A General Notion of Realizability

L. Birkedal*
The IT University of Copenhagen
Denmark

Abstract

We present a general notion of realizability encompassing both standard Kleene style realizability over partial combinatory algebras and Kleene style realizability over more general structures, including all partial cartesian closed categories. We show how the general notion of realizability can be used to get models of dependent predicate logic, thus obtaining as a corollary (the known result) that the category $\mathbf{Equ}$ *of equilogical spaces models dependent predicate logic. Moreover, we characterize when the general notion of realizability gives rise to a topos.*

1. Introduction

There has recently been a lot of interest in understanding how models based on realizability over partial combinatory algebras (PCAs) may be generalized to models based on realizability over more general structures [1, 3, 2, 18, 16, 15, 8]. For instance, in [3, 1] it is shown how how the category $\mathbf{Equ}$ of equilogical spaces can be thought of as a category of modest sets over the category of algebraic lattices and, proceeding by analogy to the category of modest sets constructed over a partial combinatory algebra, it is shown that $\mathbf{Equ}$ models dependent type theory. In this paper we suggest a common framework which covers both Kleene style realizability over PCAs and Kleene style realizability over more general "typed" structures, including the category $\mathbf{ALat}$ of algebraic lattices used for $\mathbf{Equ}$. More motivation for such a common framework can be found in [18], where it is explained how such a framework may lead to more direct comparisons of different computational models.

At the core of standard realizability models over PCA's is the basic Kleene realizability interpretation of implicational logic. Our general notion of realizability is based on the observation that for this interpretation of implicational logic, one only needs a few of all the definability properties a PCA has. Indeed, intuitively speaking, we just need some kind of pairing operation for conjunction and some kind of functional abstraction for implication. Therefore, our general notion of realizability is based on the notion of a *weakly closed partial cartesian* (WCPC) category $\mathbb{C}$, equipped with a functor $U\colon \mathbb{C} \to \mathbf{Ptl}$ into the category of sets and partial functions (see the following sections for precise definitions). We may think of a WCPC-category $\mathbb{C}$ as a general universe of realizers as follows. The objects are thought of as sets of realizers and the morphisms are thought of as realizable partial functions mapping one set of realizers into another; partial because realizers may be only partially defined. A WCPC-category is equipped with a kind of pairing operation (hence the word "cartesian") and with a kind of weak functional abstraction (hence the phrase "weakly closed"). Since a WCPC-category may have many different objects, one may also think of a WCPC-category as a *typed* universe of realizers. Any cartesian closed category is a WCPC-category and any PCA gives rise to a WCPC-category in a natural way.

To explain how our general notion of realizability is related to standard realizability over PCAs, and what the main results of the paper are, we briefly recall some basic facts about standard realizability over PCAs. To this end, let A be a PCA. Given A, one may construct the following well known categories (see [13] for a textbook treatment with references to the literature): the standard realizability tripos $\mathsf{p}\colon \mathbf{UFam}(A) \to \mathbf{Set}$, a fibred (or indexed) category; the category of partitioned assemblies $\mathbf{PartAsm}(A)$; the category of modest sets $\mathbf{Mod}(A)$; the category of assemblies $\mathbf{Asm}(A)$; and the standard realizability topos $\mathrm{RT}(A)$. For applications, one is chiefly interested in the latter three; the first two are mainly used to construct and reason about the latter three. Recall further that $\mathbf{Mod}(A)$ is a full subcategory of $\mathbf{Asm}(A)$; that $\mathbf{Asm}(A)$ is equivalent to the regular completion $(\mathbf{PartAsm}(A))_{\text{reg/lex}}$ of $\mathbf{PartAsm}(A)$; and that $\mathrm{RT}(A) = \mathbf{Set}[\mathsf{p}] \simeq (\mathbf{PartAsm}(A))_{\text{ex/lex}} \simeq (\mathbf{Asm}(A))_{\text{ex/reg}}$, *i.e.*, $\mathrm{RT}(A)$ is the standard realizability topos obtained by the tripos to topos construction applied to the tripos p, and it is equivalent to the exact completion of the left exact category $\mathbf{PartAsm}(A)$ and to the exact completion of the regular category $\mathbf{Asm}(A)$. The categories $\mathbf{Mod}(A)$ and $\mathbf{Asm}(A)$ are both locally cartesian closed and both model dependent predicate logic (see, *e.g.*, [13]).

We show how any WCPC-category $\mathbb{C}$ gives rise to a *pre-*

*Postal Adress: Lars Birkedal, The IT University of Copenhagen, Glentevej 67, DK–2400 Copenhagen NV, Denmark. Email: birkedal@itu.dk. The work reported here was carried out while the author was a PhD-student at Carnegie Mellon University, Pittsburgh, USA.

tripos $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ (almost a tripos, but without a weak generic object), and a category $\mathcal{F}(\mathbb{C})$ corresponding to the category of partitioned assemblies and how these may be used to construct categories of modest sets $\mathbf{Mod}(\mathbb{C})$ and assemblies $\mathbf{Asm}(\mathbb{C})$, and also to a realizability category $\mathbf{Set}[\mathsf{p}] \simeq (\mathbf{Asm}(\mathbb{C}))_{\text{ex/reg}} \simeq (\mathcal{F}(\mathbb{C}))_{\text{ex/lex}}$. The main results are that

- The categories $\mathbf{Mod}(\mathbb{C})$ and $\mathbf{Asm}(\mathbb{C})$ are locally cartesian closed and provide a split model of dependent predicate logic. As a corollary we get that $\mathbf{Equ} \simeq \mathbf{Mod}(\mathbf{ALat})$ models dependent predicate logic as shown explicitly in [1].

- The category $\mathbf{Set}[\mathsf{p}] \simeq (\mathbf{Asm}(\mathbb{C}))_{\text{ex/reg}} \simeq (\mathcal{F}(\mathbb{C}))_{\text{ex/lex}}$ is a topos if and only if the category $\mathbb{C}$ has a universal object, *i.e.*, an object of which all other objects are retracts. Intuitively, this tells us that we only get an impredicative model (a topos) if we start out with an impredicative (untyped) universe of realizers. In particular, since $\mathbf{ALat}$ does not have a universal object, we conclude that $(\mathbf{Asm}(\mathbf{ALat}))_{\text{ex/reg}}$ is not a topos, so $\mathbf{Equ}$ does not embed into a realizability topos constructed over $\mathbf{ALat}$ (this was known before; the point is that our result *explains* why this particular example does not produce a topos).

- For a PCA A, which gives rise to a WCPC-category $\mathbb{C}(A)$, realizability in the sense of the WCPC-category $\mathbb{C}(A)$, is indeed equivalent to realizability in the sense of the PCA A. Thus our framework is a proper generalization of standard realizability over PCAs, and hence it improves earlier published work such as [3].

There are several routes one can take to establish that $\mathbf{Asm}(\mathbb{C})$ provides a model of dependent predicate logic—we give a fairly direct proof, which allows us to show that we get a *split* model; this is important for giving the actual interpretation of the dependent type theory as discussed in, *e.g.*, [19, 22, 20, 24, 10]).

