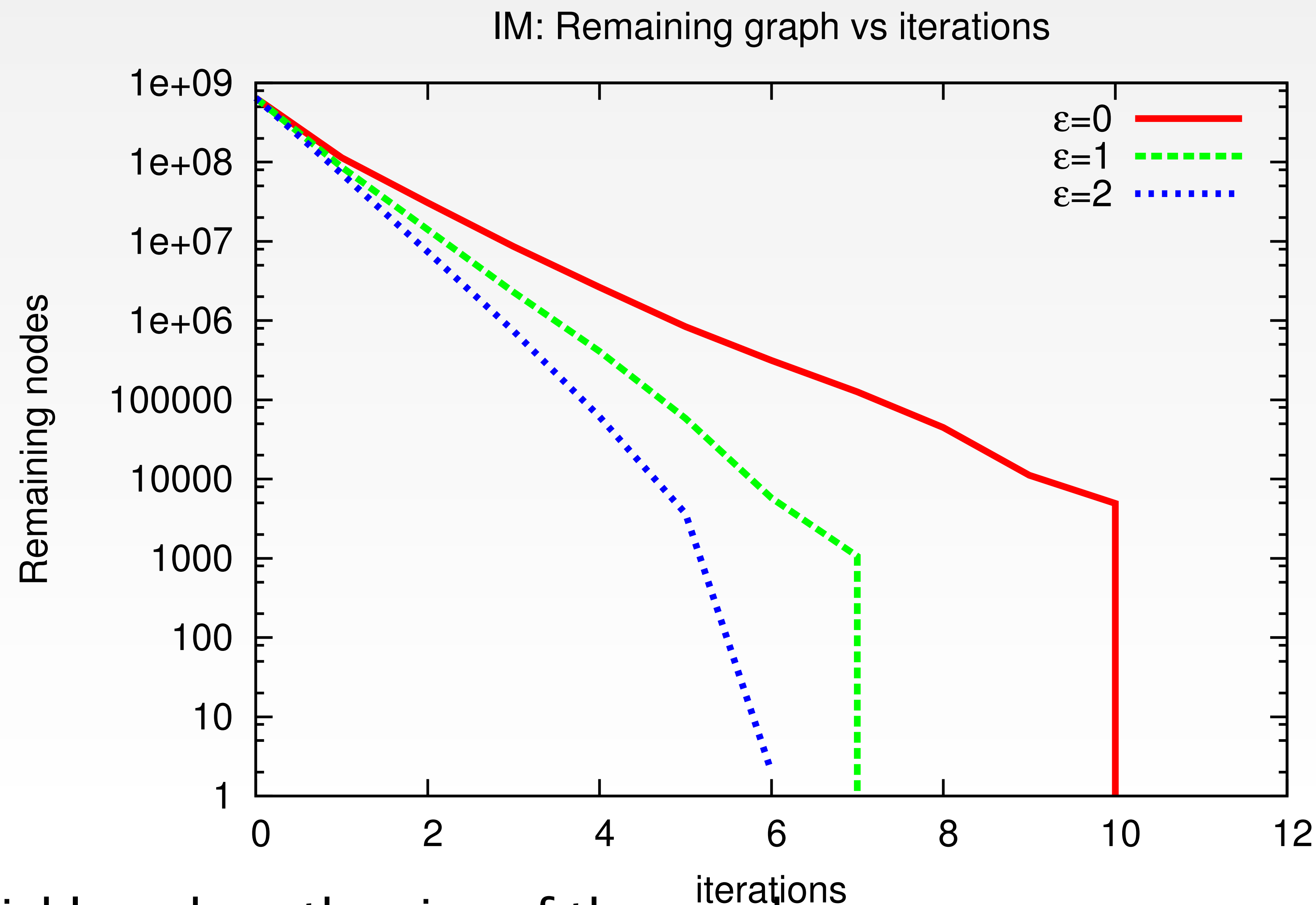
Approximation Ratio:

- Achieves a $(2+\epsilon)$ approximation in the worst case
  - Only look at the degree of the nodes removed as compared to average. in

How well does it work?

IM Network graph: 650M nodes, 6.1B edges



- Quickly reduce the size of the graph.
- Approximation ratio between 1.06 and 1.4 at $\epsilon = 1$

Overall

Improving the sequential algorithm:

- Original algorithm: O(m) heap updates:
  - Update vertex degrees every time an edge is removed.
- New algorithm O(n) heap updates:
  - Number of vertices decreases geometrically every round

Wrap Up

Graphs:

- At the core of many large data computations
- Many follow heavy tailed degree distirbutions
- Dense: Sample & Prune leads to fast algorithms
- Sparse: Adapt PRAM Algorithms

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Next Up:

- Clustering & Machine Learning

References

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