In Section 2 we define the notion of a WCPC-category and show how PCA's give rise to WCPC-categories. Then in Section 3 we define the notion of a pretripos and show that any WCPC-category gives rise to what we call a realizability pretripos. We further show that the pretripos resulting from the WCPC-category induced by a PCA is equivalent to the standard realizability tripos over the PCA, and we give a general explanation of this fact. Further, we discuss coproducts for WCPC-categories and pretriposes, and we show that a realizability pretripos gives rise to a topos iff the WCPC-category has a universal object. In Section 4 we go on to define categories of assemblies over suitable fibrations and we show that the category of assemblies over a realizability pretripos models dependent predicate logic. Finally, in Section 5 we discuss how our general notion of realizability is related to other recently suggested (as yet unpublished) frameworks and give some concluding remarks.

Acknowledgements The work reported here forms part of the author's PhD-thesis [2], written under the guidance of Prof. Dana Scott. We refer the reader to [2] for more details than we can include here. I am also grateful for useful discussions with Steven Awodey, Andrej Bauer, Peter Lietz, John Longley, Pino Rosolini, and Thomas Streicher.

2. WCPC-categories

In this section we define the notion of a *weakly closed partial cartesian category* (WCPC-category). We shall show how such a category can be seen as a general universe of realizers. In particular, we show that any PCA gives rise to a WCPC-category.

Our notion of a WCPC-category is just a weak version of a category of partial maps on a cartesian closed category. We thus begin by recalling the definition of a partial cartesian category from [25]. The wording "partial cartesian" is a shortening of "partial maps on a cartesian category." There is an excellent overview of categories of partial maps in [25] to which we refer the reader for much more information on categories of partial maps and their history than we can include here. See also [27].

The following definitions are from [25, Page 101].

Definition 2.1. A **p-category** is a category $\mathbb{C}$ endowed with a bifunctor $\times\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ which is called **product**, a natural transformation $\Delta\colon (-) \to (- \times -)$ which is called the **diagonal** and two families of natural transformations $\{\, p_{-,Y}\colon (-\times Y) \to (-) \mid Y \in \mathbb{C}\,\}$ and $\{\, q_{X,-}\colon (X\times -) \to (-) \mid X \in \mathbb{C}\,\}$ which are called **projections**, satisfying the identities

$$
\begin{aligned}
&p_{X,X}\Delta_X = id_X = q_{X,X}\Delta_X && (p_{X,Y}\times q_{X,Y})\Delta_{X\times Y} = id_{X\times Y}\\
&p_{X,Y}(id_X\times p_{Y,Z}) = p_{X,Y\times Z} && p_{X,Z}(id_X\times q_{Y,Z}) = p_{X,Y\times Z}\\
&q_{X,Y}(p_{X,Y}\times id_Z) = q_{X\times Y,Z} && q_{X,Z}(q_{X,Y}\times id_Z) = q_{X\times Y,Z}.
\end{aligned}
$$

Finally we require that the associativity and commutativity isomorphisms α and τ defined as below by

$$
\begin{aligned}
\alpha_{X,Y,Z} &= ((id_X\times p_{Y,Z})\times q_{Y,Z}q_{X,Y\times Z})\Delta_{X\times(Y\times Z)}\\
&: X\times(Y\times Z) \to (X\times Y)\times Z
\end{aligned}
$$

and

$$
\tau_{X,Y} = (q_{X,Y}\times p_{X,Y})\Delta_{X\times Y}\colon X\times Y \to Y\times X
$$

are natural in all variables (the components need not be so).

A p-functor $F\colon \mathbb{C}\to\mathbb{D}$ between p-categories is a functor preserving products up to a natural isomorphism which is coherent with projections and diagonals.

Given a map $f\colon X \to Y$ in the p-category $\mathbb{C}$, the **domain** $\operatorname{dom} f\colon X \to Y$ of f is the composite map $p_{X,Y}(id_X\times f)\Delta_X\colon X\to X$. A map $f\colon X\to Y$ in a p-category $\mathbb{C}$ is **total** if $\operatorname{dom} f = id$. One shows that total maps form a subcategory $\mathbb{C}_t$.

Example 2.2. Let $\mathbf{Ptl}$ denote the category of sets and partial functions. It is a p-category with total category the usual category $\mathbf{Set}$ of sets and total functions.

For all objects X and Y in a p-category $\mathbb{C}$, the maps id_X, $p_{X,Y}$, $q_{X,Y}$, and Δ_X are total. If morphisms $f: X \to Y$ and $g: Y \to Z$ are both total, then their composite $gf: X \to Z$ is total. The subcategory $\mathbb{C}_t$ of total maps has binary cartesian products. A p-functor maps total maps to total maps.

A p-category $\mathbb{C}$ is said to be a **partial cartesian category** if the subcategory $\mathbb{C}_t$ of total maps has a terminal object (and so has finite products). A partial cartesian category $\mathbb{C}$ is also called a p-category with a **one-element** object. The following definition is a weak version of the standard notion of a closed partial cartesian category [25].

Definition 2.3. Let $\mathbb{C}$ be a partial cartesian category. Then we say that $\mathbb{C}$ is **weakly closed** if, for all X and Z in $\mathbb{C}$, there exists an object $[X \rightharpoonup Z]$ and a morphism $e: [X \rightharpoonup Z] \times X \to Z$ in $\mathbb{C}$ such that, for all objects Y, all morphisms $f: Y \times X \to Z$ in $\mathbb{C}$, there exists a (not necessarily unique) map $f': Y \to [X \rightharpoonup Z]$ in $\mathbb{C}_t$ (note that f' is total!), such that the following diagram commutes in $\mathbb{C}$:

$$\begin{tikzcd}
{[X \rightharpoonup Z] \times X} \arrow[r, "e"] & Z \\
Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We refer to $([X \rightharpoonup Z], e)$ as the **weak partial exponential of X and Z**.

For simplicity we refer to a weakly closed partial cartesian category as a WCPC-category.

Example 2.4. The category $\mathbf{Ptl}$ of sets and partial functions is a WCPC-category. (In fact, it is not only weakly closed but closed.)

We may think of a WCPC-category $\mathbb{C}$ as a general universe of realizers as follows. The category $\mathbb{C}$ is a p-category because realizers may be only partially defined. Intuitively, $[X \rightharpoonup Z]$ is a set of realizers of functions from X to Z. There may be more than one realizer for each function from X to Z, hence f' is only required to exist, not to be unique.

Example 2.5. Observe that, trivially, any cartesian closed category is a WCPC-category. Since the category $\mathbf{ALat}$ of algebraic lattices and Scott continuous functions is cartesian closed [7, 9] it is a WCPC-category.

Next, we will show that any partial combinatory algebra gives rise to a WCPC-category, that is *not* necessarily closed.

Let A be a PCA. By an **A-definable (total) function** we mean a function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, $a \cdot b \downarrow$ and $a \cdot b = f(b)$. The set of A-definable functions form a monoid under composition; we refer to this monoid as the **monoid of A-definable functions**. By an **A-definable partial function** we mean a partial function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, we have that $a \cdot b \simeq f(b)$.

Definition 2.6. Let $(A, \cdot)$ be a PCA.

1. We define the category $\mathbb{C}(A)_t$ to be the idempotent splitting of the monoid of A-definable total functions. That is, objects are total A-definable idempotents and a morphism $f: X \to Y$ is an A-definable total function such that $YfX = f$. The identity on X is X itself and composition is composition of total functions.

2. We define the **category** $\mathbb{C}(A)$ **induced by** A to be the category with objects total A-definable idempotents (*i.e.*, same objects as $\mathbb{C}(A)_t$) and with morphisms $f: X \to Y$ A-definable partial functions f satisfying that $YfX = f$ (as partial functions). The identity on X is X and composition is composition of partial functions.

This definition is related to and inspired by the work of Dana Scott [28] and the treatment of C-monoids in [14].

Proposition 2.7. *Let A be a PCA. Then the category $\mathbb{C}(A)$ induced by A is a WCPC-category and its subcategory of total maps is equivalent to $\mathbb{C}(A)_t$.*

See [2] for the proof, which involves some of the usual calculations with idempotents as in [28, 14]. We will often refer to the category $\mathbb{C}(A)$ induced by a PCA A as **the WCPC-category induced by** A, thus implicitly referring to the above proposition.

For a PCA A, we define a p-functor $U: \mathbb{C}(A) \to \mathbf{Ptl}$ as follows. On an object $X \in \mathbb{C}(A)$, we set

$$U(X) = \{\, X(a) \mid a \in A \,\},$$

and for a morphism $f: X \to Y$ in $\mathbb{C}(A)$, we set $U(f)$ to be f (appropriately restricted and corestricted). Note that U indeed is a p-functor; in particular U maps $\mathbb{C}(A)_t$ into $\mathbf{Set}$ and U applied to the terminal object in $\mathbb{C}(A)_t$ is a terminal object in $\mathbf{Set}$.

3. Pretriposes

Definition 3.1. A fibred preorder $p: \mathbb{E} \to \mathbf{Set}$ is called a **pretripos** if it is fibred cartesian closed, that is, each fibre is cartesian closed and reindexing preserves this structure (for $\top$, $\wedge$, $\supset$); it has left and right adjoints $\exists_u \dashv u^* \dashv \forall_u$, which satisfy the Beck-Chevalley condition, along all maps $u: I \to J$ in the base category $\mathbf{Set}$. If, in addition, $p: \mathbb{E} \to \mathbf{Set}$ has fibred finite coproducts (for $\perp$, $\vee$), we say that the fibration is a **pretripos with disjunction**.

The name "pretripos" has been chosen to reflect the fact that such a fibration is almost a tripos [12, 21]; the essential difference is that a pretripos with disjunction is not required to have a weak generic object. A pretripos with disjunction models predicate logic and is a first-order hyperdoctrine in the sense of Lawvere, see [23], and a first-order fibration in the sense of Jacobs [13]. The only difference between a first-order fibration over $\mathbf{Set}$ in the sense of Jacobs [13] and a pretripos with disjunction is that we require products and coproducts along all morphisms, not only projections. As mentioned in [13, 4.3.7, Page 253], for a first-order fibration one may in fact define left and right adjoints to u^* for all maps $u\colon I \to J$, but they will not necessarily satisfy the Beck-Chevalley condition. We require explicitly that these adjoints do satisfy the Beck-Chevalley condition. This extra requirement is used in the proof of Proposition 4.3, it is met in all our applications and, moreover, it makes the connection with the standard definition of tripos [12] very simple: a tripos is a pretripos with disjunction with a weak generic object. Thus there are many examples of pretriposes; in particular, any tripos over $\mathbf{Set}$ is a pretripos with disjunction.

Realizability Pretriposes Let $\mathbb{C}$ be a WCPC-category and $U\colon \mathbb{C} \to \mathbf{Ptl}$ a p-functor. We now show how to define a pretripos $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ over $\mathbf{Set}$ from $\mathbb{C}$ and U. We refer to it as the **realizability pretripos over** $\mathbb{C}$, thus omitting explicit mentioning of U (which we think of as a forgetful functor). The idea is as for realizability triposes over partial combinatory algebras [12], the difference being that here the realizers will be drawn from $\mathbb{C}$ instead of from a PCA. By a realizer we shall mean a morphism f in $\mathbb{C}$. We present the construction in some detail in order to explicate how the defining properties of a WCPC-category are used.

We define a functor $\Psi\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ as follows. For a set I, $\Psi(I)$ is the preorder with objects pairs of the form (A, φ), where A is an object of $\mathbb{C}$ and $\varphi\colon I \to P(UA)$ in $\mathbf{Set}$. The less-than-or-equal relation is denoted $\vdash$ and the objects are preordered by decreeing that $(A, \varphi) \vdash (B, \psi)$ if and only if

$$\begin{array}{l}\exists g \in \mathbb{C}(A, B).\ \forall i \in I.\ \forall a \in \varphi(i).\\ \qquad U(g)(a)\downarrow \text{ and } U(g)(a) \in \psi(i).\end{array}$$

We refer to the objects of $\Psi(I)$ as **predicates on** I or just as **predicates**. When the object A is clear from context we sometimes just write φ for (A, φ). Further, we sometimes write $(A, \varphi)(i)$ for $\varphi(i)$. We refer to the object A in (A, φ) as the **underlying object of realizers for** φ.

For a morphism $u\colon I \to J$, $\Psi(u)$ is essentially composition:

$$\Psi(u)(A, \varphi\colon J \to P(UA)) = (A, \varphi \circ u\colon I \to P(UA)).$$

Note that $\Psi(u)$ is indeed monotone, hence a well-defined functor. Moreover, Ψ is clearly a functor, *i.e.*, Ψ is a split indexed category. The resulting split fibration (obtained by the Grothendieck construction) is written $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$.

The fibred cartesian closed structure is given as follows. In the fibre over I,

$$\top = (1, i \mapsto U1),$$

where 1 is the terminal object in the category of total maps $\mathbb{C}_t$. The object $\top$ is indeed the terminal object in the fibre over I, because for any other object $(A, \varphi\colon I \to P(UA))$, there exists a map $g\colon A \to 1$ in $\mathbb{C}_t$ which U maps to a total function $U(g)$ so that g realizes $(A, \varphi) \vdash \top$.

For predicates (A, φ) and (B, ψ) over I, define $(A, \varphi) \wedge (B, \psi)$ to be

$$(A \times B, i \mapsto \{\, (a, b) \mid a \in \varphi(i) \text{ and } b \in \psi(i) \,\}).$$

Note that this definition makes sense because U preserves products. It is straightforward to see that $\wedge$ so defined gives binary products in the fibre over I.

For exponentials (implication), we define $(A, \varphi) \supset (B, \psi)$ to be

$$\begin{array}{l}([A \rightharpoonup B],\\ \; i \mapsto \{\, g \in U[A \rightharpoonup B] \mid \forall a \in \varphi(i).\ U(e)(g, a)\downarrow \text{ and}\\ \qquad\qquad\qquad\qquad U(e)(g, a) \in \psi(i) \,\}),\end{array}$$

where $([A \rightharpoonup B], e)$ is the weak partial exponential of A and B. (Note that this definition makes sense because U preserves products.) Let us verify the adjunction:

$$\frac{(A, \varphi_1) \wedge (B, \varphi_2) \vdash (C, \varphi_3)}{(A, \varphi_1) \vdash ((B, \varphi_2) \supset (C, \varphi_3))}$$

Suppose $(A, \varphi_1) \wedge (B, \varphi_2) \vdash (C, \varphi_3)$ via realizer $h\colon A \times B \to C$. Then there exists an $h'\colon A \to [B \rightharpoonup C]$ in $\mathbb{C}_t$ such that the diagram

$$\begin{tikzcd} {[B \rightharpoonup C] \times B} \arrow[dr, "e"] & \\ A \times B \arrow[u, "h' \times id_B"] \arrow[r, "h"'] & C, \end{tikzcd}$$

commutes, where $([B \rightharpoonup C], e)$ is the weak exponential of B and C. We claim that h' is a realizer for $(A, \varphi_1) \vdash (B, \varphi_2) \supset (C, \varphi_3)$. Indeed, let $i \in I$ be arbitrary and let $a \in \varphi_1(i)$ be arbitrary. Note that $U(h')(a)$ is defined because $U(h')$ is total since h' is so. It remains to show that $U(h')(a) \in ((B, \varphi_2) \supset (C, \varphi_3))(i)$. To this end, let $b \in \varphi_2(i)$ be arbitrary. Using that U preserves products, one easily verifies that $U(e)(U(h')(a), b) = U(h)(a, b)$, which is in $\varphi_3(i)$ by assumption. The other direction is easy.

It is easy to verify that, for any $u\colon I \to J$ in the base category $\mathbf{Set}$, the reindexing functor u^* preserves the fibred cartesian closed structure on the nose. In summa, we now have that $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ is split fibred cartesian closed.

Let $u\colon I \to J \in \mathbf{Set}$ and suppose (A, φ) is a predicate in the fibre over I. We then define

$$\exists_u(A, \varphi) = (A, j \mapsto \bigcup_{i \in I} \{ \varphi(i) \mid u(i) = j \}).$$

(Note that in the typical case, where $u = \pi'\colon I \times J \to J$, then $\exists_u(A, \varphi)$ simplifies to $(A, j \mapsto \bigcup_{i \in I} \varphi(i, j))$.)

It is easy to verify that $\exists_u \dashv u^*$ and that the Beck-Chevalley condition holds.

We define $\forall_u(A, \varphi)$ to be

$$\left([1 \rightharpoonup A], j \mapsto \bigcap_{i \in I} \big((u(i) =_J j) \supset \varphi(i) \big) \right),$$

where

- $[1 \rightharpoonup A]$ is the weak partial exponential of the terminal object 1 in $\mathbb{C}_t$ and A
- $\big(u(i) =_J j\big) = \bigcup \{ U1 \mid u(i) = j \}$
- $\big((u(i) =_J j) \supset \varphi(i)\big)$ equals $\{ g \in U[1 \rightharpoonup A] \mid \forall b \in (u(i) =_J j).\ U(e)(g, b)\downarrow \text{ and } U(e)(g, b) \in \varphi(i) \}$

(Note that in the typical case, where $I \neq \emptyset$ and $u = \pi'\colon I \times J \to J$, then $\forall_u(A, \varphi)$ simplifies to $(A, j \mapsto \bigcap_{i \in I} \varphi(i, j))$.) We omit the verification of adjointness; Beck-Chevalley holds since it holds for $\exists$.

Summarizing we have the following theorem.

Theorem 3.2. *Let $\mathbb{C}$ be a WCPC-category. Then $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ is a split pretripos.*

For a WCPC-category $\mathbb{C}$ and a morphism $U\colon \mathbb{C} \to \mathbf{Ptl}$ of WCPC-categories, we refer to the induced pretripos $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ as the **realizability pretripos over** $\mathbb{C}$.

We now present our two main example of realizability pretriposes.

Example 3.3. For the category $\mathbf{ALat}$ of algebraic lattices, we define $U\colon \mathbf{ALat} \to \mathbf{Ptl}$ to be the composition of the forgetful functor from $\mathbf{ALat}$ to $\mathbf{Set}$ and the inclusion functor from $\mathbf{Set}$ into $\mathbf{Ptl}$. When we refer to the realizability pretripos $\mathbf{UFam}(\mathbf{ALat}) \to \mathbf{Set}$ over $\mathbf{ALat}$ it is always understood that we refer to this functor $U\colon \mathbf{ALat} \to \mathbf{Ptl}$ just defined.

Example 3.4. By Proposition 2.7 we know that a PCA A generates a WCPC-category. When we refer to the realizability pretripos $\mathbf{UFam}(\mathbb{C}(A)) \to \mathbf{Set}$ over the WCPC-category induced by A, the functor $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ is always assumed to be the functor U from Page 3. In this case, the realizability pretripos $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}(A)) \to \mathbf{Set}$ is equivalent as a preorder fibration over $\mathbf{Set}$ to the standard realizability tripos $\mathbf{UFam}(PA) \to \mathbf{Set}$ over the partial combinatory algebra A (as defined in [12]). In other words, there are fibred functors $F\colon \mathbf{UFam}(\mathbb{C}(A)) \to \mathbf{UFam}(PA)$ and $G\colon \mathbf{UFam}(PA) \to \mathbf{UFam}(\mathbb{C}(A))$ over $\mathbf{Set}$ such that $FG \cong id$ and $GF \cong id$, both vertically. Over 1, the functor F is defined by

$$F(X, \varphi \in P(UX)) = \varphi$$

(recall that $UX = \{ X(a) \mid a \in A \} \subseteq A$, so φ can indeed be viewed as a subset of A and the definition thus makes sense). Over 1, the functor G is defined by

$$G(\psi) = (id_A, \psi).$$

The isomorphism $G(F(X, \varphi)) \cong (X, \varphi)$ is realized by X, which is both a morphism $X \to id_A$ and a morphism $id_A \to X$ in $\mathbb{C}(A)$. The isomorphisms $F(G(\psi)) \cong \psi$ is realized by id_A (in both directions).

The preceding example is a special case of the following observation. Let $\mathbb{C}$ be a category and let $U\colon \mathbb{C} \to \mathbf{Ptl}$ be a functor. Define $\mathrm{Split}(\mathbb{C}, U)$ to be the category with objects the idempotents of $\mathbb{C}$ that are mapped by U into $\mathbf{Set}$ and with morphisms $f\colon X \to Y$ morphisms f in $\mathbb{C}$ satisfying that $YfX = f$. The identity on X is X itself and composition of morphisms is composition in $\mathbb{C}$. The functor U induces a functor[1] $\mathrm{Split}(U)\colon \mathrm{Split}(\mathbb{C}, U) \to \mathbf{Ptl}$ defined by setting $\mathrm{Split}(U)(X)$ equal to the image of $U(X)$ for an object X and by setting $\mathrm{Split}(U)(f) = f$ for a morphism f. Observe that $\mathbb{C}(A)$ is $\mathrm{Split}(\mathrm{M}(A), U_0)$ for $\mathrm{M}(A)$ the monoid of A-definable partial functions and $U_0\colon \mathrm{M}(A) \to \mathbf{Ptl}$ the inclusion functor, and that $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ as defined on Page 3 is $\mathrm{Split}(U_0)$.

Proposition 3.5. *Let $\mathbb{C}$ be a category and let $U\colon \mathbb{C} \to \mathbf{Ptl}$ be a functor. Then $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ is equivalent, as a preorder fibration, to $\mathbf{UFam}(\mathrm{Split}(\mathbb{C})) \to \mathbf{Set}$.*

The proof is essentially as in the example above.

Definition 3.6. Let $\mathbb{C}$ be a p-category. Then we say that $\mathbb{C}$ has a **weak initial object** if it has a weak initial object in the traditional sense, that is, if there exists an object 0 in $\mathbb{C}$ such that for all objects $X \in \mathbb{C}$, there exists a morphism $f\colon 0 \to X$ in $\mathbb{C}$. We further say that $\mathbb{C}$ has **weak binary coproducts** if, for any pair of objects X and Y in $\mathbb{C}$, there exists an object $X + Y$ in $\mathbb{C}$ and a diagram $X \xrightarrow{\kappa} X + Y \xleftarrow{\kappa'} Y$ *in* $\mathbb{C}_t$ such that, for all diagrams $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathbb{C}$, there exists a morphism $u\colon X + Y \to Z$ in $\mathbb{C}$ such that

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[dr, "f"'] & X + Y \arrow[d, "u"] & Y \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes in $\mathbb{C}$. We refer to the morphisms κ and κ' as **injections**. Note that they are required to be total. We say that a p-category has **weak finite coproducts** if it has a weak initial object and weak binary coproducts.

[1] The functor $\mathrm{Split}(U)$ really arises because $\mathrm{Split}(-)$ in a suitable sense is a 2-functor and $\mathrm{Split}(\mathbf{Ptl}, id)$ is equivalent to $\mathbf{Ptl}$.

In case a WCPC-category $\mathbb{C}$ has weak finite coproducts, then the weak realizability tripos over $\mathbb{C}$ has disjunction:

Theorem 3.7. *Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts. Then $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ is a split pretripos with disjunction.*

One can show that both of our main examples, the WCPC-category $\mathbb{C}(A)$ induced by a PCA A and the category of algebraic lattices $\mathbf{ALat}$ have weak finite coproducts. The category $\mathbf{ALat}$ does *not* have true finite coproducts because it is cartesian closed and it has the fixed point property (*i.e.*, every endomorphism has a fixed point), see [11]. Thus, to get the desired Corollary (3.8), it really is important that Theorem 3.7 only requires *weak* coproducts.

Corollary 3.8. *The realizability pretriposes over the WCPC-category induced by a PCA and over the WCPC-category $\mathbf{ALat}$ both have split disjunction.*

Realizability Pretriposes and Universal Objects Example 3.4 is an instance of a more general phenomenon, which we now describe. Recall that a category $\mathbb{C}$ has a **universal object** V if all objects X in $\mathbb{C}$ are retracts of V. Observe that $\mathbb{C}(A)$ has a universal object, namely id_A (as implicitly explained in Example 3.4). We can now show that, when U is faithful, the realizability pretripos over such a category $\mathbb{C}$ has a weak generic object just in case $\mathbb{C}$ has a universal object.

Theorem 3.9. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor. Then $\mathbb{C}$ has a universal object if and only if $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ has a weak generic object.*

Proof. Suppose $\mathbb{C}$ has a universal object V. We then claim that

$$(V, id\colon\ P(UV) \to P(UV))$$

is a weak generic object for $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$. To show the claim, suppose that $(A, \varphi\colon I \to PUA)$ is any predicate in the fibre over I. Then by the assumption on $\mathbb{C}$, the object A is a retract of V, that is, there are morphisms r and s in $\mathbb{C}$

$$A \underset{s}{\overset{r}{\leftrightarrows}} V$$

such that $rs = id_A$. Since $U(rs) = U(r)U(s) = U(id) = id$, we have that $U(s)$ is a total function. Thus $P(Us)$ is a morphism from $P(UA)$ to $P(UV)$ in $\mathbf{Set}$. It is now straightforward to show that $(A, \varphi) \cong (P(Us) \circ \varphi)^*(V, id)$ in the fibre over I, thus completing the proof of the claim.

For the other direction suppose that $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ has a weak generic object $(W, \psi\colon V \to P(UW)$, over some $V \in \mathbf{Set}$. Let $A \in \mathbb{C}$ be arbitrary. We show that A is a retract of W to conclude that W is a universal object in $\mathbb{C}$. Consider the predicate (A, φ) over $U(A)$, with $\varphi\colon U(A) \to P(UA)$ the function $a \mapsto \{a\}$. Since (W, ψ) is a weak generic object, there exists a morphism $f\colon U(A) \to V$ in $\mathbf{Set}$, such that

$$(A, \varphi) \cong f^*(W, \psi)$$

in the fibre over UA. Hence there are morphisms $s\colon A \to W$ and $r\colon W \to A$ in $\mathbb{C}$ such that

$$\forall a \in UA.\ \forall b \in \varphi(a).\ U(s)(b)\downarrow \text{ and } U(s)(b) \in \psi(f(a))$$

and

$$\forall a \in UA.\ \forall b \in \psi(f(a)).\ U(r)(b)\downarrow \text{ and } U(r)(b) \in \varphi(a).$$

Recalling that $\varphi(a) = \{a\}$ we get that

$$\forall a \in UA.\ U(r)(U(s)(a))\downarrow \text{ and } U(r)(U(s)(a)) \in \varphi(a) = \{a\}$$

so

$$\forall a \in UA.\ U(rs)(a) = a$$

and thus $U(rs) = id_{UA}$. Since U is faithful by assumption, we conclude that $rs = id_A$ in $\mathbb{C}$, thus completing the proof that A is a retract of W. □

Corollary 3.10. *Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor. Then $\mathbb{C}$ has a universal object if and only if $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ is a tripos.*

Proof. Immediate from Theorems 3.9 and 3.7. □

In fact, we can strengthen the above results to characterize exactly when the tripos-to-topos construction applied to $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ yields a topos. To do so we apply a result of Pitts who characterized (in 1982) when the tripos-to-topos construction applied to a pretripos with disjunction yields a topos [23]. For $\mathsf{p}\colon \mathbb{E} \to \mathbb{B}$ a pretripos, we denote by $\mathbb{B}[\mathsf{p}]$ the category obtained by applying the tripos-to-topos construction to p. Pitts showed in particular that the tripos-to-topos construction applied to the realizabilty pretripos $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ (for $\mathbb{C}$ a WCPC-category) yields a topos iff the following axiom holds:[2]

Axiom 3.11 (CA). *For all sets X there is a set PX and a predicate $In_X \in \mathbf{UFam}(\mathbb{C})_{X\times PX}$ such that, for any set I and $R \in \mathbf{UFam}(\mathbb{C})_{X\times I}$, the sentence*

$$\forall i\colon I.\ \exists s\colon PX.\ \forall x\colon X.\ \mathrm{In}_X(x, s) \supset\subset R(x, i)$$

is satisfied in $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$.

Theorem 3.12. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor. Then $\mathbb{C}$ has a universal object if and only if $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ satisfies the axiom (CA).*

[2] In fact, Pitts assumed that the pretripos has disjunction, but going through his proof one sees that disjunctions are not needed.

Proof. By Theorem 3.9, if $\mathbb{C}$ has a universal object, then $\mathbf{UFam}(\mathbb{C}) \to \mathbb{C}$ is a tripos and thus it also satisfies the axiom (CA).

For the other direction, suppose that $\mathbf{UFam}(\mathbb{C}) \to \mathbb{C}$ satisfies the axiom (CA). Let $X = 1$ and let Σ denote the object PX in (CA). Let $V \in \mathbb{C}$ be the underlying object of $\mathrm{In} = \mathrm{In}_1$, which exists *qua* (CA). We show that V is a universal object in $\mathbb{C}$. Let $A \in \mathbb{C}$ be arbitrary, let $I = UA$ and let $R(a) = \{a\}$. By (CA) the following sentence is valid in the realizability pretripos:

$$\forall a\colon UA.\ \exists s\colon \Sigma.\ \mathrm{In}(s) \asymp R(a).$$

Thus there is a realizer in

$$\bigcap_{a \in UA} \left(\bigcup_{s \in \Sigma} \mathrm{In}(s) \asymp R(a) \right) \subseteq U([V \rightharpoonup A] \times [A \rightharpoonup V]).$$

It follows, as in the proof of Theorem 3.9, that A is a retract of V. $\square$

Corollary 3.13. *For $\mathbb{C}$ be a WCPC-category and $U : \mathbb{C} \to \mathbf{Ptl}$ a faithful p-functor, the tripos-to-topos construction applied to $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ yields a topos iff $\mathbb{C}$ has a universal object.*

A further corollary is that the tripos-to-topos construction applied to the realizability tripos induced by the category of algebraic lattices does not produce a topos, since clearly $\mathbf{ALat}$ does not have a universal object (for cardinality reasons).

4. Assemblies

In this section we define a category of assemblies over a suitable fibration and then show that if the fibration is a pretripos, then the induced category of assemblies is locally cartesian closed, and if it is a realizability pretripos, then we get a split model of dependent predicate logic. We give the definition of the category of assemblies for any regular fibration (our definition appears to be new) and show that under mild conditions the resulting category of assemblies is regular. We hope that proceeding in stages like this will help to highlight which properties of the fibration are needed to establish properties of the category of assemblies.

Assemblies over Regular Fibrations A **regular fibration** [13] is a fibred preorder $\mathbb{E} \to \mathbb{B}$ whose base category $\mathbb{B}$ has finite products and which has fibred finite products (for $\top$ and $\wedge$); fibred equality ($Eq_{I,J} \dashv \delta(I,J)^*$) satisfying Frobenius (for $=$); and simple coproducts ($\exists_{(I,J)} \dashv \pi^*_{I,J}$) satisfying Frobenius (for $\exists$) and Beck-Chevalley. See [13] for more details. A regular fibration models **regular logic**, the fragment of first-order (intuitionistic) logic using only $=$, $\wedge$, $\top$, and $\exists$. Note that a pretripos is a regular fibration (since it has left adjoints along all maps in the base satisfying Beck-Chevalley; Frobenius holds since a pretripos has fibred exponentials).

For any regular fibration, we may define a category of assemblies over it as follows. For the pretripos over $\mathbf{ALat}$ the construction specializes to assemblies over algebraic lattices as in [1] and for the tripos over a PCA A it specializes to the usual category of assemblies $\mathbf{Asm}(A)$. The definition below is phrased using the internal language of a regular fibration; in the remark following the definition we recall what those logical definitions mean in categorical terms.

Definition 4.1. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a regular fibration. Define $\mathbf{Asm}(p)$ to be full subcategory of $\mathbb{E}$ on the objects E satisfying that, for all global elements $c\colon 1 \to pE$, in $\mathbb{B}$,

$$\emptyset \mid \emptyset \vdash E(c)$$

is valid in the logic of p.

Remark 4.2. The condition on objects in the above definition can be expressed categorically as follows. For all global elements $c\colon 1 \to pE$ in $\mathbb{B}$, $\top_1 \leq_1 c^*E$ in the fibre $\mathbb{E}_1$ over 1. We often write an objects E in $\mathbf{Asm}(p)$ as (X, E) with $X = pE$ to explicate the fibre of E. Note that a morphism from (X, E_X) to (Y, E_Y) is a morphism $f\colon X \to Y$ in $\mathbb{B}$ for which

$$x\colon X \mid E(x) \vdash E'(f(x))$$

is valid in the logic of p.

Proposition 4.3. *Let $p\colon \mathbb{E} \to \mathbb{B}$ be a regular fibration. If $\mathbb{B}$ is regular, 1 is regular projective in $\mathbb{B}$, and p has coproducts ($\exists_u \dashv u^*$) along all maps in the base category, satisfying the Frobenius property, then $\mathbf{Asm}(p)$ is a regular category.*

Proof Sketch. The terminal object is $(1_{\mathbb{B}}, T_{1_{\mathbb{B}}})$. The product of (X, E_X) and (Y, E_Y) is $(X \times Y, E)$ where

$$(x, y)\colon X \times Y \vdash E(x, y) \stackrel{\mathrm{def}}{=} E_X(x) \wedge E_Y(y)$$

with projections π and π' the projections of $X \times Y$. Categorically, E is $\pi^*(E_X) \wedge \pi'^*(E_Y)$. The equalizer of $f, g\colon (X, E_X) \to (Y, E_Y)$ is (Z, E_Z), where

$$\begin{tikzcd} Z \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is an equalizer diagram in $\mathbb{B}$, and E_Z is defined as follows:

$$z\colon Z \vdash E_Z(z) \stackrel{\mathrm{def}}{=} E_X(m(z)).$$

Categorically, $E_Z = m^* E_X$. The image factorization of

$$f\colon (X, E_X) \to (Y, E_Y)$$

in $\mathbf{Asm}(p)$ is obtained as follows. Since $f\colon X \to Y$ in $\mathbb{B}$ and $\mathbb{B}$ is regular, we can consider the image factorization $X \overset{e}{\twoheadrightarrow} W \overset{m}{\rightarrowtail} Y$ of f in $\mathbb{B}$. Define

$$w\colon W \vdash E_W(w) \stackrel{\mathrm{def}}{=} \exists x\colon X.\ e(x) = w \wedge E_X(x).$$

Categorically, $E_W = \exists_e E_X$. One can then show that (W, E_W) is an object of $\mathbf{Asm}(p)$ and that $(X, E_X) \xrightarrow{e} (W, E_W) \xrightarrow{m} (Y, E_Y)$ forms a stable image factorization of f in $\mathbf{Asm}(p)$. (Frobenius and Beck-Chevalley for $\exists_u$ along any map u in the base is used for stability.) □

Assemblies over Pretriposes

Theorem 4.4. *Let $p\colon \mathbb{E} \to \mathbf{Set}$ be a pretripos. Then $\mathbf{Asm}(p)$ is locally cartesian closed.*

Proof Sketch. By Proposition 4.3 we already know that $\mathbf{Asm}(p)$ is regular, so we only need to consider local closure. It suffices to show, see, *e.g.*, [13, Proposition 1.9.8], that for each morphism $u\colon (I, E_I) \to (J, E_J)$, the pullback functor

$$u^*\colon \mathbf{Asm}(p)/(J, E_J) \to \mathbf{Asm}(p)/(I, E_I)$$

has a right adjoint $\prod_u$. We define $\prod_u$ as follows. Let $\varphi\colon (X, E_X) \to (I, E_I)$ be an object over (I, E_I). Then set $\prod_u(\varphi) \stackrel{\text{def}}{=} \psi$, where the domain of ψ is

$$\Big(\coprod_{j\in J} \{ f\colon \{ i \in I \mid u(i) = j \} \to X \mid \varphi(f(i)) = i \text{ and } E'_j(f) \text{ valid in } p \}, E \Big),$$

and

$$\begin{aligned} E'_j(f) &\stackrel{\text{def}}{=} \forall i\colon \{ i \in I \mid u(i) = j \}.\ E_I(i) \supset E_X(f(i)) \\ E(j, f) &\stackrel{\text{def}}{=} E_J(j) \wedge E'_j(f) \\ \psi(j, f) &\stackrel{\text{def}}{=} j. \end{aligned}$$

□

Assemblies over Realizability Pretriposes Let $\mathbb{C}$ be a WCPC-category and let $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ be the realizability pretripos over $\mathbb{C}$. Note that an object in $\mathbf{Asm}(\mathsf{p})$ is of the form $(X, (A, E))$, where X is an object in $\mathbf{Set}$ and (A, E) is an object in $\mathbf{UFam}(\mathbb{C})_X$, that is, A is an object in $\mathbb{C}$ and E is a function $X \to P(UA)$. We often write objects $(X, (A, E)) \in \mathbf{Asm}(\mathsf{p})$ as triples (X, A, E), leaving out a pair of parentheses.

Note that the condition on objects (X, A, E) in $\mathbf{Asm}(\mathsf{p})$ regarding global elements now simplifies to the familiar [17, 1] condition $\forall x \in X.\ E(x) \neq \emptyset$. Indeed the following two examples show how our setup specializes to the cases of assemblies over PCA's and over algebraic lattices.

Example 4.5. Let A be a PCA. Recall from Example 3.4 that the induced realizability pretripos $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}(A)) \to \mathbf{Set}$ is equivalent to the standard realizability tripos over a A. Since the first-order structure is defined categorically, it is preserved by the equivalence functors. From the preceeding it is easy to verify that the category $\mathbf{Asm}(\mathsf{p})$ of assemblies over p is equivalent to the usual category of assemblies $\mathbf{Asm}(A)$ over A.

Example 4.6. Let $\mathsf{p}\colon \mathbf{UFam}(\mathbf{ALat}) \to \mathbf{Set}$ be the realizability pretripos over $\mathbf{ALat}$. Then $\mathbf{Asm}(\mathsf{p})$ is equivalent to the category of assemblies over $\mathbf{ALat}$ defined in [1].

For $\mathbb{C}$ a WCPC-category, we often write $\mathbf{Asm}(\mathbb{C})$ for $\mathbf{Asm}(\mathsf{p})$ where p is the realizability pretripos over $\mathbb{C}$. (This choice of notation is in accordance with [1], where we write $\mathbf{Asm}(\mathbf{ALat})$ for the category of assemblies over $\mathbf{ALat}$.)

We now proceed to show how to define a split closed comprehension category [13] for the category of assemblies over a realizability pretripos, thus obtaining a model of dependent type theory. Our definitions and results are generalizations of corresponding results for assemblies over partial combinatory algebras and assemblies over algebraic lattices, see [13] and [1].

For the remainder of this section, let $\mathbb{C}$ be a WCPC-category and let $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ be the realizability pretripos over $\mathbb{C}$.

Definition 4.7. Define $\mathbf{UFam}(\mathbf{Asm}(\mathsf{p}))$ to be the category with objects triples $(I, A, (X_i, E_i)_{i\in X_I})$, where

$$I = (X_I, A_I, E_I) \in \mathbf{Asm}(\mathsf{p}),$$

and $(X_i, A, E_i) \in \mathbf{Asm}(\mathsf{p})$, for all $i \in X_I$. Morphisms $(I, A, (X_i, E_i)_{i\in X_I}) \to (J, B, (Y_j, E'_j)_{j\in X_J})$, with

$$I = (X_I, A_I, E_I) \qquad \text{and} \qquad J = (X_J, A_J, E_J),$$

are pairs $(f, (f_i)_{i\in X_I})$ such that $f\colon I \to J$ in $\mathbf{Asm}(\mathsf{p})$ and such that

$$\emptyset \mid \emptyset \vdash \forall i\colon X_I.\ \forall x\colon X_i.\ E_I(i) \supset (E_i(x) \supset E'_{f(i)}(f_i(x))) \tag{1}$$

is valid in the logic of p. The identity on object $(I, A, (X_i, E_i)_{i\in X_I})$ with $I = (X_I, A_I, E_I)$, is $(id, (id_i)_{i\in X_I})$. The composition of $(f, (f_i)_{i\in X_I})$ and $(g, (g_j)_{j\in X_J})$ is $(g \circ f, (g_{f(i)} \circ f_i)_{i\in X_I})$.

Remark 4.8. Categorically, the second quantifier in (1) is given as $\forall_\pi$ with π the projection $\coprod_{i\in X_I} X_i \to X_I$ in $\mathbf{Set}$. The $\mathbf{U}$ in $\mathbf{UFam}(\mathbf{Asm}(\mathsf{p}))$ refers to the fact that a family $(I, A, (X_i, E_i)_{i\in X_I})$ over an object I is *uniform* in the sense that all the existence predicates E_i have the *same* underlying object $A \in \mathbb{C}$ as object of realizers.

Proposition 4.9. *Taking the forgetful functor from $\mathbf{UFam}(\mathbf{Asm}(\mathsf{p}))$ to $\mathbf{Asm}(\mathsf{p})$ given by $(I, A, (X_i, E_i)_{i\in X_I}) \mapsto I$ and $(f, (f_i)_{i\in X_I}) \mapsto f$ is a split fibration which is equivalent, as a fibration, to the codomain fibration over $\mathbf{Asm}(\mathsf{p})$.*

Proof Sketch. Define the functor $\mathcal{P}$ as in

$$\begin{tikzcd} \mathbf{UFam}(\mathbf{Asm}(\mathsf{p})) \arrow[rr, "\mathcal{P}"] \arrow[dr] & & \mathbf{Asm}(\mathsf{p})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Asm}(\mathsf{p}) & \end{tikzcd}$$

by mapping an object $(I, A, (X_i, E_i)_{i\in X_I})$, with $I = (X_I, A_I, E_I)$, to

$$(\coprod_{i\in X_I} X_i, A_I \times A, E) \xrightarrow{\pi} I,$$

with

$$(i,x)\colon \coprod_{i\in X_I} X_i \vdash E(i,x) \stackrel{\mathrm{def}}{=} E_I(i) \wedge E_i(x).$$

One can then verify that $\mathcal{P}$ establishes the postulated equivalence. $\square$

See [2] for the proofs of the following two theorems.

Theorem 4.10. *The functor* $\mathcal{P}\colon \mathbf{UFam}(\mathbf{Asm}(\mathsf{p})) \to \mathbf{Asm}(\mathsf{p})^{\to}$ *is a split closed comprehension category. Hence, we have a split model of dependent type theory.*

Theorem 4.11. *Let* $\mathbb{C}$ *be a WCPC-category with weak finite coproducts and let* $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ *be the induced realizability pretripos with disjunction. Then* $\mathbf{Asm}(\mathsf{p})$ *has stable and disjoint finite coproducts.*

It follows by Proposition 4.3 and Theorems 4.4 and 4.11 that, for $\mathbb{C}$ a WCPC-category with weak finite coproducts, $\mathbf{Asm}(\mathbb{C})$ is a logos and hence, by Theorem 4.10, that it models dependent predicate logic, that is,

$$\begin{tikzcd} \mathrm{Sub}(\mathbf{Asm}(\mathbb{C})) \arrow[dr] & \mathbf{UFam}(\mathbf{Asm}(\mathbb{C})) \arrow[r, "\mathcal{P}"] \arrow[d] & \mathbf{Asm}(\mathbb{C})^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathbf{Asm}(\mathbb{C}) & \end{tikzcd}$$

forms a DPL-structure in the sense of Jacobs [13, Chapter 11] (except that there is no generic object since the logic is not fully higher-order). In fact, one can also model dependent subset types and dependent quotient types, see [2].

Remark 4.12. The category $\mathbf{Mod}(\mathbb{C})$ is naturally defined to be the full subcategory of $\mathbf{Asm}(\mathbb{C})$ on those objects (X, A, E) for which $E(x) \cap E(x') \neq \emptyset \implies x = x'$, for all $x, x' \in X$. This category also provides a model of dependent predicate logic; in the case where $\mathbb{C} = \mathbf{ALat}$, it specializes to the model for $\mathbf{Equ}$ [1].

5. Related Work and Concluding Remarks

In our work with Carboni, Rosolini and Scott [3] we have developed a complementary approach to a general notion of realizability for type theory, based on the theory of exact categories and exact completions. The approach in [3], see also [6], is a generalization of "the exact-completion approach to realizability toposes" as found in [26], see also [4], whereas the approach in the present paper is a generalization of "the tripos-theoretic approach to realizability toposes" as found in [12, 21].

We briefly relate the approach in [3] to the approach in this paper. Recall the following definition from [6] — it is slightly different from a related definition in [3], which is geared more towards situations where all realizers are *total*, and which therefore does not encompass standard realizability over PCA's.

Definition 5.1. The category $\mathcal{F}(\mathbb{C})$ is the category with objects triples (X, A, σ), where $X \in \mathbf{Set}$, $A \in \mathbb{C}$, and $\sigma\colon X \to U(A)$ in $\mathbf{Set}$. Morphisms $f\colon (X, A, \sigma) \to (Y, B, \tau)$ are functions $f\colon X \to Y$ in $\mathbf{Set}$ such that there exists a $g\colon A \to B$ in $\mathbb{C}$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"] \\ U(A) \arrow[r, "U(g)"'] & U(B) \end{tikzcd}$$

commutes in $\mathbf{Ptl}$ (note that only $U(g)$ may be partial).

Let A be a PCA and let $\mathbb{C}(A)$ be the WCPC-category induced by A. Then $\mathcal{F}(\mathbb{C}(A))$ is equivalent to the category of *partitioned assemblies* over A, as defined in [5, 26].

It is straightforward to show that, for $\mathbb{C}$ a WCPC-category and $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a p-functor, $\mathcal{F}(\mathbb{C})$ has all finite limits. The following is an easy generalization of [4, Lemma 6.1].

Proposition 5.2. *The category* $\mathbf{Asm}(\mathbb{C})$ *is equivalent to* $(\mathcal{F}(\mathbb{C}))_{\mathit{reg/lex}}$.

It follows that, for $\mathbb{C}$ a WCPC-category, $U\colon \mathbb{C} \to \mathbf{Ptl}$ a p-functor, and $\mathsf{p}\colon \mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ the induced realizability pretripos, the categories $\mathbf{Set}[\mathsf{p}]$, $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$, and $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$ are all equivalent. Hence we have the following corollary:

Corollary 5.3. *If* $\mathbb{C}$ *is a WCPC-category and* $U\colon \mathbb{C} \to \mathbf{Ptl}$ *is a faithful p-functor, then* $(\mathcal{F}(\mathbb{C}))_{\mathit{ex/lex}}$ *is a topos iff* $\mathbb{C}$ *has a universal object.*

In a recent manuscript [8], de Marchi, Robinson and Rosolini has analyzed the $\mathcal{F}$-construction in more detail, one goal being to investigate exactly what conditions on $\mathbb{C}$ are forced for $\mathcal{F}(\mathbb{C})$ to be a topos (*i.e.*, they do not a priori assume that $\mathbb{C}$ is WCPC-category). What they find is that $\mathbb{C}$ *essentially* has to be a WCPC-category (there is a mild relaxation on the underlying p-category structure) with a universal object. They show that this means that $\mathbb{C}$ is essentially equivalent to $\mathbb{C}(A)$ for some PCA A (obtained from the universal object).

Independently of the work reported here, John Longley has recently suggested another general framework encompassing both typed and untyped realizability [16]. Longley's definition of a *partial combinatory type structure* (PCTS) is essentially a typed version of a PCA, with $\mathbf{K}$ and $\mathbf{S}$ combinator for all appropriate types. For any such PCTS, Longley then defines a category of assemblies and shows that it is regular and locally cartesian closed. We leave a detailed analysis of the relationship between Longley's notion of PCTS and our notion of WCPC-category and the induced notions of categories of assemblies for future work.

Peter Lietz and Thomas Streicher [15] have, in the context of Longley's PCTS's, shown some results closely related to our Theorems 3.9 and 3.12 characterizing when the tripos-to-topos construction applied to the realizability tripos over a WCPC-category yields a topos. In particular, Lietz and Streicher show that a PCTS has a universal type iff it gives rise to a topos and iff it is equivalent (in a suitable sense) to a PCA (obtained from the universal object). This result in fact indicates that our framework of WCPC-categories may be slightly more general than Longley's PCTS's because one can define a WCPC-category which gives rise to a topos, which is *not* a standard realizability topos over a PCA [2, Remark 6.0.4].

There are many interesting questions for future work. One of the most interesting is "what is a suitable notion of morphism between WCPC-categories $\mathbb{C}$ and $\mathbb{D}$?" Such a morphism should induce morphisms between the induced realizability pretriposes over $\mathbb{C}$ and $\mathbb{D}$ (and then functors between the induced categories of assemblies over $\mathbb{C}$ and $\mathbb{D}$). The right notion should include Longley's notion of applicative transformation between partial combinatory algebras [17]. We believe that a good notion of morphism from $\mathbb{C}$ to $\mathbb{D}$ would be a profunctor from $\mathbb{C}$ to $\mathbb{D}$, possibly satisfying some extra conditions. The result should be a (bi)category of WCPC-categories, generalizing Longley's category of partial combinatory algebras and applicative transformations [17].

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